

Optimal Consumption and Portfolio Choice with Dynamic Labor Income

DISSERTATION
of the University of St. Gallen,
School of Management,
Economics, Law, Social Sciences
and International Affairs
to obtain the title of
Doctor of Economics

submitted by

Daniel Moos

from

Bettwil (Aargau)

Approved on the application of

Prof. Dr. Heinz Müller

and

Prof. Paul Söderlind, PhD

Dissertation no. 3919

Difo-Druck GmbH, Bamberg 2011

The University of St. Gallen, School of Management, Economics, Law, Social Sciences and International Affairs hereby consents to the printing of the present dissertation, without hereby expressing any opinion on the views herein expressed.

St. Gallen, May 16, 2011

The President:

Prof. Dr. Thomas Bieger

Preface

First and foremost I would like to express my sincere gratitude to my academic advisor Prof. Dr. Heinz Müller. He gave me all the time and flexibility to complete my thesis and provided me the necessary guidance. I also want to thank Prof. Paul Söderlind, PhD for his willingness to be my co-advisor.

I thank Dr. David Schiess for proofreading some parts of the thesis. For the careful English correction, many thanks go to Jack Corrigall. Clearly, all mistakes remain my own.

Generally, I am grateful to everybody who accompanied me during the years working on my thesis. I would like to express explicitly my gratitude to everybody at the “Fachbereich für Mathematik & Statistik” and my colleagues during my time at Kraus Partner Investment Solutions AG.

Finally, I would like to thank my family and close friends for bringing me joy and happiness and their support during the past years of completing my thesis.

St. Gallen, June 2011

Daniel Moos

Contents

Preface	i
Abstract	vi
Zusammenfassung	vii
List of Figures	x
List of Tables	xi
Notation	xiii
1 Introduction	1
1.1 Literature Review and Field of Research	1
1.2 Motivation	6
1.3 Contents and Results of Thesis	7
2 Mean-Reverting Returns and Labor Income Growth	11
2.1 Model with Utility over Consumption	12
2.1.1 Separation of the HJB by A	15
2.1.2 Separation of the HJB by Y	16
2.1.3 Separation the HJB by the Constant Terms	18
2.1.4 Optimal Policies	18
2.1.5 Dynamics of Optimal Total Wealth	20
2.1.6 Main Results	20
2.2 Model with Utility over Terminal Wealth	21
2.3 Long-Horizon Stability of the Solution	23
2.4 Illustration of Results	24
2.4.1 Locally Riskfree Labor Income	27
2.4.2 Sensitivity of the Results to the Stability Parameter l_1	34
2.4.3 Constant Investment Opportunities	37
2.4.4 Risky Labor Income	43
2.5 Conclusion	47
2.A Appendix	49

2.A.1	Solution of the Wachter Model	49
2.A.2	Solution of the HJB-Equation for the Consumption Problem	50
2.A.3	The Dynamics of Total Wealth	54
2.A.4	Solution of the HJB-Equation for the Terminal Wealth Problem	55
2.A.5	Invariant Affine Transformation	58
2.A.6	Valuation of the Labor Income Stream	58
3	Stochastic Labor Income Volatility	61
3.1	Model	62
3.2	Long-Horizon Stability of the Solution	64
3.3	Illustration of the Results	69
3.4	Conclusion	79
3.A	Appendix	80
3.A.1	Valuation of the Labor Income Stream with the Martingale Approach	80
3.A.2	A Special System of Ordinary Differential Equations - Solutions	81
3.A.3	A Special System of Ordinary Differential Equations - Derivations	84
4	Labor Income and a Volatility Premium	87
4.1	Model with Utility over Consumption	88
4.1.1	Separating the HJB by A	90
4.1.2	Separating the HJB by Y	91
4.1.3	Separating the HJB by the Constant Terms	93
4.1.4	Optimal Policies	94
4.1.5	Dynamics of Optimal Total Wealth	96
4.1.6	Main Results	96
4.2	Model with Utility over Terminal Wealth	97
4.3	Long-Horizon Stability of the Solution	99
4.4	Illustration of the Results of the Basic Model	102
4.4.1	Locally Riskfree Labor Income	104
4.4.2	Risky Labor Income	109
4.5	Life-Cycle Model	113
4.5.1	Illustration of the Results of the Life-Cycle Model	115
4.6	Conclusion	118
4.A	Appendix	120
4.A.1	Solution of the HJB-Equation	120
4.A.2	The Dynamics of Total Wealth	121
4.A.3	A System of Two Ordinary Differential Equations	122
5	Non-Constant Labor Income Parameter Values	125
5.1	Model	126
5.2	Time Dependence in y_0 only	127

5.2.1	Illustration of the Results	129
5.3	Time Dependence in y_1 and σ_y	132
5.3.1	Illustration of the Results	135
5.4	Conclusion	138
5.A	Appendix	140
5.A.1	Riccati Differential Equation with $y(0) \neq 0$	140
6	Conclusion	141
6.1	Summary	141
6.2	Open Issues and Future Research	142
	Bibliography	145

Abstract

The thesis' focus is on consumption/portfolio optimization under time-varying investment opportunities and dynamic non-financial (labor) income using analytical methods.

The most striking result is that counter-cyclical non-financial income growth (income growth is low when expected returns are high) or pro-cyclical income volatility (income volatility is high when expected returns are high) lead to a strong reduction of optimal risky investment and consumption. Hence, it can be stated that dynamic labor income is a simple and comprehensible instrument to explain why some people do not participate in the stock market.

From a technical point of view, the assumption of complete markets allows for a separation of the complicated HJB-equation into ordinary differential equations. It will be shown that certain combinations of parameter values of the financial assets, the state variable and non-financial income lead to solutions of the differential equations that do not converge in the long-run. These settings are in favor of extreme results and this should be considered as a warning for numerical studies of the consumption-investment problem with labor income that are calibrated on empirical results.

Zusammenfassung

Die vorliegende Dissertation befasst sich mit der Konsum-/Portfoliooptimierung unter zeitvariablen Investitionsmöglichkeiten und dynamischem externen Einkommen (wie beispielsweise Arbeitseinkommen) unter der Anwendung von analytischen Methoden.

Als wichtigstes Resultat wird gezeigt, dass antizyklisches Einkommenswachstum (Einkommenswachstum ist tief, wenn die erwarteten Prämien hoch sind) oder prozyklische Einkommensvolatilität (Einkommensvolatilität ist hoch, wenn die erwarteten Prämien hoch sind) zu einer starken Reduktion der Investitionen in risikoreiche Anlagen und zu tieferem Konsum führt. Deshalb kann die Berücksichtigung von dynamischen Arbeitseinkommen als ein einfaches und verständliches Mittel betrachtet werden, um zu erklären weshalb gewisse Individuen nicht in Aktien investieren.

Aus analytischer Sicht führt die Annahme von vollständigen Märkten dazu, dass die komplizierte HJB-Gleichung in gewöhnliche Differentialgleichungen zerlegt werden kann. Es wird gezeigt, dass gewisse Parameterkonstellationen der Finanzanlagen, der Zustandsvariable und des Arbeitseinkommens nicht konvergente Lösungen implizieren. Diese Konstellationen begünstigen extreme Lösungen und sollten für numerische Studien des Konsum-/Portfoliooptimierungsproblems, welche auf empirische Daten kalibriert werden, als warnender Hinweis dienen.

List of Figures

2.1	Phase Plane Analysis I	23
2.2	Total Wealth - Locally Riskfree Labor Income	28
2.3	Optimal Risky Investment - Locally Riskfree Labor Income	31
2.4	Optimal Consumption - Locally Riskfree Labor Income	33
2.5	Total Wealth - Stability Analysis	35
2.6	Optimal Risky Investment - Stability Analysis	36
2.7	Optimal Consumption - Stability Analysis	37
2.8	Total Wealth - Constant Investment Opportunities	40
2.9	Optimal Risky Investment - Constant Investment Opportunities	42
2.10	Optimal Consumption - Constant Investment Opportunities	43
2.11	Total Wealth - Risky Labor Income	44
2.12	Optimal Risky Investment - Risky Labor Income	45
2.13	Optimal Consumption - Risky Labor Income	47
2.14	Phase Plane Analysis of $c_2(s)$	49
3.1	Phase Plane Analysis I	65
3.2	Phase Plane Analysis II	66
3.3	Phase Plane Analysis III	68
3.4	Total Wealth - Stochastic Labor Income Volatility	71
3.5	Optimal Risky Investment - Stochastic Labor Income Volatility	73
3.6	Optimal Consumption - Stochastic Labor Income Volatility	75
3.7	Total Wealth - Sensitivity	76
3.8	Optimal Risky Investment - Sensitivity	77
3.9	Optimal Consumption - Sensitivity	78
4.1	Phase Plane Analysis I	99
4.2	Phase Plane Analysis II	100
4.3	Distribution of the State Variable	103
4.4	Total Wealth - Locally Riskfree Labor Income	105
4.5	Optimal Risky Investment - Locally Riskfree Labor Income	106
4.6	Optimal Excess Consumption - Locally Riskfree Labor Income	108
4.7	Total Wealth - Risky Labor Income	109

4.8	Optimal Risky Investment - Risky Labor Income	110
4.9	Optimal Excess Consumption - Risky Labor Income	112
4.10	Total Wealth - Life-Cycle Model	117
4.11	Optimal Policies - Life-Cycle Model	118
5.1	Growth Profiles	130
5.2	Total Wealth - Locally Riskfree Labor Income	131
5.3	Optimal Risky Investment - Locally Riskfree Labor Income	132
5.4	$k(X, \tau)$ Piecewise Constant y_1	136
5.5	$\partial k(X, \tau) / \partial X$ Piecewise Constant y_1	137

List of Tables

2.1	Stability Analysis	24
2.2	Financial Market Parameter Values - Wachter and Campbell et al.	24
2.3	Financial Market Parameter Values - Normalized	25
2.4	Parameter Values	26
2.5	Sign of ψ	29
3.1	Stability Analysis	66
3.2	Parameter Values	69
4.1	Stability Analysis	100
4.2	Parameter Values	102
5.1	Parameter Values	129
5.2	$y_0(t)$ Growth Parameters	129
5.3	Parameter Values - Piecewise Constant y_1	135

Notation

We tried to adhere to the following convention on notation. Depending on the chapter and the context, the same letter or symbol can have a different meaning. Such occurrences are highlighted appropriately in the thesis. We tried to avoid such polyvalent situations; however, we did not completely succeed simply due to the sheer number of quantities for which we needed some notation.

Uppercase Letters

A	financial wealth
\hat{A}	total wealth
\bar{A}	subsistence wealth
H	human capital
G	stochastic part of human capital
J	value function
N	value of future subsistence consumption
R	value of net reserves
S_0	riskless asset
S_1	risky asset
T, T_i	point in time
X	state variable
\bar{X}	long-run mean of state variable
Y	stochastic part of labor income
\hat{Y}	total labor income
\bar{Y}	constant labor income

Lowercase Letters

c	consumption
\bar{c}	subsistence level of consumption
c_i, d_i	solution of ordinary differential equation

f, g, h	function of time
k	value of one unit of stochastic income
k_i, l_i, m_i	coefficient of ordinary differential equation
r_0	riskless rate
s, t, u	time variable
y_i	labor income growth parameter
\bar{y}	labor income growth at $X = \bar{X}$

Uppercase Greek Letters

Δ, Ψ	auxiliary constants
----------------	---------------------

Lowercase Greek Letters

γ	risk aversion coefficient
δ	time discount rate
λ_i	expected return parameter of risky asset
μ	mean
κ_x	mean-reversion parameter of the state variable
π	risky investment
θ, θ_i	market price of risk
ρ_{ij}	correlation
σ	standard deviation
σ_i	diffusion term
$\bar{\sigma}_i$	diffusion term level at $X = \bar{X}$
τ, τ_i	time until end of planning horizon
ψ, ψ_i	auxiliary constant
ω, ω_i	auxiliary constant

Various Symbols

\wedge	and
a^+, a^-	positive and negative roots of a quadratic equation
a^*	optimal value
$ a $	absolute value
$E_t[\dots]$	conditional expectation
\lim	limes
$\sec(a)$	secant: $1/\cos(a)$
\sup	supremum

Abbreviations

CAPM	capital asset pricing model
CRRA	constant relative risk aversion
FOC	first order condition
HARA	hyperbolic absolute risk aversion
HJB	Hamilton-Jacobi-Bellman
IAT	invariant affine transformation
i.e.	id est, that is
IES	intertemporal elasticity of substitution
LHS, RHS	left (right) hand side
ODE	ordinary differential equation
PDE	partial differential equation
SODE	system of ordinary differential equations

Chapter 1

Introduction

Consumption and portfolio optimization under time-varying investment opportunities and dynamic non-financial income has not received much attention until now. Two exceptions are Lynch and Tan (2009)¹ and Munk and Sørensen (2010). These papers focus on rather complicated life-cycle models and have to rely on numerical methods. Their results imply that dynamic labor income matters for consumption and portfolio decisions. In this thesis we will approach the question with analytical methods. We will be able to reproduce results from the aforementioned studies and offer considerably more theoretical insights and implications for empirical research.

1.1 Literature Review and Field of Research

The starting point of modern portfolio choice coincides with the seminal work of Markowitz (1952), who introduced the concept of diversification in a clear mathematical form by studying the trade-off between return and variance. The static one-period model was extended by Tobin (1965) to a multiple period model. Based on the work of Markowitz and Tobin, Sharpe (1964) and Lintner (1965) developed an equilibrium model for asset pricing, the well-known capital asset pricing model (CAPM). An important contribution of these studies is that all individuals should hold only one portfolio of risky assets independent of their risk aversion² - the so called myopic portfolio.

In 1969, portfolio theory made an important step. Samuelson (1969) and Merton (1969) analyzed combined consumption-investment decision problems in a multi-period framework in discrete and continuous time respectively. Samuelson and Merton introduced the method of dynamic programming (stochastic control) into the portfolio choice literature. Under the assumption of constant relative risk aversion (CRRA) utility functions and lognormally distributed risky assets, Merton was able to derive closed-form solutions for optimal consumption and investment. The work of Samuelson and Merton confirmed the major result from the static literature, namely

¹The paper by Lynch and Tan has been accepted for future publication in the *Journal of Financial Economics*, <http://jfe.rochester.edu/forth.htm> (10th January 2011).

²Of course, individuals with different risk aversions should hold different fractions of their wealth in the risky asset portfolio.

that all investors should hold the same risky portfolio. Furthermore, the work of Merton implied that all individuals should consume a fixed fraction of their wealth³.

The major result was challenged by Merton (1973) himself. While precedent models implied constant investment opportunities, i.e. constant expected returns and (co-)variances for the risky assets, Merton introduced stochastic investment opportunities. In particular, he assumed that investment opportunities vary with a certain number of state variables. The two most important results are the following. Firstly, in addition to the myopic portfolio, all investors should hold a second portfolio of risky assets that hedges the changes in the investment opportunity set. Basically, this hedging portfolio consists of assets that are most correlated with the state variables. Secondly, consumption is not a fixed fraction of wealth, but varies with the state variables. Although the work of Merton was presented in a rather abstract way, the results are quite intuitive. Loosely speaking, investment opportunities can either be good or bad. To invest in a portfolio that delivers a good return when investment opportunities turn bad and vice versa is conceptually identical to an insurance and seems a reasonable choice.

Pliska (1986), Karatzas et al. (1987) and Cox and Huang (1989) introduced the martingale representation technique into the field of portfolio choice and offered an alternative method to the well-established dynamic programming method. Although the martingale method is theoretically and practically appealing, most papers still rely on the dynamic programming method⁴.

The importance of non-financial income as labor income was recognized early. Merton (1971) introduced a constant wage and his work was extended in several dimensions. Duffie et al. (1997) discuss properties of the value function and the optimal policies under stochastic labor income. Koo (1998) and He and Pagès (1993) rule out short selling and/or impose borrowing constraints under non-insurable labor income risk. Heaton and Lucas (2000a, b) point out the importance of entrepreneurial risk and show that individuals with high business risk should hold less stocks. Munk (2000) and Viceira (2001) calibrate labor income models to realistic financial and labor market data. Bodie et al. (1992) introduce flexible labor supply. Cocco et al. (2005) examine the effect of different labor income risk on the optimal portfolio policies of a life cycle investor under a realistic calibration of the labor income process.

The most important result of these models is that labor income matters for the consumption-investment decision of an individual. In fact, the future income stream implies additional wealth to the investor and hence, the behavior of the individual generally becomes more extreme. Moreover, the presence of labor income generates an additional hedging portfolio in the demand for risky assets. Similarly to state variable hedging demand, hedging demand for labor income consists of risky assets that are most correlated with labor income. From a technical point of view, it is important to know that with stochastic labor income, closed-form solutions are generally not available. As shown by Koo (1998) and Duffie et al. (1997), without additional assumptions, the ratio of financial wealth to labor income becomes essential in order to determine the value

³However, the fraction does not have to be constant unless the planning horizon is infinite. If not, the fraction varies with the planning horizon.

⁴A good example for an application of the martingale method in a consumption-investment problem under time-varying investment opportunities is Wachter (2002).

of the future income stream and this leads to highly non-linear partial differential equations that have no explicit solution. That financial wealth becomes important in the valuation of the income stream is intuitive as well. Imagine an individual with a labor income stream that is risky and not perfectly correlated with the financial market. In this case, financial wealth of the individual can go to zero and the income stream still can have a high value; the individual might start to borrow in order to invest in risky assets. In this situation, the wage could suffer from a severe negative shock and go to zero as well. This results in a situation where the individual is left without labor income and debts and thus she is not able to afford any consumption, which is clearly not optimal⁵. As shown in, for example, Huang and Milevsky (2008), solutions to labor income problems are available under the assumption of perfect correlation between labor income and the financial market or riskfree labor income.

With the influential work by Kim and Omberg (1996) a new branch of portfolio choice literature emerged. Kim and Omberg presented closed-form solutions for a portfolio choice problem with stochastic investment opportunities. In particular, they assumed that there is only one state variable that follows an Ornstein-Uhlenbeck process and that the equity premium is affine in the state variable. While Kim and Omberg focused on hyperbolic absolute risk aversion (HARA) utility over terminal wealth, Wachter (2002) extended the model to CRRA utility over consumption but has to assume complete markets. From a technical view, the solution of the optimization problem involves separating a non-linear partial differential equation into a system of ordinary differential equations. Some of the resulting ordinary differential equations are already known from the term structure literature and can be solved by similar formulas⁶.

Models with an affine equity premium were extended in several ways. Most importantly, Campbell et al. (2004) extended the model to stochastic differential utility⁷ (SDU). Herzog et al. (2004) included multiple state variables. The numerical calibration of the Wachter and the Campbell et al. models to real data showed that state variable hedging demand has the same sign as myopic demand and is important compared to the size of myopic demand.

Work in dynamic portfolio choice is not limited to models with an affine equity premium. Chacko and Viceira (2005) studied models with time-varying volatility. Munk et al. (2004) incorporate stochastic interest rates and inflation uncertainty. Buraschi et al. (2010) analyzed the effects of stochastic correlation. Liu (2007) presented a general model that includes those of Wachter, Kim and Omberg and some aspects of those of Chacko and Viceira and Munk et al. as special cases. He was able to state conditions for the return and the state variable processes that

⁵Assuming a standard utility function as power utility with a coefficient of risk aversion greater than one. For these utility functions $\lim_{c \rightarrow 0+} u(c) = -\infty$.

⁶For example, some parts of the solution of the Kim and Omberg (1996) and Wachter (2002) model are similar to parts in the solution of the Cox-Ingersoll-Ross term structure model. Compare Kim and Omberg's $C(\tau)$ (p. 158), Wachter's $A_1(\tau)$ (p. 71) with Ingersoll (1987, p. 397) or Duffie (2001, p. 142).

⁷SDU was introduced by Duffie and Epstein (1992a, b). A major advantage of SDU is that it allows the separation of relative risk aversion from intertemporal elasticity of substitution (IES). This is a well-known drawback of HARA utility functions. See, for example, Bommier (2007). For an early application of SDU in the portfolio choice literature see Schroder and Skiadas (1999). For recent applications of SDU in the asset pricing literature see Avramov and Hore (2007) or Hore (2008).

have to be satisfied in order to separate the resulting partial differential equation into ordinary differential equations.

The combination of labor income and stochastic opportunity set models is a rather new development. Lynch and Tan (2009) consider a model where labor income growth, and in an extension labor income volatility, depend on the dividend yield. Calibrated on empirical data they find that time-varying labor income can have an important impact on the optimal policies. In fact, they show that the highly positive investment in risky assets implied by the Wachter and Campbell et al. model can be reversed. Munk and Sørensen (2010) present a combined stock-bond allocation problem with stochastic short interest rates and assume that labor income growth depends on the short rate. The focus of their work is on the joint implications of stochastic interest rates and labor income for the valuation of human capital and optimal policies.

Other important contributions in the field of portfolio choice include the impact of limited/no short selling and borrowing constraints⁸, the effect of trading costs⁹ and the influence of information uncertainty¹⁰.

Furthermore, it should be mentioned that the notion of time variation in investment opportunities is still under challenge. In fact, a vast literature has focused on the search for state variables that can explain time-variation in the expected return of the stock market. Research goes back to Dow (1920), who analyzed the predictive power of the dividend yield on expected returns. Goyal and Welch (2008) give a critical overview of the empirical evidence of popular state variables used in previous studies. Their evidence suggests that most examined state variables have lost some or all of their in-sample predictive power in the last years. Furthermore, all tested state variables have no or only low out-of-sample predictive power. Lettau and Van Nieuwerburgh (2007) point out that financial ratios (as the dividend yield, price-earnings ratio, etc.) have predictive power, but shifts in the steady state of the state variable make the in-sample return forecastability hard to exploit. Another critical study is Liu and Zhang (2008) who question the predictive power of the so-called value spread, the difference in the price-earnings ratio of value and growth stocks.

Recent work in favor of predictability are Inoue and Kilian (2004), Ang and Bekaert (2007), Campbell and Thompson (2007), Cochrane (2005, 2008) and Cooper and Priestley (2009). Inoue and Kilian (2004) show that in-sample tests for predictability are more reliable than out-of-sample tests. Ang and Bekaert (2007) find that the dividend yield and the short rate are good predictors and show that the results are robust to international data. Campbell and Thompson (2007) point out that prediction quality can be improved by imposing economically meaningful restriction on the signs of the estimated coefficients and return forecasts. Cochrane (2005, 2008) states that the variation in the dividend yield implies that if returns are not predictable, dividend growth must be predictable. He finds that the absence of dividend growth predictability gives stronger evidence than does the presence of return predictability. Cooper and Priestley (2009)

⁸Koo (1998), He and Pagès (1993).

⁹Liu and Loewenstein (2002), Balduzzi and Lynch (1999).

¹⁰Brennan (1998), Xia (2001), Garlappi et al. (2007), Kan and Zhou (2007), Wachter and Warusawitharana (2009). In addition, for a recent overview of Bayesian portfolio models see Avramov and Zhou (2010).

show impressive in- and out-of-sample predictive power for the so-called output gap. This state variable measures the deviation of industrial production from its long-term trend level.

The brief review shows the wide range of the portfolio choice literature. As stated by Wachter (2010):

“Ultimately, the goal of academic work on asset allocation is the conversion of the time series of observable returns and other variables of interest into a single number: Given the preferences and horizon of the investor, what fraction of her wealth should she put in stock¹¹? The aim is to answer this question in a “scientific” way, namely by clearly specifying the assumptions underlying the method and developing a consistent theory based on these assumptions. The very specificity of the assumptions and the resulting advice can seem dangerous, imputing more certainty to the models than the researcher can possibly possess. Yet, only by being so highly specific, does the theory turn into something that can be clearly debated and ultimately refuted in favor of an equally specific and hopefully better theory.”

Thus, for the sake of clarity we will restrict the models in this thesis to include the following properties:

1. The individual has HARA utility¹² over consumption or terminal wealth.
2. The planning horizon is known, i.e. there is no lifetime uncertainty.
3. The investor chooses between a risky (a broad stock portfolio) and a riskless asset.
4. The individual faces outside (non-financial) income.
5. The risky asset and non-financial income are subject to time variation.
6. The time variation is driven by a single state variable.
7. There are no barriers to trading in the assets, such as leverage or short-sale constraints.
8. There is no parameter uncertainty.

From these characteristics it can be stated that our models are closest to that of Lynch and Tan (2009). Another related model is that of Munk and Sørensen (2010) but their focus is on combined bond-stock problems and they assume that stock returns are not subject to time-variation. What the models have in common is that they combine two important branches of the portfolio choice literature, namely time-varying investment opportunities and labor income. Moreover, the exclusion of model frictions as described in (6.) and (7.) allows us to study the

¹¹The quote should not be misunderstood: the distinction between equity and (risky) long-term bonds is an important issue for asset allocation as well. Nevertheless, for the sake of simplicity it is standard to deal with only one risky asset and think of it as a broad stock portfolio unless one intends to study optimal stock-long-term bond allocation problems explicitly.

¹²With the exception of exponential utility, HARA utility implies a subsistence level of consumption/wealth.

effects of dynamic labor income clearly. However, for future research several additional features could be added to the presented models.

Finally, it should be mentioned that the thesis provides an answer to the question of how an individual should behave (given a certain set of assumptions). The issue of how individuals really behave is an empirical one and not part of this thesis¹³.

1.2 Motivation

As mentioned above, the literature that combines labor income models with time-varying investment opportunities and time variation in labor income is rather undeveloped. The aforementioned work of Munk and Sørensen and Lynch and Tan focuses on the solution of rather complicated models, which demands a reliance upon numerical methods¹⁴. As a consequence, theoretical insights and the sensitivity of results on the parameters and on the states of the economy are largely neglected.

The primary intention of this thesis is to fill this gap. We will develop models that are not as realistic as the cited models. In fact, we have to impose more assumptions in order to solve the models in explicit form and, admittedly, certain assumptions are not completely in line with reality. However, during the thesis we will point out the critical assumptions and will discuss their importance and implications for the results.

Labor income must be an important dimension in portfolio-consumption problems. In our opinion, the two most crucial arguments in favor of this statement are the following. Firstly, for a young individual at the beginning of her working life, financial wealth is generally low compared to the value of future income stream. This suggests that labor income characteristics as time-varying growth rates become important in the consumption-investment decision process. Secondly, labor income allows for individuality. While it is natural to assume that financial markets, and thus the relevant parameters of financial markets, are identical to all individuals, labor income brings diversity in a natural way¹⁵. To be more precise, the classical work of Merton and the extension to time-varying investment opportunities, as for example Wachter (2002) and Campbell et al. (2004), provide only few channels for individuality. In fact, different risk aversion and intertemporal elasticity of substitution give only limited possibilities for distinct investment and consumption strategies. This is not the case in the presence of labor income. Well-educated people are likely to have different exposure to changes in the investment opportunity set than low-educated people. The importance of the labor income stream is not the same to rich and poor individuals. The duration of the employment phase until retirement brings diversity between young and old individuals.

Furthermore, time variation in labor income growth is not only a theoretical concept. The estimates of Lynch and Tan (2009) in Table 1 are significant on the 1 percent level. Stock

¹³As stated by Campbell (2006), the availableness of high quality data in this field is still limited.

¹⁴Munk and Sørensen (2010) pay attention to special cases that could be solved in closed-form. Nevertheless, they do not analyze these cases in depth.

¹⁵See, for example, Dynan et al. (2004).

and Watson (1999) show clear variation in wages and employment at business cycle frequency. Moreover, assuming time variation not only in the financial, but also in the labor market seems conceptually reasonable.

Because of these arguments, it clearly worthwhile to study the implications of time-varying labor income and to understand the sensitivity of the results for a reasonable range of parameters. In the course of the thesis, the following three important issues are highlighted and possible explanations derived:

1. The dynamic models of Kim and Omberg (1996), Wachter (2002) and Campbell et al. (2004) imply that an individual should have an even stronger position in the risky asset through the state variable hedging portfolio. In fact, the unconstrained solutions look like hedge fund portfolios, not typical investor portfolios. With a single risky asset, the result is a highly leveraged position, unless one assumes extremely risk-averse individuals¹⁶. As shown in Kim and Omberg or Wachter, hedging demand decreases with decreasing horizon. This implies that young people should hold more stocks. Although this is in line with financial advice¹⁷, empirical surveys do not show such clear age effects¹⁸. By the inclusion of labor income, the results of Lynch and Tan show a different picture, i.e. individuals with a long horizon have no/negative exposure in the risky asset. The explicit solutions enable an understanding of the forces that drive these results.
2. The work of Wachter (2002) and Campbell et al. (2004) shows that optimal portfolios and consumption should vary strongly with the state variable. For example, Figure 3 of Wachter shows clearly that the optimal allocation should vary from no equity exposure to a leveraged¹⁹ position within two standard deviations from the long-run mean of the state variable. Neither Lynch and Tan nor Munk and Sørensen pay any attention to this dimension. The analytical solution will allow this dimension to be studied in detail.
3. The valuation of the future income stream has an essential impact on the optimal policies. As will be shown below, the valuation of the income stream asks for the solution of ordinary differential equations (ODEs). The analytical expression will provide important insights as to which parameter combinations will result in solutions of the ODEs that do not converge in the long term.

1.3 Contents and Results of Thesis

After this introductory chapter, we will present a model that is closely related to the model of Lynch and Tan (2009). To be more precise, the main feature of this model is time variation in the expected return of the risky asset and time variation in labor income growth. The time variation is driven by a single state variable that follows an Ornstein-Uhlenbeck process. Compared to

¹⁶Special thanks to Edward Omberg, who stressed this critical issue.

¹⁷See, for example, Munk et al. (2004) Table 2 (p. 158).

¹⁸Campbell (2006).

¹⁹More than 100 percent of financial wealth.

Lynch and Tan, we will impose certain assumptions in order to find explicit solutions. With the help of these expressions we are able to interpret our results clearly and we can provide theoretical insights. The most important result is that

- The inclusion of time-varying labor income growth leads to highly individual optimal policies. For realistic parameters it is shown that some individuals do not want to hold a positive amount of risky assets. Hence, the fact that some individuals do not hold any equity at all²⁰ can be confirmed.

Further essential results are the following:

- Under the assumptions of perfect correlation between the risky asset and labor income or locally riskfree labor income, the separation of the Hamilton-Jacobi-Bellmann (HJB) equation into ordinary differential equations is still possible.
- The inclusion of time variation in labor income leads to an adaption of state variable hedging demand. In fact, state variable hedging demand can be separated into the usual part that arises in the absence of labor income and a new part²¹. This part grows monotonically with the planning horizon and can have either sign.
- In contrast to myopic and classical state variable hedging demand, indirect labor hedging demand does not depend on the level of financial wealth. Hence, it remains important even if financial wealth is low.
- A negative sensitivity of labor income growth on the state variable can induce falling risky investment and consumption even if expected returns are increasing in the state variable. This is in contrast to the models of Wachter (2002) and Campbell et al. (2004). Moreover, the level of risky investment can be reduced as well.
- From a technical point of view, the valuation of the labor income stream involves solving ordinary differential equations. Certain combinations of state variable and financial market parameters lead to solutions which do not converge for long horizons. In these cases, the valuation of the income stream leads to extreme results even if the sensitivity of labor income growth to the state variable is low.
- If the sensitivity of the risky asset and labor income on the state variable are in a particular relation, indirect labor hedging demand is zero and the valuation of the labor income stream is independent of the state variable and similar to the case with constant income growth.

In the subsequent chapter the setting of the basic model is extended to stochastic labor income volatility. The specification is an adaption of a model extension examined by Lynch and Tan. Additional important results are given by:

²⁰See, for example, Figure 2 of Campbell (2006).

²¹We will refer to this part as the “indirect labor hedging demand”. The choice of this expression will become clear in the derivation of the optimal policies in Appendix 2.A.2.

- The inclusion of stochastic volatility in labor income always drives a wedge between the risky asset and labor income. Hence, the valuation of the labor income stream will always depend on the state variable and indirect labor hedging demand is never zero.
- If labor income and the risky asset are positively correlated, then the part of hedging demand that is due to stochastic volatility converges to a stable solution even for an infinite horizon. Nevertheless, a highly persistent state variable can lead to extreme results.
- The addition of stochastic labor income volatility allows for more interesting patterns of hedging demand. In fact, indirect and direct labor hedging demand become non-monotone in the state variable.
- Labor income volatility can generate risky investment that is rather insensitive to changes in the state variable. Furthermore, optimal policies that include both decreasing risky investment and consumption are possible even if the expected return of the risky asset is rising.
- The system of ordinary differential equation for the valuation of the labor income stream becomes more complicated. In particular, it includes a Riccati differential equation. If the labor income process has some advantageous properties, the value of the future labor income stream can be infinite even for a finite horizon. However, these cases ask for unrealistically extreme parameter values.

The fourth chapter introduces a model with stochastic volatility for the risky asset and an affine volatility premium. It is assumed that the labor income stream faces the same characteristics. Stochastic volatility follows a CIR-process and a similar model without labor income is presented by Liu (2007). As an extension, the model is integrated in a life cycle model that includes a period of retirement without any non-financial income. The most important additional insights are:

- The system of ordinary differential equation for the valuation of the labor income stream includes a Riccati differential equation. If the labor income process has some advantageous properties, the value of the future labor income stream can be infinite even for a finite horizon. As before, these cases ask for unrealistically extreme parameter values.
- The extension of the basic model to a life-cycle model with a phase of retirement is a simple and comprehensible instrument to reduce the value of total wealth. As a consequence, risky investment and excess consumption are reduced to a realistic level.
- The reduction of total wealth in the life-cycle model implies that the importance of myopic and state variable hedging demand is reduced compared to the two labor income hedging demands.

In the fifth chapter, the assumption of constant labor income parameters over the life-cycle is weakened. While constant parameters are a reasonable choice for the financial market, this is not

the case for labor income, as with growing age the skills of an individual change. The inclusion of time-variation shows that:

- Time-dependence in the part of labor income growth that is not related to the state variable is rather simple to implement.
- The inclusion of high labor income growth at the beginning of the working period leads to a higher valuation of the future income stream. As a consequence, the importance of labor income on the optimal policies increases.
- Time-dependence in the parts that measure the sensitivity of the labor income dynamics to changes in the state variable are difficult to implement because closed-form solutions to Riccati differential equations with time-varying coefficients only exist in a few special cases. Nevertheless, closed-form solutions can be found for piecewise constant parameters.
- The analytical results show that non-constant labor income parameters allow for more sophisticated patterns of the function, which values the future income stream labor. This can lead to an indirect labor hedging demand that changes sign over the life-cycle.
- Even if labor income from young individuals is not exposed to changes in the state of the economy, the valuation of the future income stream depends on the state if income is exposed to changes in the state at a later time period. In this case, the sensitivity of the value of the future income stream of a young individual is rather stable over time.

Chapter 2

Portfolio and Consumption Decisions under Mean-Reverting Returns and Labor Income Growth

In this chapter, the consumption investment problem of an individual facing a dynamic financial market and dynamic non-financial income (labor income) is solved. The financial market consists of two assets. One is a riskless bond and a risky asset with mean-reverting returns. Thus, the financial market setting is up to an invariant affine transformation identical to Kim and Omberg (1996) and Wachter (2002). In addition, it is assumed that the individual faces outside labor income that has a mean-reverting growth rate. There is a single state variable that drives both the risky asset and labor income.

A similar model is discussed by Lynch and Tan (2009)¹. The model of Lynch and Tan is more realistic with respect to the model assumptions, but has to rely on numerical methods. For this reason, the reported effects can only be interpreted with a certain depth and sensitivity analysis is neglected largely. In fact, Lynch and Tan focus exclusively on the development of the optimal policies over the life-cycle and omit the sensitivity of the results over states. This chapter aims to fill this gap and points out critical issues within this setting that are not only relevant on a analytical dimension, but also have implications for empirical research. However, for the sake of closed-form solutions more restrictive assumptions have to be taken. In fact, it has to be assumed that labor income is either locally riskfree or perfectly correlated with the risky asset. Nevertheless, these assumptions come with an advantage. In particular, it is shown that the assumption allows the inclusion of a subsistence level of consumption (HARA preferences) without having to assume that initial financial wealth exceeds the value of the future subsistence consumption stream. Furthermore, and similarly to Wachter (2002) it must be assumed that the state variable and the risky asset are perfectly correlated. However, for the dividend yield this assumption is not too problematic as the correlation is close to -1.

¹The paper by Lynch and Tan has been accepted for future publication in the *Journal of Financial Economics*, <http://jfe.rochester.edu/forth.htm> (10th January 2011).

The assumption of a single state variable eases the mathematical derivation and the interpretation of the results. Moreover, there are several reasons that suggest that the assumption of a single state variable is not too restrictive. Firstly, the use of only one state variable that drives both financial assets and labor income is motivated by the empirical estimation in Lynch and Tan who use the dividend yield as state variable². Secondly, from a theoretical point of view, factors that drive capital and labor markets simultaneously are reasonable since most output is produced by a combination of labor and capital. Finally, empirical macroeconomic literature as Stock and Watson (1999) shows clear variation in wages/employment and of the financial market at business cycle frequency.

The rest of this chapter is organized as follows. In Section 2.1, the model with preferences over intermediate consumption is introduced. Section 2.2 adapts the same model to utility over terminal wealth. In Section 2.3, the long-horizon stability of the solution is discussed. The subsequent section presents the results for numerically realistic parameter values. The final section concludes. Mathematical derivations as the solution of the HJB-equation and other non-trivial derivations are provided in the Appendices 2.A.1 - 2.A.6.

2.1 Model with Utility over Consumption

For the sake of simplicity, we assume that the individual works during the entire optimization horizon. Nevertheless, life cycle models with a retirement period could be included without severe problems³.

The conditional expected utility over the remaining lifetime for an individual at t is

$$E_t \left[\int_t^T \frac{e^{-\delta s}}{1-\gamma} (c(s) - \bar{c})^{1-\gamma} ds \right], \quad \gamma > 1$$

where $\bar{c} > 0$ is the subsistence level of consumption, $\delta \geq 0$ is the time discount parameter and $\tau \equiv T - t$ is the fixed and certain time horizon⁴. In this part, we assume that the risky assets' expected return is affine in a state variable and has constant volatility. In particular, we assume that

$$\frac{dS_1(t)}{S_1(t)} = (\lambda_1 X(t) + r_0) dt + \sigma_s dW_s(t) \quad (2.1)$$

where $\lambda_1 > 0$ and $\sigma_s > 0$. r_0 is the short rate and the riskless asset follows⁵

$$\frac{dS_0(t)}{S_0(t)} = r_0 dt$$

It should be noted that in this framework, the market price of risk is linear in $X(t)$

$$\theta(t) \equiv \frac{\lambda_1}{\sigma_s} X(t)$$

²See Table 1 in Lynch and Tan (2009, p. 44).

³See, for example, Huang and Milevsky (2008), Moos and Müller (2010) or Chapter 4 for examples.

⁴For the sake of simplicity, it is assumed that the individual is not exposed to lifetime uncertainty. However, lifetime uncertainty models as presented in Pliska and Ye (2007) could be included without severe technical issues.

⁵A specification of the short rate of the form $dS_0(t)/S_0(t) = (r_0 + r_1 X(t) + r_2 X(t)^2) dt$ could be chosen without severe problems. Properties of quadratic short rate models are discussed in Leippold and Wu (2002, 2003). For the sake of simplicity, and in order to show the effects of non-financial income clearly, this is omitted.

The state variable dynamics are given by

$$dX(t) = -\kappa_x (X(t) - \bar{X}) dt + \sigma_x dW_x(t) \quad (2.2)$$

where $\kappa_x \geq 0$, $\bar{X} \geq 0$ and $\sigma_x > 0$. (2.2) is a well-known Ornstein-Uhlenbeck process⁶. The specification of the investment opportunities is in accordance with Liu (2007), who, in a general, analyzes model consumption-investment problems without labor income in a stochastic opportunity set. Moreover, the financial market setting is one-to-one similar⁷ to Wachter (2002). Hence, the effect of the inclusion of a stochastic labor income can be directly compared with the results of Wachter⁸.

It is assumed that the wage consists of two parts. In particular,

$$\hat{Y}(t) = \bar{Y} + Y(t)$$

where $\bar{Y} \geq 0$ is a constant and thus without risk. This part can be interpreted as a minimum wage that is guaranteed by a third party. $Y(t)$ is the risky part of labor income and follows

$$\frac{dY(t)}{Y(t)} = (y_0 + y_1 X(t)) dt + \sigma_y dW_y(t) \quad (2.3)$$

where y_0 is the constant part of labor income, y_1 is the sensitivity of labor income growth on the state variable and $\sigma_y \geq 0$. Since $Y(t)$ can not become negative, \bar{Y} is the minimum income of the investor.

With the specified income, the financial wealth dynamics of an investor are as follows

$$\begin{aligned} dA(t) = & \left[\pi(t) A(t) \lambda_1 X(t) + A(t) r_0 + \hat{Y}(t) - c(t) \right] dt \\ & + \pi(t) A(t) \sigma_s dW_s(t) \end{aligned} \quad (2.4)$$

The HJB is given by

$$\begin{aligned} 0 = & J_t + \sup_c \left[e^{-\delta t} \frac{(c_t - \bar{c})^{1-\gamma}}{1-\gamma} - J_A c_t \right] \\ & + \sup_\pi \left[J_A \pi(t) A(t) \lambda_1 X(t) + \frac{1}{2} J_{AA} \pi(t)^2 A(t)^2 \sigma_s^2 \right. \\ & \left. + J_{AX} \pi(t) A(t) \rho_{sx} \sigma_s \sigma_x + J_{AY} \pi(t) A(t) Y(t) \rho_{sy} \sigma_s \sigma_y \right] \\ & + J_A \left[A(t) r_0 + \hat{Y}(t) \right] - J_X \kappa_x (X(t) - \bar{X}) + J_Y Y(t) (y_0 + y_1 X(t)) \\ & + \frac{1}{2} J_{XX} \sigma_x^2 + \frac{1}{2} J_{YY} Y(t)^2 \sigma_y^2 + J_{XY} Y(t) \rho_{xy} \sigma_x \sigma_y \end{aligned} \quad (2.5)$$

⁶The process is well-known in the literature of mathematical finance. See Vasicek (1977) for an early example in the term structure literature and Lo and Wang (1995) and Schöbel and Zhu (1999) for examples in the option pricing literature.

⁷Of course, there are some changes in notation.

⁸We would like to thank Jessica Wachter for providing us with her original Matlab code. With the help of her code we were able to verify our results and our code.

where $dW_s dW_x = \rho_{sx} dt$, $dW_s dW_y = \rho_{sy} dt$ and $dW_x dW_y = \rho_{xy} dt$. The first order conditions (FOCs) are given by

$$c_t^* = \left(e^{\delta t} J_A \right)^{-\frac{1}{\gamma}} + \bar{c} \quad (2.6)$$

and

$$A(t) \pi_t^* = -\frac{J_A}{J_{AA}} \frac{\lambda_1}{\sigma_s^2} X(t) - \frac{J_{AX}}{J_{AA}} \frac{\rho_{sx} \sigma_x}{\sigma_s} - \frac{J_{AY}}{J_{AA}} \frac{\rho_{sy} \sigma_y}{\sigma_s} Y(t) \quad (2.7)$$

Plugging in the FOCs (2.6) and (2.7) into the HJB equation (2.5) yields

$$\begin{aligned} 0 = & J_t + \frac{\gamma}{1-\gamma} e^{-\frac{\delta}{\gamma} t} J_A^{1-\frac{1}{\gamma}} - J_A \bar{c} + J_A A(t) r_0 + J_A \bar{Y} + J_A Y(t) \\ & - J_X \kappa_x (X(t) - \bar{X}) + J_Y Y(t) (y_0 + y_1 X(t)) \\ & + \frac{1}{2} J_{AA} A(t) \pi_t^* \lambda_1 X(t) + \frac{1}{2} J_{AX} A(t) \pi_t^* \rho_{sx} \sigma_x \sigma_s \\ & + \frac{1}{2} J_{AY} A(t) Y(t) \pi_t^* \rho_{sy} \sigma_y \sigma_s \\ & + J_{XY} Y(t) \rho_{xy} \sigma_x \sigma_y + \frac{1}{2} J_{YY} Y(t)^2 \sigma_y^2 + \frac{1}{2} J_{XX} \sigma_x^2 \end{aligned} \quad (2.8)$$

One tries a value function of the following form

$$J = \frac{e^{-\delta(T-\tau)} \left[\int_0^\tau e^{\frac{1}{\gamma}(c_0(s)+c_1(s)X+\frac{1}{2}c_2(s)X^2)} ds \right]^\gamma (A + k(\tau, X)Y - R(\tau))^{1-\gamma}}{1-\gamma} \quad (2.9)$$

where $\tau \equiv T - t$, $k(X, \tau)$ is a function of the state variable and the time horizon and $R(\tau)$ is a function of the time horizon. Both will be parameterized below.

For the sake of readability, the solution of the HJB (2.8) is shown in Appendix 2.A.2. As we focus on closed-form solutions some assumptions have to be implemented. As in Wachter (2002), it must be assumed that the risky asset and the state variable have to be perfectly correlated.

$$\rho_{sx} \in \{-1, 1\} \quad (2.c.1)$$

Furthermore, it has to be assumed that either

$$\rho_{sy} \in \{-1, 1\} \Rightarrow \rho_{xy} = \rho_{sx} \rho_{sy} \in \{-1, 1\} \quad (2.c.2)$$

or

$$\sigma_y = 0 \quad (2.c.3)$$

Admittedly, these assumptions do not match reality one-to-one. Nevertheless, several papers have shown that the results of exactly solvable special cases are qualitatively similar to cases with non-perfect correlation⁹. Hence, we expect that the qualitative results hold for more general cases.

Furthermore, in Campbell and Viceira (1999), Barberis (2000), Wachter (2002), Campbell et al. (2004) and Lynch and Tan (2009) the dividend yield was chosen as the state variable. As

⁹See Cocco et al. (2005), Huang et al. (2008), Huang and Milevsky (2008), Bick et al. (2009) and Dybvig and Liu (2010).

shown below, the dividend yield has a correlation to equity close to -1 and thus the assumption $\rho_{sx} = -1$ is, in this case, not too restrictive. Finally, if labor income volatility is rather low, the locally riskfree labor income case $\sigma_y = 0$ is certainly a reasonable approximation. Hence, it can be stated that despite these assumptions, the results of the model are not only of theoretical interest, but have implications for realistic cases.

Finally, these assumptions come with an advantage besides the interpretability of closed-form solutions. In fact, in the case of $\rho_{sy} \notin \{-1, 1\}$ and $\sigma_y > 0$, current financial wealth has to be higher than the reserves for the future subsistence consumption¹⁰. This would be an unrealistic assumption, especially for young individuals who generally have a low financial wealth.

Similarly to the models without labor income, the final HJB (2.37) of Appendix 2.A.2 can be separated into ordinary differential equations.

2.1.1 Separation of the HJB by A

Separating the HJB (2.37) of Appendix 2.A.2 by A gives the following equation

$$0 = \int_0^\tau e^{C(X,s)} \left\{ \begin{aligned} & -\delta - \left(\frac{\partial c_0(s)}{\partial s} + \frac{\partial c_1(s)}{\partial s} X + \frac{1}{2} \frac{\partial c_2(s)}{\partial s} X^2 \right) + (1 - \gamma) r_0 \\ & -\kappa_x (c_1(s) X + c_2(s) X^2) + \kappa_x \bar{X} (c_1(s) + c_2(s) X) \\ & + \frac{1}{2} \frac{1-\gamma}{\gamma} \frac{\lambda_1^2}{\sigma_s^2} X^2 + \frac{1-\gamma}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 X (c_1(s) + c_2(s) X) \\ & + \frac{1}{2} \frac{1}{\gamma} \sigma_x^2 \left(\gamma c_2(s) + c_1^2(s) + 2c_1(s) c_2(s) X + c_2(s)^2 X^2 \right) \end{aligned} \right\} ds \quad (2.10)$$

which can be separated by X^2 , X and constant terms into three ordinary differential equations.

$$\frac{\partial c_2(s)}{\partial s} = k_0 + k_1 c_2(s) + k_2 c_2(s)^2 \quad (2.11)$$

$$\frac{\partial c_1(s)}{\partial s} = k_3 c_2(s) + \frac{k_1}{2} c_1(s) + k_2 c_2(s) c_1(s) \quad (2.12)$$

$$\frac{\partial c_0(s)}{\partial s} = k_5 + k_3 c_1(s) + k_4 c_2(s) + \frac{k_2}{2} c_1(s)^2 \quad (2.13)$$

with initial conditions $c_2(0) = c_1(0) = c_0(0) = 0$ and

$$\begin{aligned} k_0 &\equiv \frac{1-\gamma}{\gamma} \frac{\lambda_1^2}{\sigma_s^2}, & k_1 &\equiv 2 \left[-\kappa_x + \frac{1-\gamma}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 \right], & k_2 &\equiv \frac{1}{\gamma} \sigma_x^2 \\ k_3 &\equiv \kappa_x \bar{X}, & k_4 &\equiv \frac{1}{2} \sigma_x^2, & k_5 &\equiv -\delta + (1-\gamma) r_0 \end{aligned}$$

The system of ODEs (2.11) - (2.13) is one and the same as in the problems without income. In fact, this is exactly the solution found in Wachter (2002)¹¹. A detailed discussion is therefore omitted. Nevertheless, for the sake of completeness, Appendix 2.A.1 contains the results of the Wachter model and the two following important results should be kept in mind:

¹⁰In Koo (1998) and Munk (2000) it is shown for an individual with power utility over consumption that under non-perfect correlation between the financial asset and labor income, total wealth and risky investment go to zero as the financial wealth approaches zero. This is intuitive, as otherwise the individual risks ending up with a negative wealth and no income and hence, cannot afford a positive consumption level, which is clearly not optimal.

¹¹It should be noticed that in Wachter (2002) there are several typographical errors. Most notably, b_1 and b_3 are interchanged. All other differences arise from different notation. More important, despite the typographical errors the presented examples in Wachter are not affected and are correct.

- Because of the assumption $\gamma > 1$, it follows that $c_2(s) < 0$ and $c_1(s) < 0$ for $s > 0$. As a consequence, the sign of state variable hedging demand can be determined unambiguously for $X > 0$.
- Given $\gamma > 1$, $c_2(s)$ converges to a finite number as $s \rightarrow \infty$. In other words, the solution of the Riccati differential equation is well-defined.

2.1.2 Separation of the HJB by Y

For the Y parts¹²

$$0 = \int_0^\tau e^{C(X,s)} ds \left\{ \begin{aligned} & -\frac{\partial k}{\partial \tau} - r_0 k + 1 + k(y_0 + y_1 X) - \kappa_x X \frac{\partial k}{\partial X} + \kappa_x \bar{X} \frac{\partial k}{\partial \bar{X}} \\ & -\frac{1}{2} \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 X \frac{\partial k}{\partial X} - \frac{1}{2} \frac{\rho_{sy}\sigma_y}{\sigma_s} \lambda_1 X k - \frac{1}{2} \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 X \frac{\partial k}{\partial X} \\ & -\frac{1}{2} \frac{\rho_{sy}\sigma_y}{\sigma_s} \lambda_1 X k + \rho_{xy}\sigma_x\sigma_y \frac{\partial k}{\partial X} + \frac{1}{2} \sigma_x^2 \frac{\partial^2 k}{\partial X^2} \end{aligned} \right\} \\ + \int_0^\tau e^{C(X,s)} (c_1(s) + c_2(s) X) ds \left\{ \begin{aligned} & \sigma_x^2 \left[-\frac{1}{2} \rho_{sx}^2 - \frac{1}{2} \rho_{sx}^2 + 1 \right] \frac{\partial k}{\partial X} \\ & \sigma_x \sigma_y \left[-\frac{1}{2} \rho_{sx} \rho_{sy} - \frac{1}{2} \rho_{sx} \rho_{sy} + \rho_{xy} \right] k \end{aligned} \right\}$$

With the assumptions (2.c.1) and (2.c.2) or (2.c.3), the second part on the right hand side vanishes and the equation simplifies to

$$0 = \int_0^\tau e^{C(X,s)} ds \left\{ \begin{aligned} & -\frac{\partial k}{\partial \tau} - r_0 k + 1 + k(y_0 + y_1 X) - \kappa_x X \frac{\partial k}{\partial X} + \kappa_x \bar{X} \frac{\partial k}{\partial \bar{X}} \\ & -\frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 X \frac{\partial k}{\partial X} - \frac{\rho_{sy}\sigma_y}{\sigma_s} \lambda_1 X k + \rho_{xy}\sigma_x\sigma_y \frac{\partial k}{\partial X} + \frac{1}{2} \sigma_x^2 \frac{\partial^2 k}{\partial X^2} \end{aligned} \right\} \quad (2.14)$$

It should be noticed that the assumptions enable the complete separation of the solution of the labor income part from the results of the SODE (2.11) - (2.13) and this simplifies the solution considerably.

As $\int_0^\tau e^{C(X,s)} ds > 0$, (2.14) is zero if the part in the brackets is zero. A function of the form

$$k(X, \tau) = \int_0^\tau e^{d_0(s)+d_1(s)X} ds \quad (2.15)$$

with initial conditions $d_1(0) = d_0(0) = 0$ will solve equation (2.14). These initial conditions are the only ones that ensure that (2.16) is solved and that the solution converges to the one of the constant opportunity set ($\lambda_1 = 0$ and $y_1 = 0$). The relevant partial derivatives are as follows

$$\begin{aligned} k_\tau &= \int_0^\tau \left(\frac{\partial d_0(s)}{\partial s} + \frac{\partial d_1(s)}{\partial s} X \right) e^{d_0(s)+d_1(s)X} ds + 1 \\ k_X &= \int_0^\tau d_1(s) e^{d_0(s)+d_1(s)X} ds \\ k_{XX} &= \int_0^\tau d_1(s)^2 e^{d_0(s)+d_1(s)X} ds \end{aligned}$$

Plugging in the partial derivatives into (2.14) leads to

$$0 = \int_0^\tau e^{d_0(s)+d_1(s)X} \left\{ \begin{aligned} & -\left(\frac{\partial d_0(s)}{\partial s} + \frac{\partial d_1(s)}{\partial s} X \right) - r_0 + (y_0 + y_1 X) \\ & -\kappa_x X d_1(s) + \kappa_x \bar{X} d_1(s) - \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 X d_1(s) \\ & -\frac{\rho_{sy}\sigma_y}{\sigma_s} \lambda_1 X + \rho_{xy}\sigma_x\sigma_y d_1(s) + \frac{1}{2} \sigma_x^2 d_1(s)^2 \end{aligned} \right\} ds \quad (2.16)$$

¹²Terms similar to A are directly set to zero because of (2.10).

Matching coefficients on X and the constant term leads to a system of two ordinary differential equations.

$$\frac{\partial d_1(s)}{\partial s} = l_0 + l_1 d_1(s) \quad (2.17)$$

$$\frac{\partial d_0(s)}{\partial s} = l_2 + l_3 d_1(s) + l_4 d_1(s)^2 \quad (2.18)$$

where

$$l_0 \equiv y_1 - \frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1, \quad l_1 \equiv -\kappa_x - \frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1$$

$$l_2 \equiv y_0 - r_0, \quad l_3 \equiv \kappa_x \bar{X} + \rho_{xy}\sigma_x\sigma_y, \quad l_4 \equiv \frac{1}{2}\sigma_x^2$$

The first equation is a linear differential equation with constant coefficients, the second can be solved by integration. It should be noticed that human capital depends on the state variable even if labor income is not directly influenced by the state variable ($y_1 = 0$). The effect stems from the $\frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1$ term in l_0 . The solution of equation (2.17) with initial condition $d_1(0) = 0$ is given by

$$d_1(s) = \begin{cases} \frac{l_0}{l_1}(e^{l_1 s} - 1) & , l_1 \neq 0 \\ l_0 s & , l_1 = 0 \end{cases} \quad (2.19)$$

Because of the simple form of $d_1(s)$ and $\frac{\partial d_0(s)}{\partial s} = l_2 + l_3 d_1(s) + l_4 d_1(s)^2$, the solution of $d_0(s)$ is also available in closed-form. Simple integration yields

$$d_0(s) = \begin{cases} \left(l_2 - l_3 \frac{l_0}{l_1} + l_4 \frac{l_0^2}{l_1^2} \right) s + \left(l_3 \frac{l_0}{l_1^2} - 2l_4 \frac{l_0^2}{l_1^3} \right) (e^{l_1 s} - 1) \\ \quad + \frac{1}{2} l_4 \frac{l_0^2}{l_1^3} (e^{2l_1 s} - 1) & , l_1 \neq 0 \\ l_2 s + \frac{1}{2} l_3 l_0 s^2 + \frac{1}{3} l_4 l_0^2 s^3 & , l_1 = 0 \end{cases} \quad (2.20)$$

Remarks

- From (2.15) it can be easily seen that $k(X, \tau) > 0$ for $\tau > 0$. This is intuitive as the risky part of labor income Y cannot become negative and hence, the individual attaches a positive value to the future labor income stream¹³.
- As will become clear from the phase plane analysis of Section 2.3, in order that the solution $d_1(s)$ converges for long horizon, $l_1 = -\kappa_x - \frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1 < 0$. Thus, given l_0 , the stability of $d_1(s)$ does not depend on parameters of the labor income process, but only on parameters of the risky asset and the state variable.
- The term $y_1 - \beta_{sy}\lambda_1$ can be interpreted as a pricing formula for the wage premium similar to the CAPM, with $\beta_{sy} \equiv \frac{\rho_{sy}\sigma_y}{\sigma_s}$. In other words, if the wage compensation is in accordance

¹³To be more precise, k gives only the value of one unit of stochastic labor income Y and not of \bar{Y} . Because this is obvious, we continue to use the used terminology and do not mention this explicitly for the remainder of the thesis.

with the stock market compensation¹⁴, $y_1 - \frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1 = 0$ and $d_1(s) = 0, \forall s$. As will be shown below, if $y_1 - \frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1 \neq 0$, an adapted state variable hedging demand will arise.

- The risk aversion parameter γ is not involved in the valuation of the income stream. This is intuitive, as the assumption of complete markets allows for a perfect hedge of labor income risk.

2.1.3 Separation the HJB by the Constant Terms

Finally, for the constant parts¹⁵,

$$0 = \int_0^\tau e^{C(X,s)} ds \left\{ \frac{\partial R}{\partial \tau} + \bar{Y} - \bar{c} + r_0 R \right\} \quad (2.21)$$

By the same arguments as above, the equation is zero if the term in the brackets is zero. The equation in the brackets is a linear differential equation with constant coefficients and initial condition $R(0) = 0$

$$R(\tau) = \frac{\bar{c} - \bar{Y}}{r_0} (1 - e^{-r_0 \tau}) \quad (2.22)$$

Since $\frac{\bar{c} - \bar{Y}}{r_0}$ is the value of a perpetual bond that pays $\bar{c} - \bar{Y}$ as its coupon, it becomes clear that (2.21) can be interpreted as the reserves necessary to cover the subsistence level of consumption net of the minimum wage that is guaranteed.

Remarks

- From the derivation in Appendix 2.A.2 and (2.22) it can be noticed that the constant part of labor income enters the HJB in the same way as the (constant) subsistence level of consumption. For this reason, it can be stated that only the difference $\bar{c} - \bar{Y}$ really matters for valuation of the net reserves. Moreover, the optimal investment decision depends only on the difference $\bar{c} - \bar{Y}$ as well. Nevertheless, optimal consumption is directly affected by \bar{c} and hence, individuals with the same $\bar{c} - \bar{Y}$ but different \bar{c} hold an identical portfolio, have same excess consumption $c_t^* - \bar{c}$, but have different consumption levels.
- Given the solution of k and R , total wealth $\hat{A} \equiv (\dots) = A + kY - R$ can be structured in a more interpretable form¹⁶. In fact, $\hat{A} = A + H - N$. A is financial wealth of the individual, $H \equiv kY + \frac{\bar{Y}}{r_0} (1 - e^{-r_0 \tau})$ is human capital and $N \equiv \frac{\bar{c}}{r_0} (1 - e^{-r_0 \tau})$ are the reserves covering the subsistence level of consumption.

2.1.4 Optimal Policies

As shown in Appendix 2.A.2, plugging in the relevant partial derivatives into the FOCs leads to

$$c_t^* = \frac{\hat{A}}{\int_0^\tau e^{C(X,s)} ds} + \bar{c} \quad (2.23)$$

¹⁴In this case, the solution of $k(X, \tau)$ collapses to $k(\tau) = \frac{1}{y_0 - r_0} (e^{(y_0 - r_0)\tau} - 1)$ and is similar to the constant labor income growth case.

¹⁵Terms that are similar to A are directly neglected because they are equal to zero because of (2.10).

¹⁶A similar interpretation is common, see, for example, Koo (1998) or Huang and Milevsky (2008).

and

$$\begin{aligned}
A\pi_t^* = & \frac{1}{\gamma} \frac{\lambda_1}{\sigma_s^2} X \hat{A} + \frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \frac{\int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} \hat{A} \\
& - \frac{\rho_{sx}\sigma_x}{\sigma_s} \frac{\partial k}{\partial X} Y - \frac{\rho_{sy}\sigma_y}{\sigma_s} k Y
\end{aligned} \tag{2.24}$$

where \hat{A} is total wealth.

Remarks

- *Optimal consumption (2.23) consists of two parts. Only the first part varies over time, the subsistence part is constant. As a consequence, consumption varies less strongly than total wealth. Indeed, in the classical Merton (1969) model, consumption has the same variation as financial wealth, which is implausible¹⁷. In Wachter (2002), the relation is not one-to-one, but since the variation in $\int_0^\tau e^{C(X,s)} ds$ is low, the relation is close. Adding a subsistence level of consumption eases this issue.*
- *For individuals close to the margin of subsistence ($\hat{A} \rightarrow 0$), optimal consumption converges to \bar{c} and its variation disappears.*
- *The first two terms of the optimal investment rule (2.24) are identical to Wachter (2002).*
- *For individuals close to the margin of subsistence ($\hat{A} \rightarrow 0$), the first two parts vanish.*
- *The third term of optimal risky investment (2.24) is state variable hedging demand that arises under labor income. It does not vanish for individuals close to the margin of subsistence. Furthermore, it even exists if labor income is locally riskfree ($\sigma_y = 0$) or the correlation between labor income and the risky asset is zero ($\rho_{sy} = 0$). Of course, it is necessary that the risky asset and the state variable are correlated ($\rho_{sx} \neq 0$). It is shown below that this part is negative for individuals with unfavorable income characteristics, which helps to explain the low equity exposure of low-educated and poor individuals.*
- *Furthermore, the third term of optimal risky investment (2.24) has a natural interpretation. In fact, partitioning the third term into*

$$\underbrace{-\rho_{sx}}_{i)} \underbrace{\frac{\sigma_x}{\sigma_s}}_{ii)} \underbrace{\frac{\partial k}{\partial X}}_{iii)} Y$$

allows the following interpretation. Most importantly, iii) is the first derivative of the value per unit of labor income on X . In other words, this part gives the change in the value of one unit of labor income when the state variable moves. ii) is a multiplier that relates the strength of the shocks of the risky asset and the state variable. i) is simply plus or minus one and gives the direction the state variable moves in relation to the risky asset. Thus, it can be summarized that this third term is a hedge for the value of the future income stream to changes in the state of the economy.

¹⁷ Cochrane (2007, p.76).

- The last term is hedging demand for labor income risk. This part does not vanish for individuals close to the margin of subsistence¹⁸. For positive (negative) ρ_{sy} it will decrease (increase) the amount invested in the risky asset. Moreover, it will vanish if labor income returns are uncorrelated with the risky asset or if labor income is locally riskfree.

2.1.5 Dynamics of Optimal Total Wealth

In Appendix 2.A.3 it is shown that under assumptions (2.c.1) - (2.c.3), optimal total wealth follows

$$\begin{aligned} \frac{d\hat{A}^*}{\hat{A}^*} = & \left(r_0 + \frac{1}{\gamma} \frac{\lambda_1^2}{\sigma_s^2} X^2 \right) dt \\ & + \left(\frac{1}{\gamma} \frac{\rho_{sx} \sigma_x}{\sigma_s} \lambda_1 \frac{\int_0^\tau (c_1(s) X + c_2(s) X^2) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} - \frac{1}{\int_0^\tau e^{C(X,s)} ds} \right) dt \\ & + \frac{1}{\gamma} \left(\frac{\lambda_1}{\sigma_s} X + \rho_{sx} \sigma_x \frac{\int_0^\tau (c_1(s) X + c_2(s) X^2) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} \right) dW_s(t) \end{aligned} \quad (2.25)$$

Remarks

- Under the assumption of perfect correlation, the individual is able to hedge the labor income risk entirely. Optimal total wealth follows a geometric Brownian motion with time-varying coefficients and will stay non-negative in all cases. Hence, given that initial total wealth $\hat{A}(0) > 0$, the individual will be able to afford the subsistence level of consumption in all cases.
- As can be seen, optimal total wealth follows the same dynamics as financial wealth in the case without labor income and a subsistence level of consumption (Wachter model). The individual takes into account the additional wealth due to human capital and the reduction in wealth due to the reserves covering the subsistence level and controls his total wealth in the same manner as the investor in the Wachter model controls financial wealth.
- In Appendix 3.A.1 of Chapter 3 the valuation of the future income stream is performed using the martingale approach. Since the assumption of complete markets implies that the market price of risk is unique and the risk-neutral valuation asks for the absence of arbitrage, it is not surprising that the value of the future labor income stream is a combination of the riskfree and the risky asset. As a consequence, the special relation of financial and non-financial assets allows them to be absorbed in one factor - total wealth.

2.1.6 Main Results

The most important results can be summarized in the following proposition.

¹⁸This part is well known, see, for example, Koo (1998) or Viceira (2001).

PROPOSITION 2.1 *Given the assumptions $\hat{A}(0) > 0$, $\rho_{sx} \in \{-1, 1\}$ and either $\rho_{sy} \in \{-1, 1\}$ or $\sigma_y = 0$ one obtains*

$$J = \frac{e^{-\delta(T-\tau)} \left[\int_0^\tau e^{\frac{1}{\gamma}(c_0(s)+c_1(s)X+\frac{1}{2}c_2(s)X^2)} ds \right]^\gamma (A + k(\tau, X)Y - R(\tau))^{1-\gamma}}{1 - \gamma}$$

with

$$k(\tau, X) = \int_0^\tau e^{d_0(s)+d_1(s)X} ds$$

where $d_0(s)$ and $d_1(s)$ are the solution to the following system of ordinary differential equations

$$\begin{aligned} \frac{\partial d_1(s)}{\partial s} &= l_0 + l_1 d_1(s) \\ \frac{\partial d_0(s)}{\partial s} &= l_2 + l_3 d_1(s) + l_4 d_1(s)^2 \end{aligned}$$

with initial conditions $d_0(0) = 0$ and $d_1(0) = 0$ and where

$$\begin{aligned} l_0 &\equiv y_1 - \frac{\rho_{sy}\sigma_y}{\sigma_s} \lambda_1, & l_1 &\equiv -\kappa_x - \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 \\ l_2 &\equiv y_0 - r_0, & l_3 &\equiv \kappa_x \bar{X} + \rho_{xy}\sigma_x\sigma_y, & l_4 &\equiv \frac{1}{2}\sigma_x^2 \end{aligned}$$

The net reserves follow

$$R(\tau) = \frac{\bar{c} - \bar{Y}}{r_0} (1 - e^{-r_0\tau})$$

The solutions of $c_0(s)$, $c_1(s)$ and $c_2(s)$ are identical to Wachter (2002).

Optimal consumption and risky investment are given by

$$c_t^* = \frac{\hat{A}}{\int_0^\tau e^{C(X,s)} ds} + \bar{c} \quad (2.26)$$

$$\begin{aligned} A\pi_t^* &= \frac{1}{\gamma} \frac{\lambda_1}{\sigma_s^2} X \hat{A} + \frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \frac{\int_0^\tau (c_1(s) + c_2(s)X) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} \hat{A} \\ &\quad - \frac{\rho_{sx}\sigma_x}{\sigma_s} \left(\int_0^\tau d_1(s) e^{d_0(s)+d_1(s)X} ds \right) Y - \frac{\rho_{sy}\sigma_y}{\sigma_s} kY \end{aligned} \quad (2.27)$$

2.2 Model with Utility over Terminal Wealth

Following from this, the similar problem for an investor with utility over terminal wealth only is solved. Kim and Omberg (1996) and Liu (2007) show that without labor income, the assumption $\rho_{sx} \in \{-1, 1\}$ is not necessary in order to obtain closed-form solutions. We will show that in the presence of labor income this assumption is necessary as well. Thus, compared to the consumption problem, the set of assumptions (2.c.1) - (2.c.3) is the same. Nevertheless, the results of this part can be compared with the consumption framework.

Expected utility is given by

$$E_t \left[\frac{(A_T - \bar{A})^{1-\gamma}}{1 - \gamma} \right], \quad \gamma > 1$$

Following the same steps as above, the following HJB-equation results

$$\begin{aligned}
0 = & J_t + J_A A(t) r_0 + J_A \bar{Y} + J_A Y(t) \\
& - J_X \kappa_x (X(t) - \bar{X}) + J_Y Y(t) (y_0 + y_1 X(t)) \\
& + \frac{1}{2} J_{AA}(t) \pi_t^* \lambda_1 X(t) + \frac{1}{2} J_{AX}(t) \pi_t^* \rho_{sx} \sigma_x \sigma_s + \frac{1}{2} J_{AY}(t) Y(t) \pi_t^* \rho_{sy} \sigma_y \sigma_s \\
& + J_{XY}(t) \rho_{xy} \sigma_x \sigma_y + \frac{1}{2} J_{YY}(t) \sigma_y^2 + \frac{1}{2} J_{XX}(t) \sigma_x^2
\end{aligned} \tag{2.28}$$

where π_t^* is also the FOC given by (2.7).

The solution to this problem can be found in Appendix 2.A.4. The results are summarized in the following proposition.

PROPOSITION 2.2 *Given the assumptions $\hat{A}(0) > 0$, $\rho_{sx} \in \{-1, 1\}$ and either $\rho_{sy} \in \{-1, 1\}$ or $\sigma_y = 0$ one obtains*

$$J = \frac{e^{c_0(\tau) + c_1(\tau)X + \frac{1}{2}c_2(\tau)X^2} (A + k(\tau, X)Y - R(\tau))^{1-\gamma}}{1-\gamma}$$

with $k(\tau, X)$ identical to Proposition 2.1. The net reserves follow

$$R(\tau) = -\frac{\bar{Y}}{r_0} (1 - e^{-r_0\tau}) + e^{-r_0\tau} \bar{A} \tag{2.29}$$

The solutions of $c_0(s)$, $c_1(s)$ and $c_2(s)$ are identical to Wachter (2002).

Optimal risky investment is given by

$$\begin{aligned}
A\pi_t^* = & \frac{1}{\gamma} \frac{\lambda_1}{\sigma_s^2} X \hat{A} + \frac{1}{\gamma} \frac{\rho_{sx} \sigma_x}{\sigma_s} (c_1(\tau) + c_2(\tau)X) \hat{A} \\
& - \frac{\rho_{sx} \sigma_x}{\sigma_s} \left(\int_0^\tau d_1(s) e^{d_0(s) + d_1(s)X} ds \right) Y - \frac{\rho_{sy} \sigma_y}{\sigma_s} kY
\end{aligned}$$

where $\tau \equiv T - t$.

Besides the analytical derivation, the necessity of the assumption $\rho_{sx} \in \{1, -1\}$ is intuitive in order to get explicit solutions. In fact, from the definition of total wealth it can be clearly recognized that the valuation of the stochastic part of labor income k depends on X . Now, if total wealth is close to zero and there is a shock to the state variable, k can fall and the individual risks ending up with negative total wealth, which is clearly not optimal. This undesirable situation can only be avoided if state variable risk to the stochastic labor income stream can be hedged perfectly. If not, the level of financial wealth becomes important for the value of the income stream and the described separation of the HJB is not possible. Hence, an extended Wachter model with non-perfect correlation between the risky asset and the state variable could be solved in closed-form with the extensions deterministic labor income and subsistence wealth¹⁹.

As can be seen from the optimal policy, only the second term of the RHS is changed. The differences for optimal investment are extensively discussed in Wachter (2002). Wachter shows that the differences in the consumption and terminal wealth case are rather small. For this reason, and for the sake of brevity, we focus for the illustration of the results on the consumption case.

¹⁹The extension subsistence wealth (HARA utility) was already solved by Kim and Omberg (1996).

2.3 Long-Horizon Stability of the Solution

The first equation of the system of ODE (2.17) - (2.18) is a linear differential equation with constant coefficients. Thus, there is no assumption that ensures that the solution is stable as $s \rightarrow \infty$. Figure 2.1 shows the phase plane analysis for the linear differential equation. Four cases can be distinguished, l_0 determines whether the axis intercept is positive or negative and l_1 defines the sign of the slope.

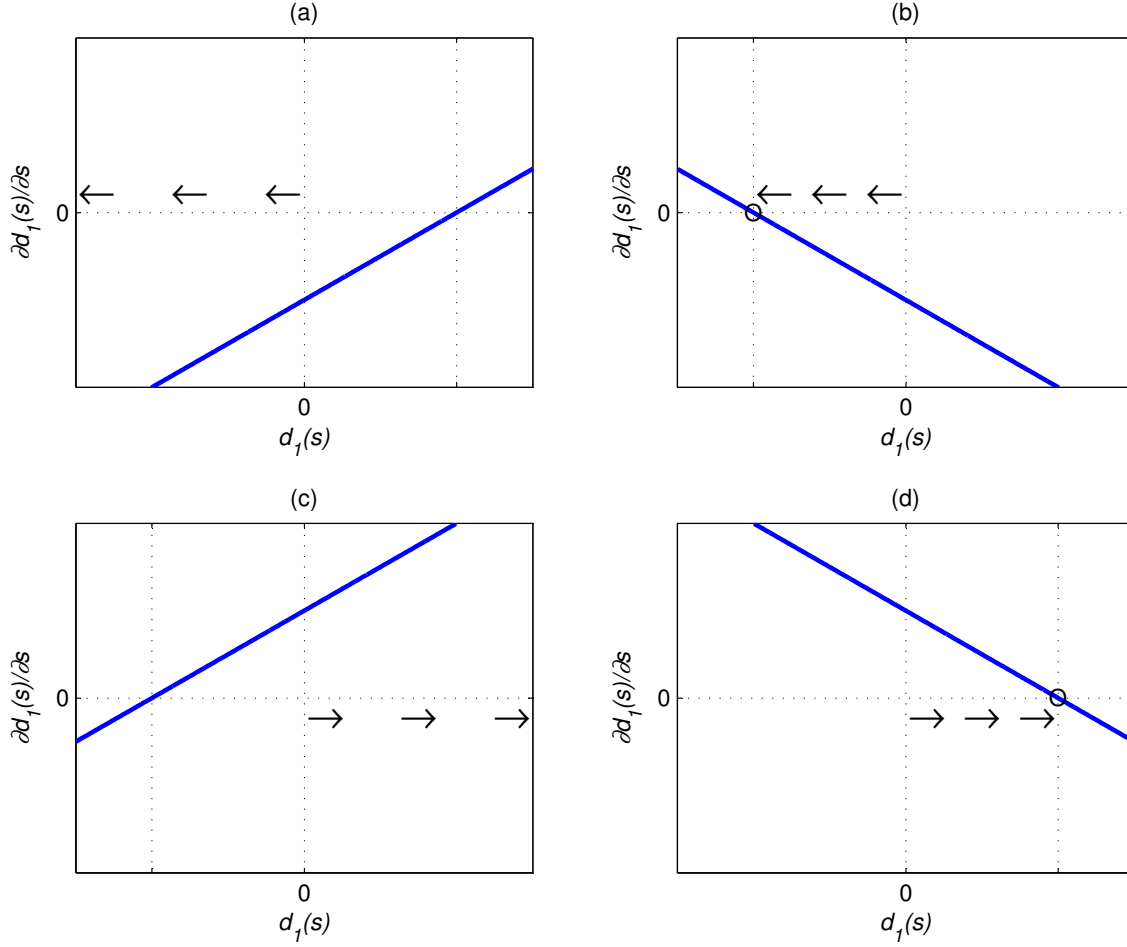


Figure 2.1: Phase Plane Analysis I

Panels (a), (b), (c) and (d) show a phase plane analysis of the equation $\frac{\partial d_1(s)}{\partial s} = l_0 + l_1 d_1(s)$. In all cases one real particular solution exists. In Panels (a) and (c), $l_1 > 0$ in Panels (b) and (d), $l_1 < 0$. Only in Panels (b) and (d) does $d_1(s)$ converge to a stable solution marked by a circle.

The following properties should be noted.

Remarks

- Figure 2.1 shows that the sign of $d_1(s)$ is equal to the sign of l_0 (vertical axis intercept).
- From Figure 2.1 it can be seen that $d_1(s)$ is either monotonically increasing or decreasing in the time horizon and does not change sign.

- *Only the slope is crucial for stability and hence the coefficient l_1 becomes important. It should be emphasized that this coefficient contains only information with respect to the financial market and no labor income parameter is involved. As it is assumed that the working period is finite, no transversality condition has to be stated. Nevertheless, it should be kept in mind that $l_1 > 0$ is in favor of extreme results as $k(\tau, X)$ (and $\partial k(\tau, X)/\partial X$) grows without bound.*

Table 2.1 summarizes the stability analysis. There is a crucial warning for numerical studies of the consumption-investment problem with labor income that are calibrated on empirical results. From the definition of l_1 , if the state variable is highly persistent (low κ_x) and the correlation between the state variable and the risky asset is close to -1 then the valuation of the income stream can be extreme even if the sensitivity of the income process on the state variable is low.

			Stable?
(a)	$l_0 < 0,$	$l_1 > 0$	no
(b)	$l_0 < 0,$	$l_1 < 0$	yes
(c)	$l_0 > 0,$	$l_1 > 0$	no
(d)	$l_0 > 0,$	$l_1 < 0$	yes

Table 2.1: Stability Analysis

2.4 Illustration of Results

As mentioned above, the model without non-financial income and CRRA utility was presented by Wachter (2002). Moreover, Campbell et al. (2004) consider a similar model for the financial market with more sophisticated utility²⁰.

Wachter			Campbell et al.		
$r_0 = 0.0168$			$r_0 = 0.0033$		
$\lambda_1 = 12$	$\sigma_s = 0.1510$		$\lambda_1 = 4$	$\sigma_s = 0.1579$	
$\kappa_x = 0.2712$	$\bar{X} = 0.0034$	$\sigma_x = 0.0029$	$\kappa_x = 0.1755$	$\bar{X} = 0.0132$	$\sigma_x = 0.0115$
$\rho_{sx} = -1$			$\rho_{sx} = -1$		

Table 2.2: Financial Market Parameter Values - Wachter and Campbell et al.

Table 2.2 shows their parameters in annualized form and adapted to our notation²¹. Wachter

²⁰Stochastic differential utility (SDU) allows for separating risk aversion from intertemporal elasticity of substitution and was introduced by Duffie and Epstein (1992b).

²¹The notations differ only by an invariant affine transformation. Hence, results are not affected. Invariant affine transformation (IAT) are well-known from the term structure literature as, see for example, Dai and Singleton (2000).

and Campbell et al. use the dividend yield as the single state variable²².

For the sake of comparability, the values in Table 2.2 are normalized to $\lambda_1 = 1$ by another IAT²³ and are displayed in Table 2.3.

Wachter			Campbell et al.		
$r_0 = 0.0168$			$r_0 = 0.0033$		
$\lambda_1 = 1$	$\sigma_s = 0.1510$		$\lambda_1 = 1$	$\sigma_s = 0.1579$	
$\kappa_x = 0.2712$	$\bar{X} = 0.0408$	$\sigma_x = 0.0348$	$\kappa_x = 0.1755$	$\bar{X} = 0.0528$	$\sigma_x = 0.0460$
$\rho_{sx} = -1$			$\rho_{sx} = -1$		

Table 2.3: Financial Market Parameter Values - Normalized

Remarks

- *The sample period and the sample frequency differ somewhat. In Wachter (Campbell et al.), the sample period is given as 1952–1995 (1947.1–1995.4) and monthly (quarterly) data is used. The estimated correlation is not exactly $\rho_{sx} = -1$ but close to this. In the Campbell et al. dataset $\rho_{sx} = -0.963$ and in the Wachter dataset $\rho_{sx} = -0.935$.*
- *The stationary distribution of the state variable is normally distributed $N(\mu, \sigma^2)$ with $\mu = \bar{X}$ and $\sigma^2 = \sigma_x^2 / (2\kappa_x)$. For the Wachter parameter set, this is $N(0.0408, 0.0022)$ and for the Campbell et al. dataset, $N(0.0528, 0.0060)$. From $\lambda_1 = 1$ and the standard deviation of the normal distributions of 4.69% and 7.75% respectively, it can be directly seen that the estimated parameters imply very strong variations in the premium.*

For the sake of brevity, we will work with an adapted Campbell et al. (2004) dataset only. In our opinion, the implied variation in the equity premium is much too strong. In fact, both datasets imply that a variation of 20 percentage points is not unusual and furthermore, that there is a considerable probability of a negative equity premium. Moreover, the estimation error in Table 1 of Campbell and Viceira (2000) and the work of Goyal and Welch (2008) suggest that assuming a lower variation can also be justified from an empirical point of view. As a consequence, we reduce this variation of X by adjusting σ_x for the basic setting. Nevertheless, we will look at the implication of the parameter set of Wachter and Campbell et al. as a part of the sensitivity analysis with respect to σ_x . The parameters for non-financial income are chosen variably in order to show the effects clearly. To sum up, the initial parameter set is given in Table 2.4.

The choice of σ_x implies that the state variable has an unconditional standard deviation of 1.5%, which yields a considerably lower variation in the equity premium.

It should be noted that the chosen parameters imply a low component of stochastic labor income at the beginning of the working period. In fact, Table 2.4 reveals that only a quarter of

²²For the predictive power of the dividend yield see, for example, Fama and French (1988), Campbell and Thompson (2007) and Cochrane (2005, 2008). On the other hand, Goyal and Welch (2008) doubt that the dividend yield is a good predictor.

²³See Appendix 2.A.5.

initial total income is non-constant. Increasing the importance of Y compared to \bar{Y} would give even more weight to the impact of stochastic labor income²⁴.

Financial Market		
$r_0 = 0.0033$		
$\lambda_1 = 1$	$\sigma_s = 0.1579$	
$\kappa_x = 0.1755$	$\bar{X} = 0.0528$	$\sigma_x = 0.0089$
$\rho_{sx} = -1$		
Individual		
$\gamma = 4$	$\delta = 0.06$	
$\bar{y} = 0.03$		
$A(0) = 50$	$Y(0) = 10$	$\bar{Y} = 40$
$\bar{c} = 45$		

Table 2.4: Parameter Values

The parameters for the labor income process (2.3) are chosen variably in order to show the effects clearly. For the sake of comparability, y_0 and y_1 are chosen, so that the growth rate at the long-run mean \bar{X} is constant. Specifically,

$$y_0 = \bar{y} - y_1 \bar{X} \quad (2.30)$$

where \bar{y} is the long-run growth rate and given in Table 2.4.

For these kind of models, a critical issue is that the parameters for the financial and the non-financial processes are chosen exogenously. In a general equilibrium model with a production side that is driven by technology and resources, the financial and labor markets should be linked in a reasonable way. For this reason, certain parameter choices for the illustrative examples might be unrealistic.

Nevertheless, several studies have shown that the labor market adapts less quickly to changes in the real economy than the stock market²⁵. In fact, institutional conditions such as long-term labor contracts, unions and so on suggest that a simple equilibrium relation between the two kinds of income do not exist. Indeed, the empirical estimation of Table 1 in Lynch and Tan (2009) suggests counter cyclical patterns, i.e. low labor income growth when expected returns are high. Hence, it seems an appropriate choice to treat the parameters freely in order to understand the sensitivity of the results for a range of reasonable parameter values.

In order to reduce the dimension of the problem, we start by discussing the case of locally riskfree labor income $\sigma_y = 0$. In a second step, risky labor income is introduced. In this case, it is assumed that²⁶ $\rho_{sy} = 1$.

²⁴Of course, the growth rate and the volatility of Y would have to be reasonable adjusted in order to have comparable income paths.

²⁵See, for example, Stock and Watson (1999).

²⁶The case $\rho_{sy} = -1$ can be derived in analogy.

The focus of the interpretation is on the state variable dimension for an individual with a given time horizon, a given wealth and a given initial income. Of course, the optimal allocation and consumption over the time horizon is an important issue as well and the work of Lynch and Tan (2009) and Munk and Sørensen (2010) focus exclusively on this dimension. Nevertheless, as has become clear from the phase plane analysis of Section 2.3, the magnitude of hedging demands decrease for shorter horizon. As a consequence, the results for shorter horizons are analogous and do not bring any surprising results²⁷.

In all figures, the center of the horizontal axis corresponds to \bar{X} , and the grid points show $(\bar{X} - 3\sigma, \bar{X} - 1.5\sigma, \bar{X}, \bar{X} + 1.5\sigma, \bar{X} + 3\sigma)$ where σ is the standard deviation of the unconditional normal distribution of the state variable as was defined above. It should be noticed that since λ_1 is normalized to one, the horizontal axis shows the (annualized) equity premium. Furthermore, in all figures the panels on the left and on the right hand side have the same scale for the vertical axis. However, the ranges of the vertical axis can differ.

For the sake of clarity, we introduce the following definitions for the components of risky investment

$$\begin{aligned}
 A\pi_t^* = & \underbrace{\frac{1}{\gamma} \frac{\lambda_1}{\sigma_s^2} X \hat{A}}_{\text{"myopic"}} + \underbrace{\frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \frac{\int_0^\tau (c_1(s) + c_2(s)X) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} \hat{A}}_{\text{"state variable hedging"}} \\
 & - \underbrace{\frac{\rho_{sx}\sigma_x}{\sigma_s} \left(\int_0^\tau d_1(s) e^{d_0(s)+d_1(s)X} ds \right) Y}_{\text{"indirect labor hedging"}} - \underbrace{\frac{\rho_{sy}\sigma_y}{\sigma_s} \left(\int_0^\tau e^{d_0(s)+d_1(s)X} ds \right) Y}_{\text{"direct labor hedging"}} \quad (2.31)
 \end{aligned}$$

As far as possible, the presented results are justified by an analytical argumentation and followed by an economic intuition.

2.4.1 Locally Riskfree Labor Income

The assumption of locally riskfree labor income simplifies the results considerably and facilitates the interpretation. In particular, direct labor hedging demand vanishes and SODE (2.17) - (2.18) reduces to

$$\begin{aligned}
 \frac{\partial d_1(s)}{\partial s} &= l_0 + l_1 d_1(s) \\
 \frac{\partial d_0(s)}{\partial s} &= l_2 + l_3 d_1(s) + l_4 d_1(s)^2
 \end{aligned}$$

where

$$\begin{aligned}
 l_0 &\equiv y_1, \quad l_1 \equiv -\kappa_x - \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 \\
 l_2 &\equiv y_0 - r_0, \quad l_3 \equiv \kappa_x \bar{X}, \quad l_4 \equiv \frac{1}{2} \sigma_x^2
 \end{aligned}$$

The results are displayed in Figures 2.2 - 2.4. The blue and red lines in the left (right) panels belong to an individual with a negative (positive) labor income sensitivity of $y_1 = -0.5$ and

²⁷See, for example, Wachter (2002, p. 76), Figure 2, for an illustration of this statement.

$y_1 = -0.25$ ($y_1 = 0.5$ and $y_1 = 0.25$) respectively²⁸. The green lines contain the results for an individual where the growth rate of labor income is constant $y_1 = 0$ and is the same in the left and the right panels. It serves as a benchmark case and eases the comparison.

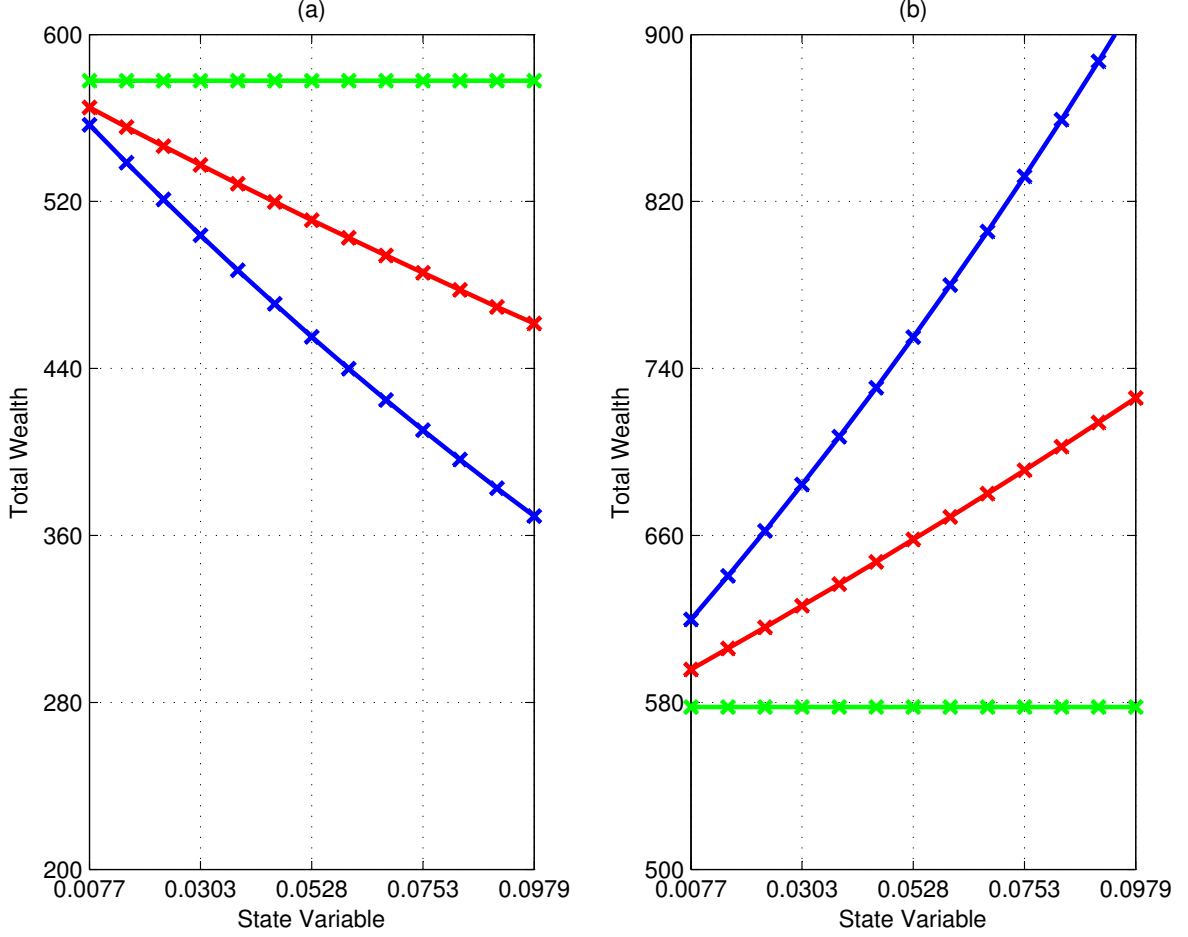


Figure 2.2: Total Wealth - Locally Riskfree Labor Income

This Figure shows total wealth \hat{A} dependent on the state variable under locally riskfree labor income $\sigma_y = 0$. Parameters are chosen as in Table 2.4. In Panel (a) the sensitivity of labor income growth is non-positive, the blue (red, green) line shows the results for an individual with $y_1 = -0.5$ ($-0.25, 0$). In Panel (b) the sensitivity of labor income growth is non-negative, the blue (red, green) line shows the results for an individual with $y_1 = 0.5$ ($0.25, 0$).

Figure 2.2 shows the result for total wealth. In the case of locally riskfree labor income, the sign of $d_1(s)$, $s > 0$ is unambiguously determined by the sign of y_1 . The *slope* of total wealth follows immediately from

$$\frac{\partial \hat{A}}{\partial X} = \frac{\partial k}{\partial X} Y = \left(\int_0^\tau d_1(s) e^{d_0(s) + d_1(s)X} ds \right) Y$$

By the positivity of Y and the exponential function, the sign of $\partial \hat{A} / \partial X$ is given by the sign of $d_1(s)$.

²⁸The value $y_1 = -0.5$ compared to λ_1 is the one that is most in line with the estimates of Lynch and Tan (2009).

This result is intuitive, as high states of X imply low labor income growth and hence the value of the future income stream declines. The results for the right panel can be interpreted in analogy²⁹.

Furthermore, it should be recognized that at $X = \bar{X}$ the *level* of total wealth is considerably lower for small values of y_1 . Since X has a symmetric distribution around \bar{X} this result is not trivial. The intuition comes from the desire to have an intertemporally favorable environment. In fact, it is assumed that $\rho_{sx} = -1$ and this implies that high states of X follow a decline in the value of the risky asset. If $y_1 > 0$ the labor income stream has a high growth rate after a decline in the financial market and this matches the aim of intertemporal hedging, i.e. to be in a good state after a negative return and vice versa.

Appendix 2.A.6 shows that strict analytical results with respect to this property are not available³⁰. Nevertheless, from the derivations it can be stated that the term

$$\Psi \equiv l_0 \psi \bar{X} = y_1 \psi \bar{X}$$

becomes crucial in order to determine the effect of the sensitivity of labor income growth onto the value of the income stream. In fact, it is shown that for $\Psi > 0$ ($\Psi < 0$) the valuation of the income stream at $X = \bar{X} > 0$ is higher (lower) compared to the value of a constant income stream with an identical growth rate (green line).

ψ is defined by

$$\psi \equiv \left[\frac{1}{l_1} (e^{l_1 s} - 1) - s \right] \left(1 + \frac{l_3 / \bar{X}}{l_1} \right) = \left[\frac{1}{l_1} (e^{l_1 s} - 1) - s \right] \left(1 - \frac{\kappa_x}{\kappa_x + \frac{\rho_{sx} \sigma_x}{\sigma_s} \lambda_1} \right)$$

From Table 2.5 it should be noticed that ψ is positive if $\rho_{sx} < 0$ (second and third columns) and negative if $\rho_{sx} > 0$ (first column).

	$l_1 < \kappa_x$	$\kappa_x < l_1 < 0$	$l_1 > 0$
$\text{sign} \left(\frac{1}{l_1} (e^{l_1 s} - 1) - s \right)$	—	—	+
$\text{sign} \left(1 + \frac{\kappa_x}{l_1} \right)$	+	—	+
$\text{sign}(\psi)$	—	+	+

Table 2.5: Sign of ψ

It can be seen that the sign of the effect depends crucially on the opportunity set of the financial market. In fact, if low returns on the financial asset are followed by high expected returns on the risky asset

$$\rho_{sx} = -1 < 0 \Rightarrow l_1 > \kappa_x \Rightarrow \psi > 0$$

²⁹For the sake of readability, if the relation is clear this statement is omitted for the remainder of the text.

³⁰As an alternative to Appendix 2.A.6, the absence of unambiguous results can be verified by looking at the equation that determines $d_0(s)$. In fact, for extreme σ_x $d_0(s)$ can become very large because of the unambiguously positive term $l_4 d_1(s)^2$ in equation (2.18). Hence, the level of future income at $X = \bar{X}$ could be higher compared to the constant income growth case even for $y_1 < 0$.

and, in addition, the individual faces low labor income growth ($y_1 < 0$), the individual attaches a lower value to the income stream.

An alternative view gives a risk-neutral valuation of the future income stream. In Appendix 3.A.1 of Chapter 3, the value of future stochastic income $G \equiv kY$ is derived by the Martingale approach. From equation (3.12) it can be recognized that the risk-neutral valuation and the no-arbitrage condition imply that G is priced in accordance with the market price of risk θ . It must be noticed that the RHS of (3.12) gives the expected premium in excess of the riskless rate that G must deliver. Since

$$y_1 < 0 \Rightarrow \frac{\partial G}{\partial X} = \frac{\partial k}{\partial X} Y < 0 \quad \wedge \quad \rho_{sx} = -1 < 0$$

implies that the expected excess return of G must be positive, the value of G must be lower compared to $y_1 = 0$ ($\partial G / \partial X = 0$). Indeed, this is similar to a financial asset that is discounted at a higher rate. The case $y_1 > 0$ can be derived in analogy and a higher value for G results.

Panels (a) and (b) of Figure 2.3 show optimal total risky investment. The components according to (2.31) are displayed in the lower panels. It can be recognized that a negative (positive) y_1 reduces (increases) the exposure in the risky asset. In the left panels, myopic demand and state variable hedging demand are reduced because of the lower total wealth, while indirect labor hedging demand is negative because of the sign of $d_1(s)$. Besides, because $\sigma_y = 0$, direct labor hedging demand is zero.

The sign of indirect labor hedging demand can be understood as follows. The situation is easiest to understand for an individual close to the margin of subsistence. In this case, myopic and state variable hedging demand are close to zero and risky investment is determined by indirect labor hedging demand. In the case $y_1 < 0$, the demand is negative. This has to be the case, since a rise in X leads to a decrease in the growth rate of Y and this has a negative impact on total wealth. The position in the risky asset must compensate the decline in future labor income in order to prevent total wealth from becoming negative.

The sign of indirect labor hedging demand can be further explained in analogy to state variable hedging demand as already described in Kim and Omberg (1996) and Wachter (2002). State variable hedging demand is positive because the individual likes high expected returns after a decline in the risky asset and this is the case for $\rho_{sx} < 0$. Now, in case of $y_1 < 0$ labor income delivers the opposite, a low growth rate after a decline in the risky asset. To compensate for this undesirable situation the individual takes a short position in the risky asset. The combination of the labor income stream and this short position creates the situation she likes. In fact, this position generates high returns followed by low income growth rates and vice versa.

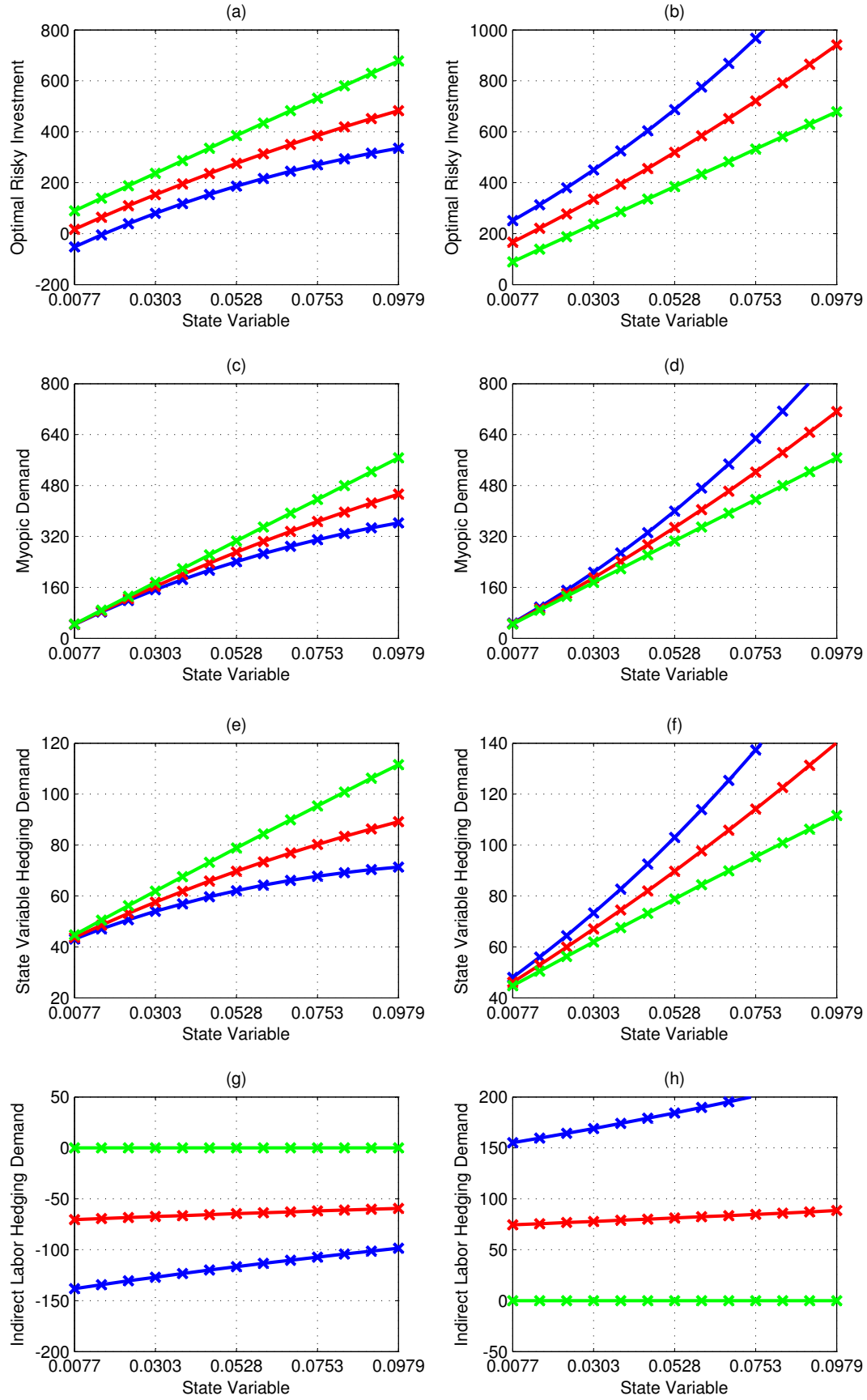


Figure 2.3: Optimal Risky Investment - Locally Riskfree Labor Income

Figure 2.3 continued: Panels (a) and (b) show optimal total investment in the risky asset $A\pi_t^*$ dependent on the state variable under locally riskfree labor income $\sigma_y = 0$. Parameters are chosen as in Table 2.4. In the panels to the left the sensitivity of labor income growth is non-positive, the blue (red, green) line shows the results for an individual with $y_1 = -0.5$ ($-0.25, 0$). In the panels to the right the sensitivity of labor income growth is non-negative, the blue (green, red) line shows the results for an individual with $y_1 = 0.5$ ($0.25, 0$). Panels (c) to (h) show the components of risky investment as described in equation (2.31).

The *slope* of the components of risky investment are indicated in Figure 2.3 and are valid for realistic parameter values; general rules are hard to find. Nevertheless, for the right hand panels it can be stated that myopic demand is unambiguously increasing in the state variable because $\frac{1}{\gamma} \frac{\lambda_1}{\sigma_s^2} > 0$ and total wealth is increasing in X . This statement must not be true for the left hand case as total wealth is decreasing in the state variable and this has a contrary effect. For $X > 0$, a similar statement is true for state variable hedging demand.

Indirect labor hedging demand is increasing in the state variable for both cases as

$$\partial^2 k(\tau, X) / \partial X^2 = \int_0^\tau d_1(s)^2 e^{d_0(s) + d_1(s)X} ds > 0 \quad (2.32)$$

As a last comment on indirect labor hedging demand it should be noticed that the sensitivity with respect to X and the level of indirect labor hedging demand are in close relation. This stems from the fact that $d_1(s)$ is important for both measures. In particular, a highly negative (positive) $d_1(s)$ implies a low (high) level of indirect labor hedging demand and a high sensitivity. This statement is important. In fact, it states that indirect labor hedging demand of high magnitude always comes with a high sensitivity of this hedging demand to changes in the state variable. Nevertheless, from Panel (a) it can be clearly recognized that the counter effects in myopic demand and state variable hedging demand can compensate the high sensitivity of indirect labor hedging demand and that total risky investment is not too sensitive. This is not so in the case displayed in Panel (b), since the effects of all components of risky investment go in the same direction and lead to a highly sensitive investment policy.

Figure 2.4 shows optimal consumption. As can be seen from (2.26), optimal consumption is determined by the numerator total wealth and the denominator

$$\int_0^\tau e^{\frac{1}{\gamma}(c_0(s) + c_1(s)X + \frac{1}{2}c_2(s)X^2)} ds.$$

As total wealth declines (increases) for negative (positive) y_1 , the effect of total wealth on the amount consumed is straightforward.

For the empirically relevant range of the state variable $X > 0$, $c_1(s) < 0$ and $c_2(s) < 0$ lead to

$$\int_0^\tau (c_1(s) + c_2(s)X) e^{\frac{1}{\gamma}(c_0(s) + c_1(s)X + \frac{1}{2}c_2(s)X^2)} ds < 0 \quad (2.33)$$

for $\tau > 0$. Thus, for $X > 0$ the denominator is unambiguously decreasing³¹. As a consequence, it can be stated that for $y_1 > 0$ the numerator is increasing and the denominator decreasing and thus optimal consumption rises with X .

³¹For $X < 0$ the statement is not valid in general since $(c_1(s) + c_2(s)X)$ becomes positive for low X .

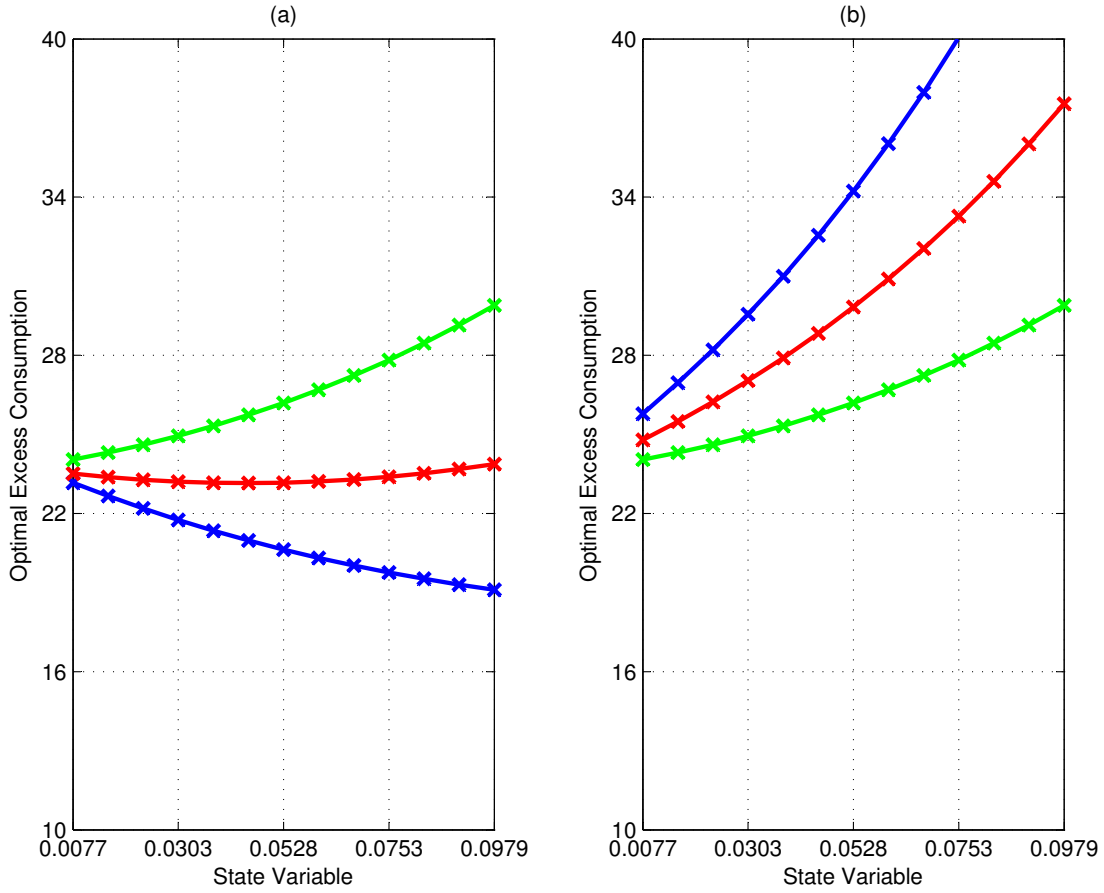


Figure 2.4: Optimal Consumption - Locally Riskfree Labor Income

Panels (a) and (b) show optimal consumption exceeding the subsistence level $c_t^* - \bar{c}$ dependent on the state variable under locally riskfree labor income $\sigma_y = 0$. Parameters are chosen as in Table 2.4. In Panel (a) the sensitivity of labor income growth is non-positive, the blue (red, green) line shows the results for an individual with $y_1 = -0.5$ ($-0.25, 0$). In Panel (b) the sensitivity of labor income growth is non-negative, the blue (green, red) line shows the results for an individual with $y_1 = 0.5$ ($0.25, 0$).

The situation for the individual to the left is more interesting. For $X > 0$, both the denominator and the numerator are decreasing and thus, for realistic parameters optimal consumption can even fall in times when X rises. Loosely speaking, optimal consumption is decreasing if the percentage decline in total wealth is stronger than the percentage decline in the denominator. Hence, it can be concluded that the possible decline in consumption is more pronounced for individuals with low financial wealth and/or low income prospects.

The insight of declining consumption for high states of X is of importance. In the Wachter model, there was a clear relation between the equity premium and the consumption-financial wealth ratio for the empirically relevant set of positive X . Specifically, the premium is high when the consumption-wealth ratio is high. This must clearly not be the case under the presence of time-varying labor growth. This result might also have implications for the asset pricing literature and might help to explain the mixed evidence on consumption-based asset pricing

models³². Moreover, Lynch and Tan (2009) interpret the dividend yield as a proxy for the business-cycle. In their view, states with high X are recessions and thus falling consumption for increasing X seems to be a desirable property.

It can be summarized that on the one hand, for $y_1 > 0$ the pro-cyclical variation of expected return and labor income growth (income growth is low when expected returns are high) leads to an extreme variation in the optimal policies, which seems implausible. On the other hand, for $y_1 < 0$ the optimal policies are on lower levels and are less sensitive across states. Thus, the case $y_1 < 0$ implies more realistic behavior.

2.4.2 Sensitivity of the Results to the Stability Parameter l_1

As indicated in the phase plane analysis, the results for the labor income part $k(\tau, X)$ depend crucially on the slope parameter l_1 . In particular, it could be stated that the long-run behavior of $k(\tau, X)$ is stable if $l_1 < 0$. It can be easily checked that this is indeed the case for the chosen parameters. The sensitivity of the results will be demonstrated by altering σ_x . Figures 2.5 - 2.7 exhibit the results.

The lines with crosses (circles, squares) show $\sigma_x = 0.0089$ (0.0134, 0.0178). The higher state variable volatility implies stronger variation in the equity premium and the labor income growth rates. In all cases, l_1 is still smaller than zero, but the direction can be clearly recognized. In fact, it can be seen from the phase plane analysis in Figure 2.1 that a higher σ_x leads to a less steep line and that $d_1(s)$ becomes greater in magnitude.

Since the results are qualitatively unchanged to Section 2.4.1, a detailed discussion is omitted. Nevertheless, two points should be noted. Firstly, it can be stated that for realistic parameter values, results become more extreme. To give an example, at $X = \bar{X}$ the *slope* of total wealth is lower (higher) for $y_1 < 0$ ($y_1 > 0$). This is intuitive as the higher σ_x implies more persistent shocks³³. In other words, it takes longer for the state variable to return to its long-run mean. Hence, in case of $y_1 < 0$ after a negative shock on the risky asset it takes longer until labor growth catches up, and this clearly reduces the value of the future labor income stream.

Secondly, state variable hedging demand in Panel (c) shows some interesting patterns. On one hand, it can be clearly recognized why Wachter and Campbell et al. show important hedging demands for the framework without labor income (here the green lines imply constant labor income and can be considered as the analogous framework). For the green case, state variable hedging demand is substantially increased for high σ_x . Nevertheless, this does not have to be the case in the presence of labor income and a negative sensitivity of labor income to the state variable. In this case, the decrease in total wealth can (over-)compensate the direct effect of σ_x on state variable hedging demand.

Finally, it should be emphasized that l_1 is crucial indeed. As shown by the parameter set of

³²Consumption based asset pricing goes back to Breeden (1979). On the one hand, Lettau and Ludvigson (2001) provide results in favor of consumption based asset pricing. On the other hand, Brennan and Xia (2005) and Goyal and Welch (2008) challenge the results of Lettau and Ludvigson.

³³Persistency is better understand if we notice that for high σ_x , κ_x is small relative to σ_x .

Wachter (2002) and Campbell et al. (2004), l_1 varies widely. In fact, from the parameter values in Table 2.2 it can be verified that in the Wachter dataset, $l_1^w = -0.0407$, while in the Campbell et al. dataset, $l_1^c = 0.1158$.

In our example, the small changes in σ_x implied $l_1 = -0.1191$ (-0.0910 , -0.0628) and even these small changes had strong effects on the results. Thus, a parameter set as in Campbell et al. implies results that are unreasonably extreme unless y_1 is close to zero.

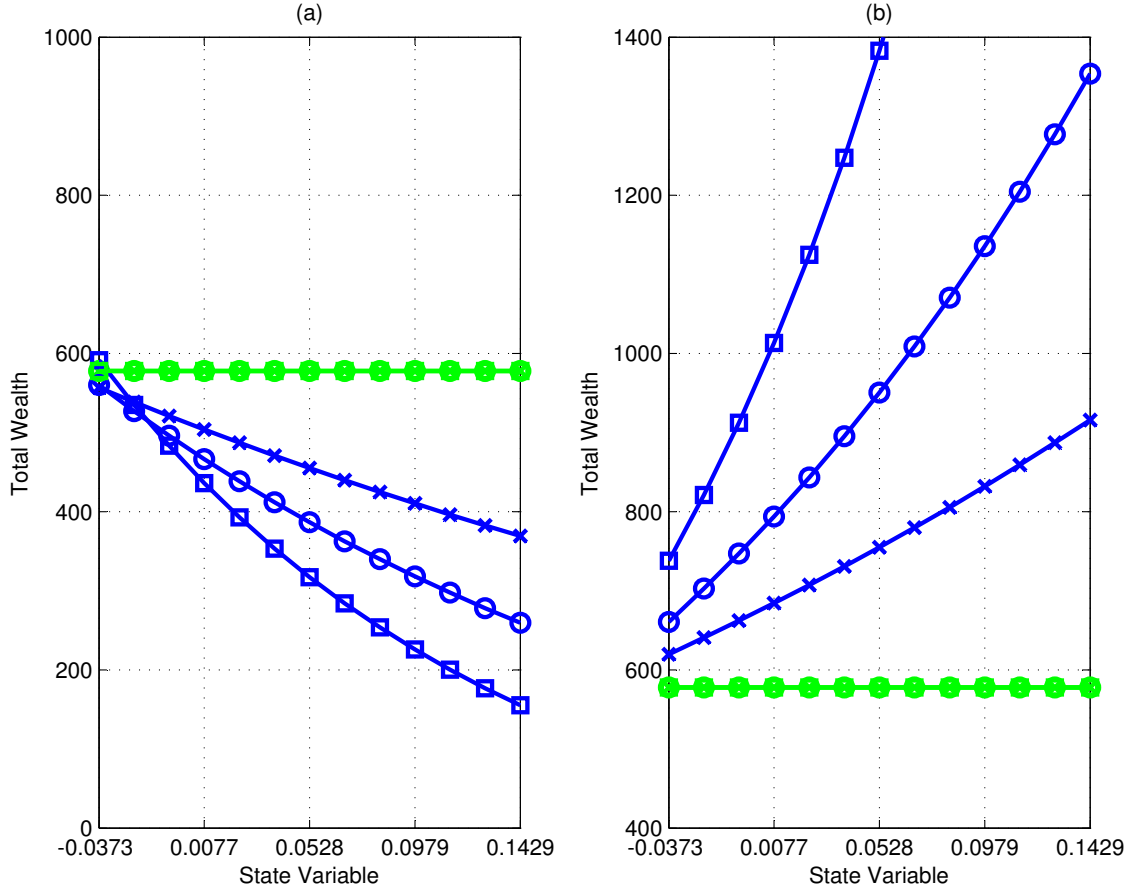


Figure 2.5: Total Wealth - Stability Analysis

This Figure exhibits the results of a stability analysis by altering the relevant parameter $l_1 \equiv -\kappa_x - \frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1$ by changing σ_x . Panels (a) and (b) show total wealth \hat{A} dependent on the state variable under locally riskfree labor income $\sigma_y = 0$. Parameters are chosen as in Table 2.4. In the panel to the left (right) the blue lines show the results for an individual with a negative (positive) sensitivity of labor income growth on $X(t)$ of $y_1 = -0.5$ ($y_1 = 0.5$). In both panels the green lines show the case of constant labor income growth $y_1 = 0$. The lines with crosses (circles, squares) display the results for an individual with $\sigma_x = 0.0089$ (0.0134, 0.0178).

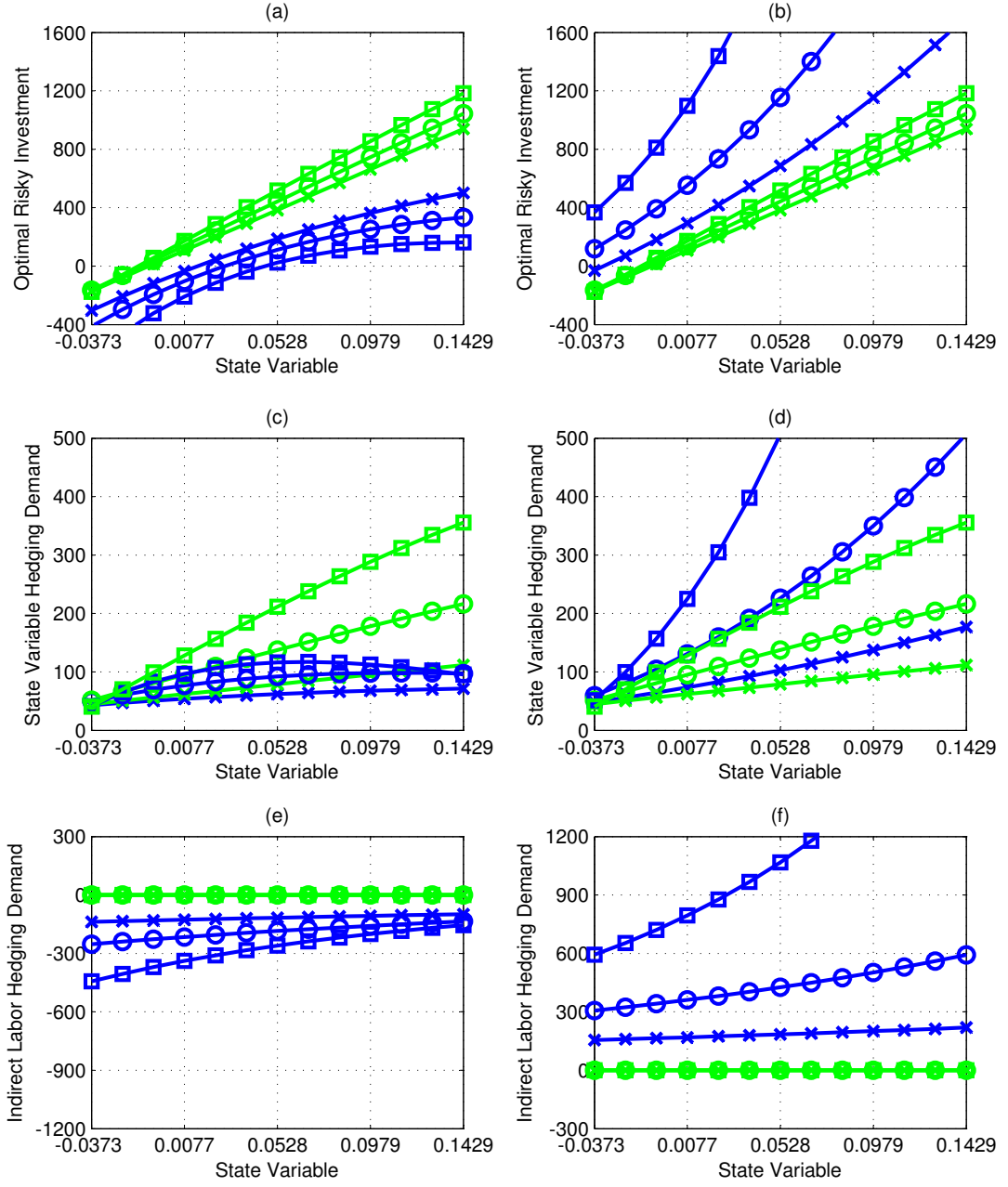


Figure 2.6: Optimal Risky Investment - Stability Analysis

This Figure exhibits the results of a stability analysis by altering the relevant parameter $l_1 \equiv -\kappa_x - \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1$ by changing σ_x . Panels (a) and (b) show optimal total risky investment $A\pi_t^*$ dependent on the state variable under locally riskfree labor income $\sigma_y = 0$. Panels (c) and (d) show state variable hedging demand, Panels (e) and (f) show indirect labor income hedging demand as described in equation (2.31). Parameters are chosen as in Table 2.4. In the panels to the left (right) the blue lines show the results for an individual with a negative (positive) sensitivity of labor income growth on $X(t)$ of $y_1 = -0.5$ ($y_1 = 0.5$). In all panels the green lines show the case of constant labor income growth $y_1 = 0$. The lines with crosses (circles, squares) display the results for an individual with $\sigma_x = 0.0089$ (0.0134, 0.0178).

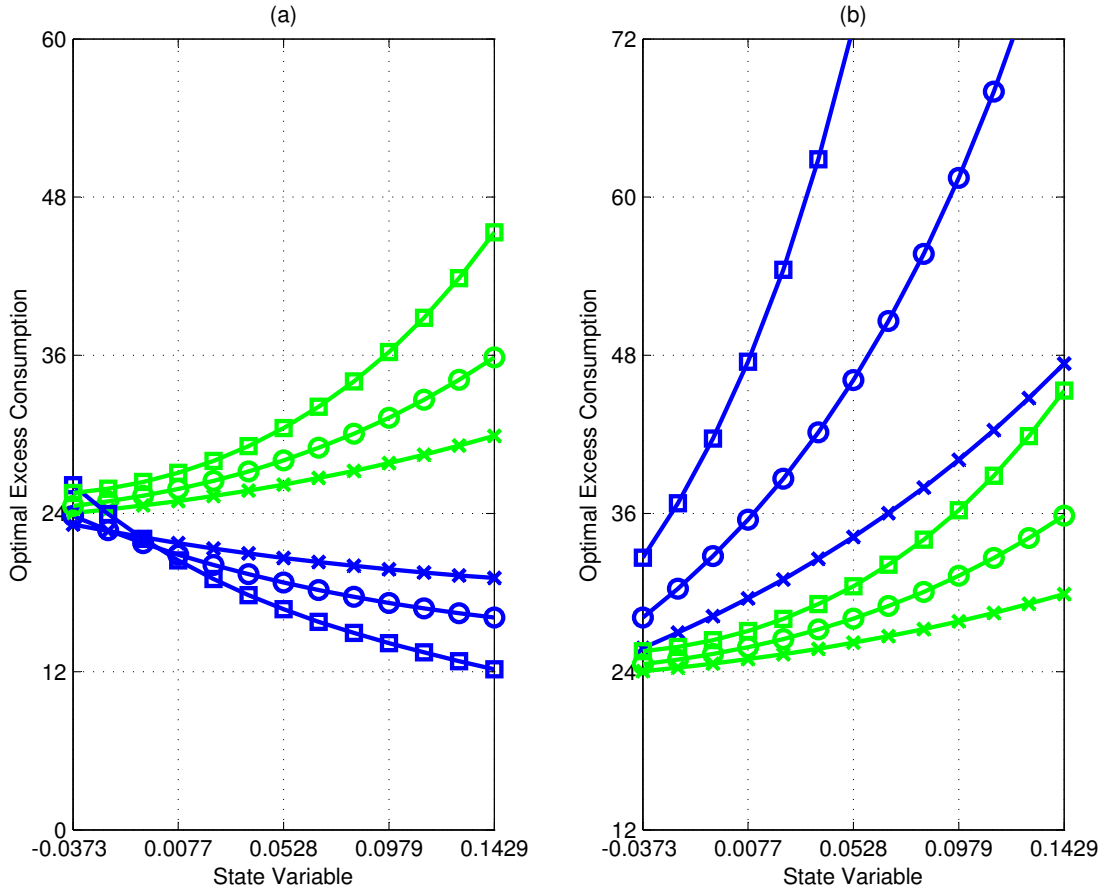


Figure 2.7: Optimal Consumption - Stability Analysis

This Figure exhibits the results of a stability analysis by altering the relevant parameter $l_1 \equiv -\kappa_x - \frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1$ by changing σ_x . Panels (a) and (b) show optimal consumption exceeding the subsistence level $c_t^* - \bar{c}$ dependent on the state variable under locally riskfree labor income $\sigma_y = 0$. Parameters are chosen as in Table 2.4. In the panel to the left (right) the blue lines show the results for an individual with a negative (positive) sensitivity of labor income growth on $X(t)$ of $y_1 = -0.5$ ($y_1 = 0.5$). In both panels the green lines show the case of constant labor income growth $y_1 = 0$. The lines with crosses (circles, squares) display the results for an individual with $\sigma_x = 0.0089$ (0.0134, 0.0178).

2.4.3 Constant Investment Opportunities

Time variation in the equity premium is still under challenge³⁴. Nevertheless, Lynch and Tan (2009) show that the dividend yield seems a good predictor of labor income growth since it is related to business-cycle fluctuations. Moreover, the dividend yield is naturally related to the stock market with a correlation close to -1 . For this reason, we will look at a model where $\lambda_1 \rightarrow 0$ and labor income is locally riskfree.

The model can also be interpreted without the connection to the dividend yield. In fact, the perfectly negative correlation of the state variable and the risky asset simply implies that the

³⁴For a general overview see Goyal and Welch (2008); Pástor and Stambaugh (2001) point out the problem of structural breaks in valuation ratios that are used as instruments.

growth rate of labor income is in close relation to the financial market. Specifically, the growth rate of labor income is low after a decline in the risky asset.

It should be noticed that IAT allow almost every combination of long run equity premium and premium sensitivity to be modeled, but the case $\lambda_1 = 0$ also implies a zero long-run premium in the specification of (2.1). The reason is that as $\lambda_1 \rightarrow 0$, $\bar{X} \rightarrow \infty$ in order to ensure a non-zero long-run premium. In order to avoid a zero equity premium, (2.1) is adapted to

$$\frac{dS_1(t)}{S_1(t)} = (\lambda_0 + \lambda_1 X(t) + r_0) dt + \sigma_s dW_s(t)$$

Taking into account the modified risky asset dynamics and following the steps described in Appendix 2.A.2. SODE (2.11) - (2.13) changes to

$$\begin{aligned} \frac{\partial c_2(s)}{\partial s} &= k_0 + k_1 c_2(s) + k_2 c_2(s)^2 \\ \frac{\partial c_1(s)}{\partial s} &= k_6 + k_3 c_2(s) + \frac{k_1}{2} c_1(s) + k_2 c_2(s) c_1(s) \\ \frac{\partial c_0(s)}{\partial s} &= k_5 + k_3 c_1(s) + k_4 c_2(s) + \frac{k_2}{2} c_1(s)^2 \end{aligned}$$

with initial conditions $c_2(0) = c_1(0) = c_0(0) = 0$ and

$$\begin{aligned} k_0 &\equiv \frac{1-\gamma}{\gamma} \frac{\lambda_1^2}{\sigma_s^2}, \quad k_1 \equiv 2 \left[-\kappa_x + \frac{1-\gamma}{\gamma} \frac{\rho_{sx} \sigma_x}{\sigma_s} \lambda_1 \right], \quad k_2 \equiv \frac{1}{\gamma} \sigma_x^2 \\ k_3 &\equiv \kappa_x \bar{X}, \quad k_4 \equiv \frac{1}{2} \sigma_x^2, \quad k_5 \equiv -\delta + (1-\gamma) r_0 + \frac{1}{2} \frac{1-\gamma}{\gamma} \frac{\lambda_0^2}{\sigma_s^2} \\ k_6 &\equiv \frac{1-\gamma}{\gamma} \frac{\lambda_0 \lambda_1}{\sigma_s^2} \end{aligned}$$

Because $\lambda_1 = 0$ and the initial conditions,

$$c_2(s) = c_1(s) = 0, \quad \forall s$$

Now, the equation for $c_0(s)$ becomes simple.

$$\frac{\partial c_0(s)}{\partial s} = \Delta \equiv -\delta + (1-\gamma) r_0 + \frac{1}{2} \frac{1-\gamma}{\gamma} \frac{\lambda_0^2}{\sigma_s^2}$$

Thus,

$$h(t) \equiv \int_0^\tau e^{C(X,s)} ds = \int_0^\tau e^{\frac{1}{\gamma} \Delta s} ds = \gamma \frac{1}{\Delta} \left(e^{\frac{1}{\gamma} \Delta \tau} - 1 \right)$$

In fact, the solution of $\int_0^\tau e^{C(X,s)} ds$ is, in this case, the well-known solution from Merton (1969).

It should be noticed that without the initial conditions equal to zero, $c_2(s)$ and $c_1(s)$ would not be zero and the solution would not be equal to the Merton solution. A similar statement is true for $\gamma \rightarrow 1$ (log utility)³⁵. Hence, the choice of the initial conditions can be justified not only because they are necessary to solve the HJB as described in Appendix 2.A.2, but for intuitive reasons as well.

³⁵ A related statement with respect to risk aversion is made in Campbell and Viceira (1999) and Chacko and Viceira (2005).

More importantly with $c_2(s) = c_1(s) = 0$, state variable hedging demand vanishes and because $\lambda_1 = 0$, myopic demand varies only with total wealth.

The SODE of the labor income part (2.17) - (2.18) changes to

$$\begin{aligned}\frac{\partial d_1(s)}{\partial s} &= l_0 + l_1 d_1(s) \\ \frac{\partial d_0(s)}{\partial s} &= l_2 + l_3 d_1(s) + l_4 d_1(s)^2\end{aligned}$$

where

$$\begin{aligned}l_0 &\equiv y_1, \quad l_1 \equiv -\kappa_x \\ l_2 &\equiv y_0 - r_0 - \frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_0, \quad l_3 \equiv \kappa_x \bar{X} - \frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_0 + \rho_{xy}\sigma_x\sigma_y, \quad l_4 \equiv \frac{1}{2}\sigma_x^2\end{aligned}$$

Remarks

- Through the simplification, the sign of l_1 is unambiguously negative and the phase plane analysis reveals that this leads to stability of $d_1(s)$ as $s \rightarrow \infty$.
- The sign of y_1 determines the sign of $d_1(s)$ and thus, the sign of indirect labor income hedging demand.

Figures 2.8 - 2.10 exhibit the results. The blue lines in the left (right) panels belong to an individual with a negative (positive) labor income sensitivity of $y_1 = -0.5$ ($y_1 = 0.5$). The green lines contain the results for an individual where the growth rate of labor income is constant $y_1 = 0$. The lines with crosses (circles, squares) belong to an individual with $\sigma_y = 0$ (0.04, 0.08), all other parameters are chosen as in Table 2.4 except $\lambda_1 = 0$ and $\lambda_0 = \bar{X}$.

Figure 2.8 shows the value of total wealth dependent on the state variable. As in the case of locally riskfree labor income, the sign of the *slope* is exclusively determined by y_1 .

It should be noticed that the *level* of total wealth declines with higher labor income volatility. This is intuitive and the primary effect stems from a lower³⁶ l_2 . In fact, the higher income volatility in combination with a positive correlation of the risky asset and labor income leads to a more precautionary valuation of the income stream. From the discussion of the dynamics of total wealth it is known that the individual controls total wealth in the same manner as an investor without labor income and subsistence consumption. As non-financial income becomes risky, the individual will need additional (short) positions in the risky asset. This is taken into account by adding a correspondingly lower value to the labor income stream.

As before, the risk-neutral valuation from Appendix 3.A.1 of Chapter 3 can give additional insights. Adapted to the market price of risk of the constant financial market setting $\theta_c(t) \equiv \frac{\lambda_0}{\sigma_s}$, the second part of the RHS of 3.12

$$\frac{\partial G}{\partial Y} Y \frac{\rho_{sy}\sigma_y}{\sigma_s} \lambda_0 = k Y \frac{\rho_{sy}\sigma_y}{\sigma_s} \lambda_0 > 0$$

is unambiguously positive. Hence, a higher σ_y asks for a higher premium and the value of k must be correspondingly lower.

³⁶See Appendix 2.A.6 for more details.

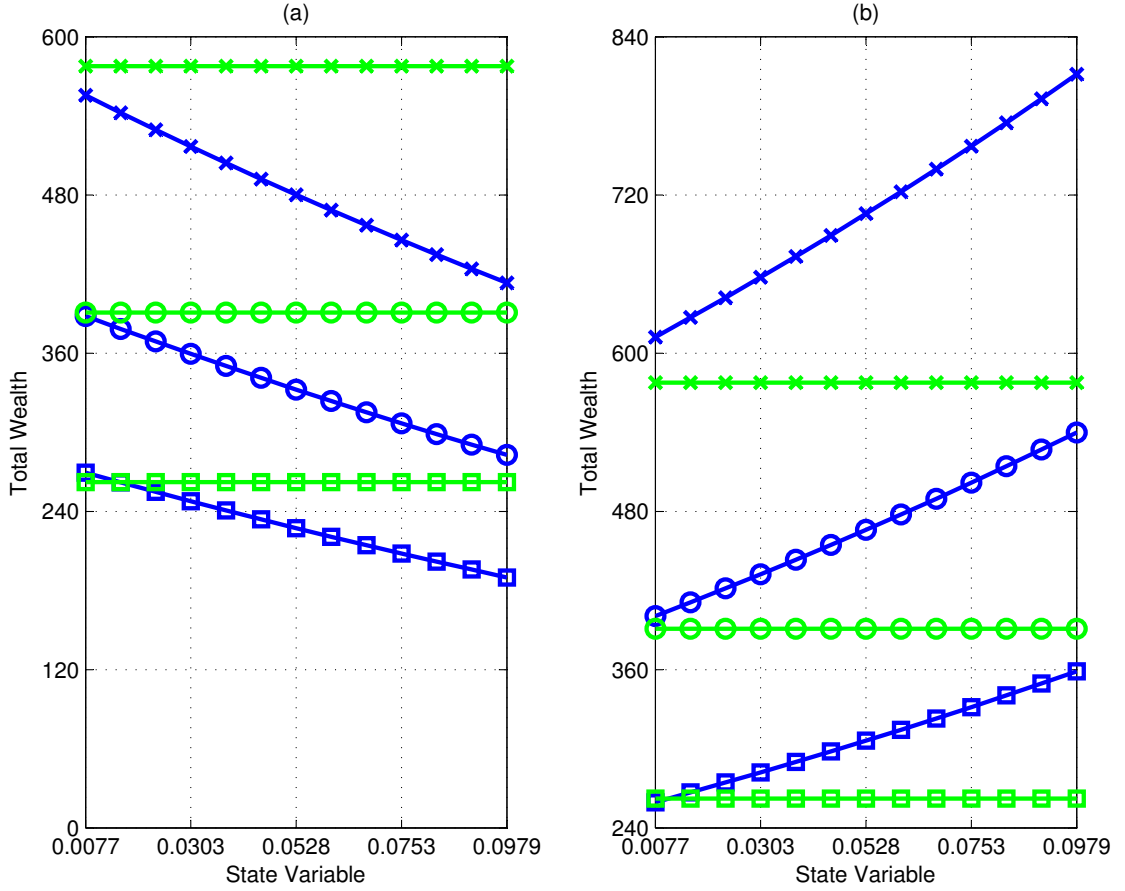


Figure 2.8: Total Wealth - Constant Investment Opportunities

This Figure exhibits total wealth \hat{A} dependent on the state variable under risky labor income. Parameters are chosen as in Table 2.4 except that the risky asset is assumed to have a constant premium $\lambda_0 = \bar{X}$, $\lambda_1 = 0$. In the panel to the left (right) the blue lines show the results for an individual with a negative (positive) sensitivity of labor income growth on $X(t)$ of $y_1 = -0.5$ ($y_1 = 0.5$). In both panels the green lines show the case of constant labor income growth $y_1 = 0$. The lines with crosses (circles, squares) display the results for an individual with $\sigma_y = 0$ (0.0400, 0.0800).

Moreover, there is a secondary effect of smaller magnitude that stems from changes in l_3 . As can be seen from Figure 2.8, at the long-run mean $X = \bar{X}$ the differences in the valuation of the income stream become lower for high values of σ_y . Reviving the discussion from above³⁷, this can be explained by the term

$$\psi_c \equiv \left[\frac{1}{l_1} \left(e^{l_1 s} - 1 \right) - s \right] \left(\frac{l_3}{l_1} + \bar{X} \right)$$

In fact, the negativity of $\rho_{xy} = \rho_{sy}\rho_{sx}$ leads to a lower l_3 for higher σ_y . As a consequence, ψ_c becomes smaller (and can even turn negative). As shown in Appendix 2.A.6, this lowers the difference to the constant growth case.

This result is not intuitive as a negative ρ_{xy} and $y_1 > 0$ imply that a decline in labor income is followed by high income growth and this seems to be a desired feature from an intertemporal

³⁷See Section 2.4.1 and Appendix 2.A.6.

point of view. An answer can be found by looking at the dynamics of total wealth. As already described, by the valuation of the income stream the individual compensates the dynamics of the non-financial income stream in order to end up with total wealth, which behaves as in a setting without labor income. The critical term $\rho_{xy}\sigma_x\sigma_y$ in l_3 can be clearly identified in the dynamics of total wealth (2.38) as $\frac{\partial k}{\partial X}\rho_{xy}\sigma_x\sigma_y$. Since this part does not originate from a first order condition but simply from the cross product of labor income and state variable diffusion, it is comprehensible that there is no connection to intertemporal hedging. Furthermore,

$$y_1 < 0 \Rightarrow \frac{\partial k}{\partial X} < 0, \quad \left(y_1 > 0 \Rightarrow \frac{\partial k}{\partial X} > 0 \right)$$

in combination with $\rho_{xy} = -1$ implies a positive (negative) drift for total wealth. This additional drift has to be taken into account by valuing the income stream.

Optimal investment is displayed in Figure 2.9. The most distinct feature is well known from the portfolio choice literature with no time variation in labor income. In particular, with $\rho_{sy} > 0$ risky investment is reduced for $\sigma_y > 0$ over direct labor hedging demand, which is described in equation (2.31).

Because state variable hedging demand is zero and myopic demand varies only with total wealth, their interpretation is easy and omitted. Indirect labor hedging demand is shown in Panels (c) and (d). This component is easy to understand. The *slopes* are unambiguously positive because of the same argument as in Section 2.4.1. More importantly, the effects are strongest in magnitude for low levels of σ_y and hence, optimal risky investment is affected even for the locally riskfree labor income case (crosses).

Direct labor hedging demand

$$-\frac{\rho_{sy}\sigma_y}{\sigma_s} \left(\int_0^\tau e^{d_0(s)+d_1(s)X} ds \right) Y$$

is displayed in Panels (e) and (f). The positive correlation between the risky asset and labor income implies negative direct labor hedging demand and the slopes are explained by the sign of $-\rho_{sy}d_1(s)$.

Figure 2.10 shows optimal consumption exceeding the subsistence level, which is given by

$$c_t^* - \bar{c} = \frac{\hat{A}(t)}{h(t)}$$

As the denominator of optimal consumption does not vary with the state variable, the amount consumed varies only with total wealth. As a consequence, consumption falls (rises) with X if $y_1 < 0$ ($y_1 > 0$).

It can be summarized that despite the simplicity of the model, it is able to reproduce realistic patterns. In particular, falling consumption in times of a high state variable (recession), and low or even negative risky asset exposure for individuals with long maturity and unfavorable labor income characteristics.

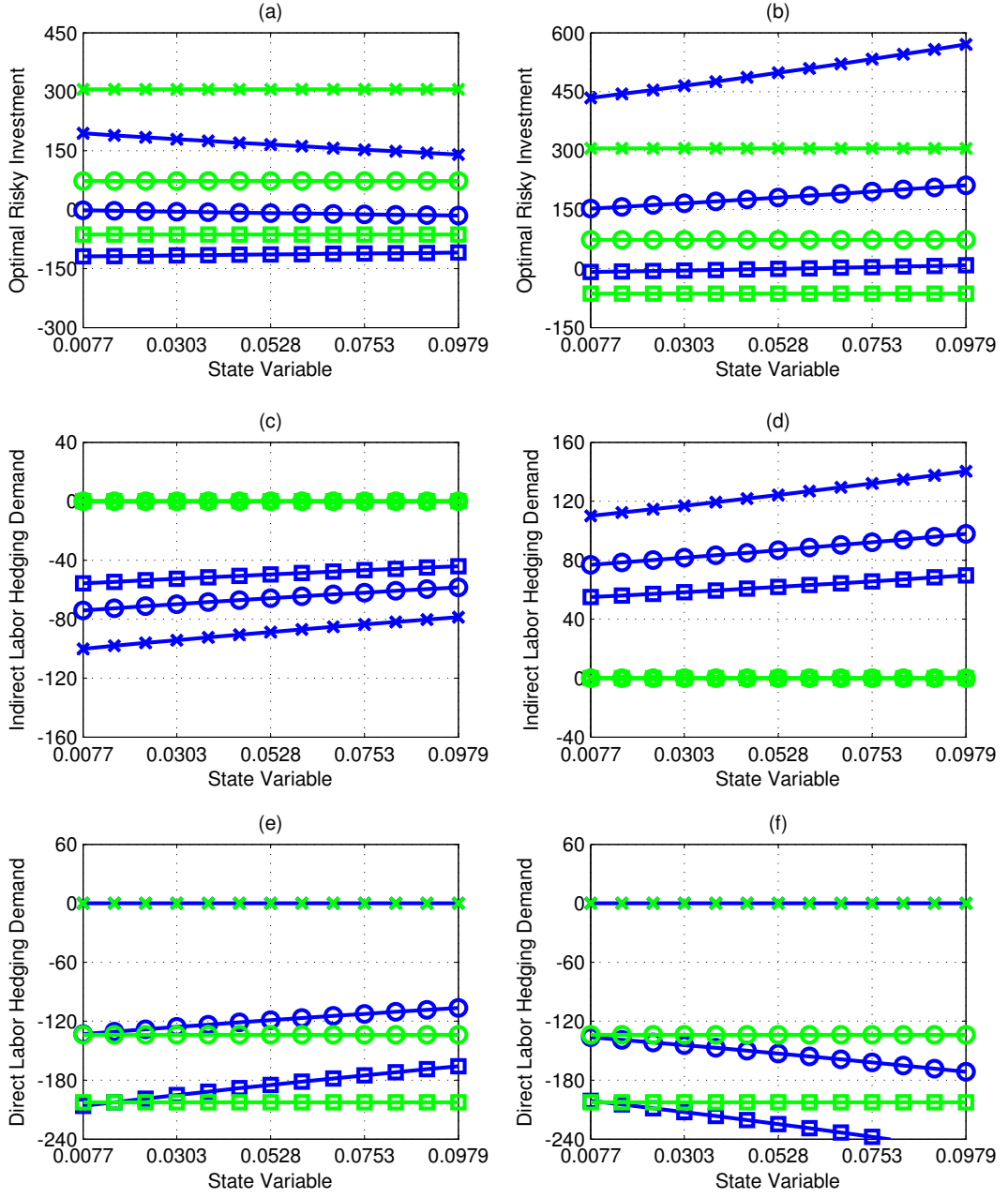


Figure 2.9: Optimal Risky Investment - Constant Investment Opportunities

Panels (a) and (b) show optimal total risky investment $A\pi_t^*$ dependent on the state variable under risky labor income. Panels (c) and (d) show indirect labor income hedging demand, Panels (e) and (f) show direct labor income hedging demand as described in equation (2.31). Parameters are chosen as in Table 2.4 except that the risky asset is assumed to have a constant premium $\lambda_0 = \bar{X}$, $\lambda_1 = 0$. In the panels to the left (right) the blue lines show the results for an individual with a negative (positive) sensitivity of labor income growth on $X(t)$ of $y_1 = -0.5$ ($y_1 = 0.5$). In all Panels the green lines show the case of constant labor income growth $y_1 = 0$. The lines with crosses (circles, squares) display the results for an individual with $\sigma_y = 0$ (0.0400, 0.0800).

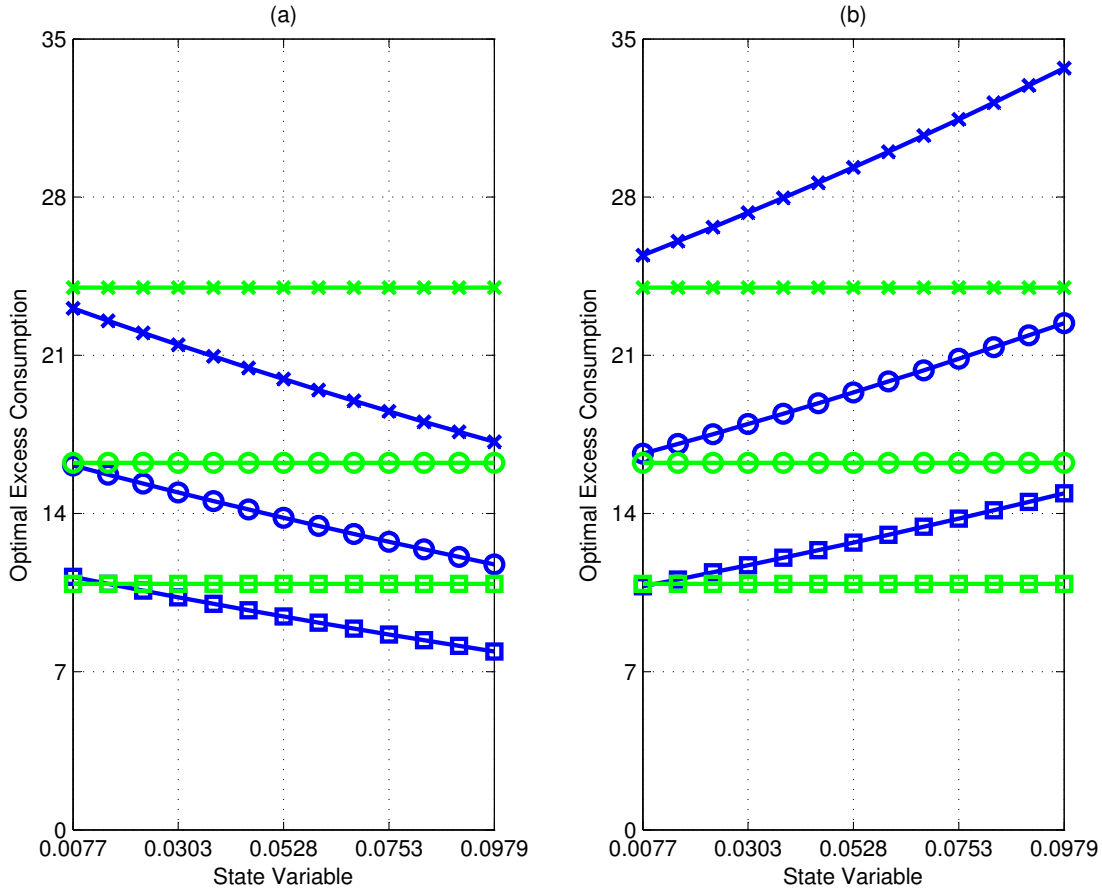


Figure 2.10: Optimal Consumption - Constant Investment Opportunities

Panels (a) and (b) show optimal consumption exceeding the subsistence level $c_t^* - \bar{c}$ dependent on the state variable under risky labor income. Parameters are chosen as in Table 2.4 except that the risky asset is assumed to have a constant premium $\lambda_0 = \bar{X}$, $\lambda_1 = 0$. In the panel to the left (right) the blue lines show the results for an individual with a negative (positive) sensitivity of labor income growth on $X(t)$ of $y_1 = -0.5$ ($y_1 = 0.5$). In both Panels the green lines show the case of constant labor income growth $y_1 = 0$. The lines with crosses (circles, squares) display the results for an individual with $\sigma_y = 0$ (0.0400, 0.0800).

2.4.4 Risky Labor Income

This section discusses the general model. Many issues have already been pointed out in special cases. For this reason, the discussion is restricted to new and/or important characteristics. As above, the blue lines in the left (right) panels belong to an individual with a negative (positive) labor income sensitivity of $y_1 = -0.5$ ($y_1 = 0.5$). The green lines contain the results for an individual where the growth rate of labor income is constant ($y_1 = 0$). The lines with crosses (circles, squares) belong to an individual with $\sigma_y = 0$ (0.04, 0.08); all other parameters are chosen as in Table 2.4.

Figure 2.11 shows total wealth dependent on the state variable. The combination of time-varying returns/income growth and risky labor income has an impact on $l_0 \equiv y_1 - \frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1$. The assumption $\rho_{sy} = 1$ leads to a reduction of l_0 that results in a more negative *slope* through the

decrease in $d_1(s)$. In fact, even for a positive sensitivity of labor income growth on X , the slope can become negative. Moreover, for realistic parameters the *level* of total wealth is lower for higher σ_y at \bar{X} . Nevertheless, as before, the differences between the blue and the green lines at $X = \bar{X}$ are narrowing. The interpretation is similar to the preceding section.

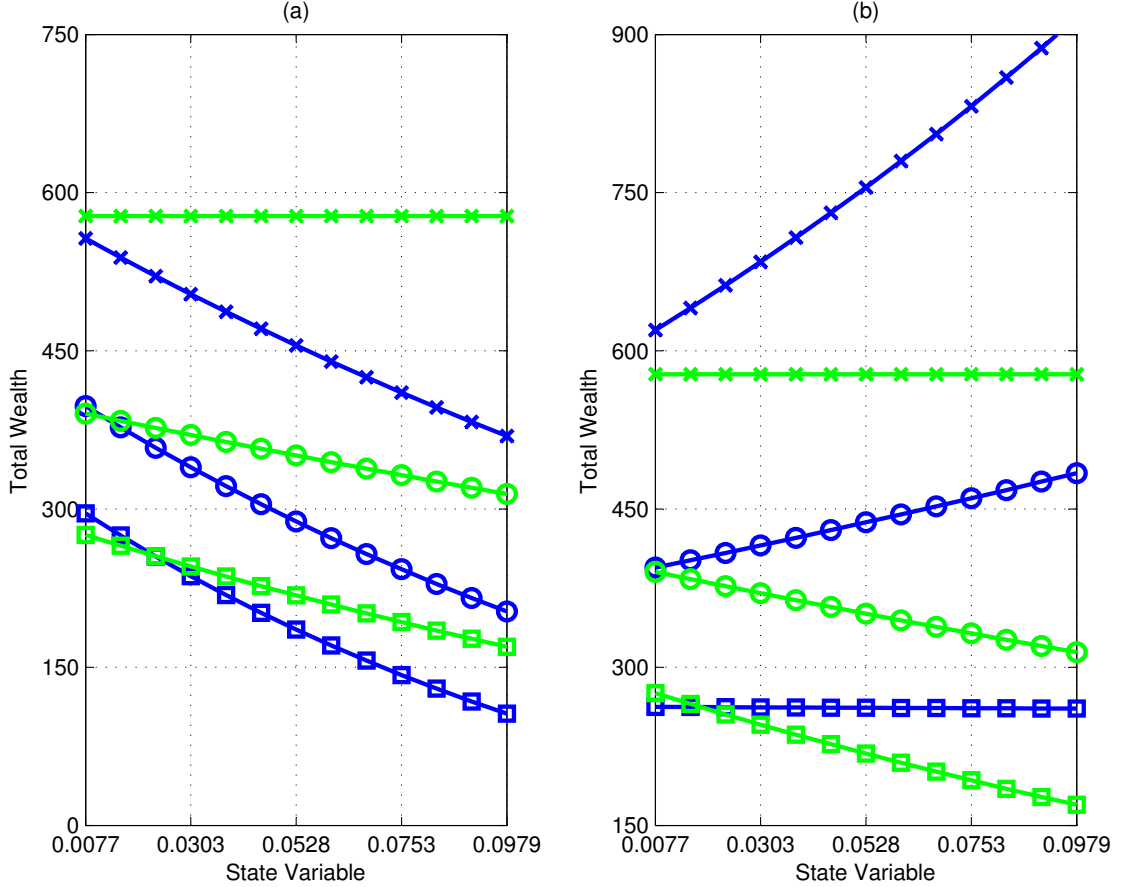


Figure 2.11: Total Wealth - Risky Labor Income

This Figure exhibits the results for total wealth \hat{A} dependent on the state variable under risky labor income. Parameters are chosen as in Table 2.4. In the panels to the left (right) the blue lines show the results for an individual with a negative (positive) sensitivity of labor income growth on $X(t)$ of $y_1 = -0.5$ ($y_1 = 0.5$). In both panels the green lines show the case of constant labor income growth $y_1 = 0$. The lines with crosses (circles, squares) display the results for an individual with $\sigma_y = 0$ (0.0400, 0.0800).

Figure 2.12 shows optimal risky investment. In Panels (a) and (b) it can be recognized that for realistic parameters the *level* of total risky investment decreases with labor income volatility. Indeed, total risky investment can become negative. Since myopic and state variable hedging demand are only affected by changes in total wealth, they are omitted.

Panels (c) and (d) show that indirect labor hedging demand is affected and shows different patterns. In particular, Panel (d) shows different levels for the hedging demand while in Panel (c) they are of similar magnitude.

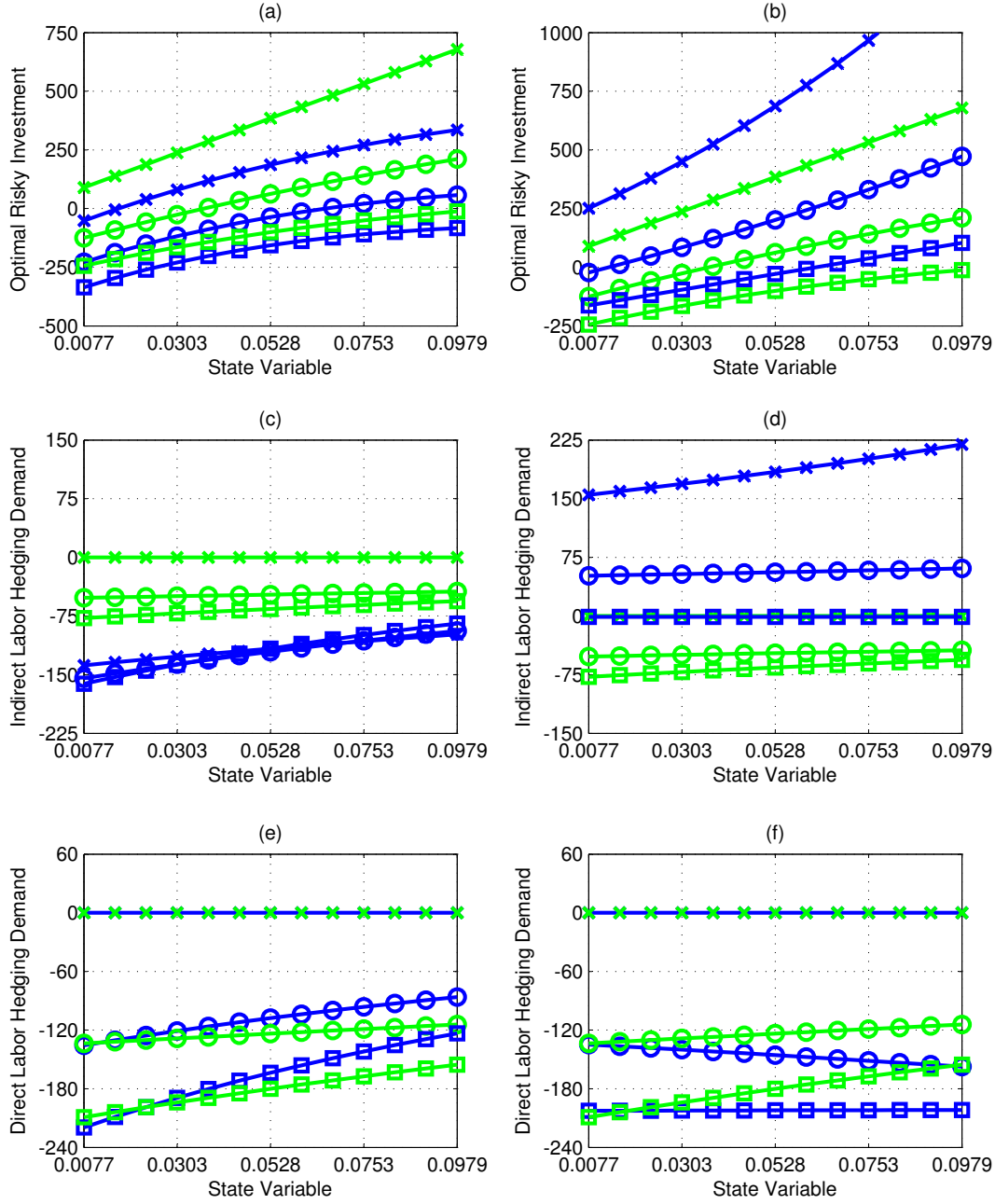


Figure 2.12: Optimal Risky Investment - Risky Labor Income

Panels (a) and (b) show optimal total risky investment $A\pi_t^*$ dependent on the state variable under risky labor income. Panels (c) and (d) show indirect labor income hedging demand, Panels (e) and (f) show direct labor income hedging demand as described in equation (2.31). Parameters are chosen as in Table 2.4. In the panels to the left (right) the blue lines show the results for an individual with a negative (positive) sensitivity of labor income growth on $X(t)$ of $y_1 = -0.5$ ($y_1 = 0.5$). In all panels the green lines show the case of constant labor income growth $y_1 = 0$. The lines with crosses (circles, squares) display the results for an individual with $\sigma_y = 0$ (0.0400, 0.0800).

The analytical reason for the case displayed in Panel (c) is as follows. For $l_0 < 0$ a higher σ_y leads to a $d_1(s)$ of higher magnitude (more negative). The positivity of l_3 leads to $l_3 d_1(s) < 0$

and hence it can be seen from (2.18) that a lower $d_0(s)$ results. Thus, from the crucial term

$$k_X = \int_0^\tau \underbrace{d_1(s)}_{\omega_1} \underbrace{e^{d_0(s)+d_1(s)X}}_{\omega_2} ds$$

it becomes evident that for a higher σ_y ω_1 is higher in magnitude but ω_2 lowers. Hence, the two effects compensate each other. As can be seen in Panel (d), this is not the case for $l_0 > 0$ as both components become lower in magnitude for rising σ_y .

The interpretation for direct labor hedging demand in Panels e) and f) is similar to the model presented in Section 2.4.3. Moreover, the changes in l_0 due to σ_y are already pointed out in the discussion of total wealth.

Figure 2.13 displays optimal consumption exceeding the subsistence consumption. It should be noticed that because of the changes in total wealth, consumption increases less strongly and can even fall with rising X for cases with a positive sensitivity of income growth on X .

It can be summarized that the full model allows for a variety of pattern as, for example, low or even negative equity exposure and falling consumption in high states of X .

This section concludes with a final note on volatility of consumption. In the numerical examples, consumption exceeding the subsistence consumption was between 20 – 100 percent of subsistence consumption at $X = \bar{X}$. As a consequence, consumption volatility will be correspondingly lower than the volatility of total wealth. The problem that consumption is as volatile as wealth is the analogous problem to the equity premium puzzle in the asset pricing literature³⁸. From the empirical literature it is well known that aggregate consumption has a low volatility. Hence, in order to be in line with reality the solution of a consumption-investment problem should imply a consumption stream that has a lower volatility than wealth as long as the fraction of wealth invested in the risky asset is high³⁹.

Moreover, lowering the time discount parameter δ would imply unambiguously lower excess consumption⁴⁰. Thus, a reduction of the discount rate would reduce consumption volatility further. In the numerical example, the discount rate of six percent was taken over from Campbell et al. (2004) and Wachter (2002). In the models without labor income, such a high value is necessary in order to ensure that the individual consumes a reasonable fraction of her (financial) wealth⁴¹. In models with labor income, such a high discount rate is clearly not needed. Finally, in Chapter 4, a similar model is extended to a life-cycle model including a phase of retirement with no non-financial income or subsistence consumption. In this model, excess consumption is reduced further because the individual has to increase her saving ratio for the phase of retirement.

³⁸See Mehra and Prescott (1985).

³⁹See Cochrane (2007, p. 76).

⁴⁰The time discount rate parameter has an impact on $c_0(s)$ only.

⁴¹In the model of Campbell et al. (2004) with stochastic differential utility, it is shown that for the (important) special case of intertemporal substitution equal to one, the consumption wealth ratio is constant and equal to the discount rate.

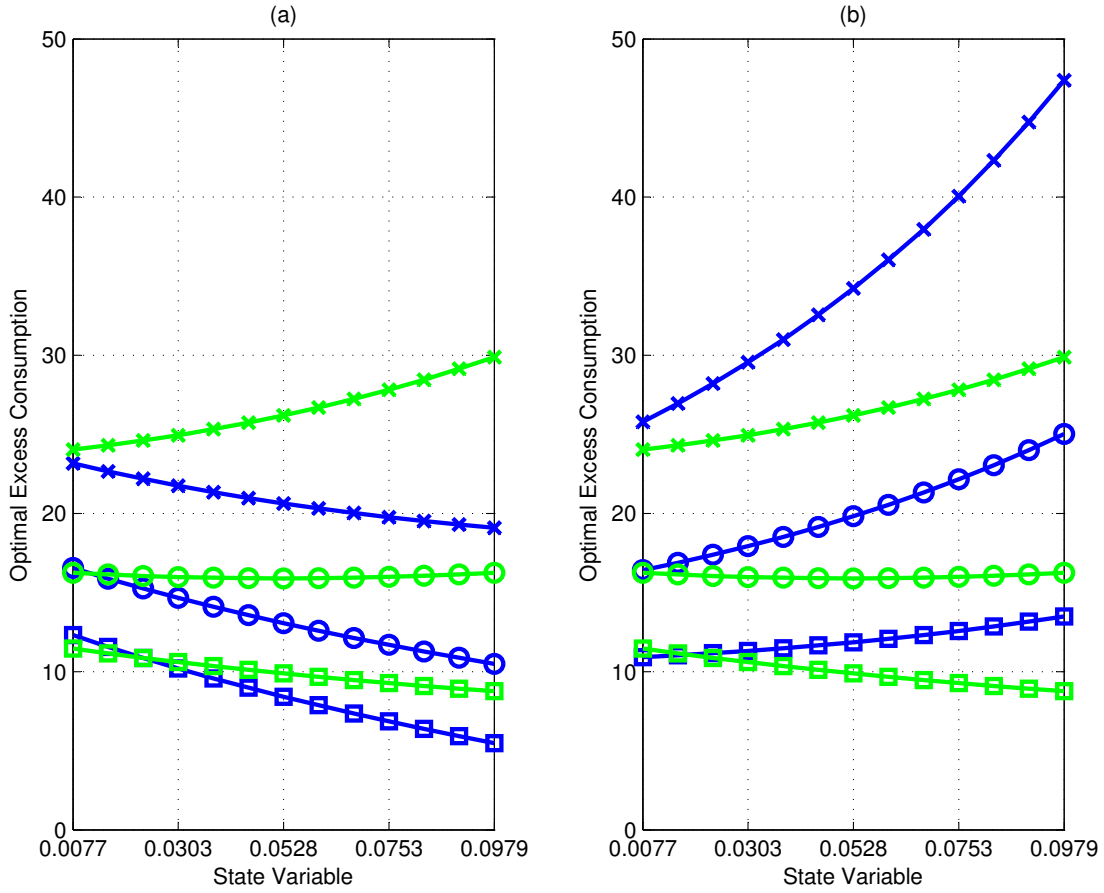


Figure 2.13: Optimal Consumption - Risky Labor Income

Panels (a) and (b) show optimal consumption exceeding the subsistence level $c_t^* - \bar{c}$ dependent on the state variable under risky labor income. Parameters are chosen as in Table 2.4. In the panel to the left (right) the blue lines show the results for an individual with a negative (positive) sensitivity of labor income growth on $X(t)$ of $y_1 = -0.5$ ($y_1 = 0.5$). In both panels the green lines show the case of constant labor income growth $y_1 = 0$. The lines with crosses (circles, squares) display the results for an individual with $\sigma_y = 0$ (0.0400, 0.0800).

2.5 Conclusion

The most important results of the basic model are the following:

1. The impact of time variation in non-financial income on optimal investment and consumption is important. Assuming time variation in the financial market and ignoring it for non-financial income leads to considerably distinct results.
2. The inclusion of time variation in labor income leads to an adaption of state variable hedging demand. In fact, state variable hedging demand can be separated into the usual part that arises in the absence of labor income and a new part. This part grows monotonically with planning horizon and can have either sign. Hence, a reduction in risky investment for individuals with a long planning horizon as reported in Lynch and Tan (2009) can be reproduced.

3. A negative sensitivity of labor income growth on the state variable can induce falling risky investment and consumption even if expected returns are increasing in the state variable. Moreover, the level of risky investment can be reduced as well.
4. Under the assumptions (2.c.1) - (2.c.3), the complicated HJB equation can be separated into ordinary differential equations which can be solved in closed-form.
5. From a technical point of view, the valuation of the labor income stream involves solving ordinary differential equations. Certain combinations of state variable and financial market parameters lead to solutions that do not converge for long horizons. In order to have optimal policies that are neither extreme in level nor highly variable over states, there are two explanations. Firstly, the parameters of the financial market and non-financial income must be in a close relation. Secondly, other important aspects as borrowing/short selling constraints⁴², trading costs⁴³ or information uncertainty⁴⁴ are neglected in the model.

In addition, it should be kept in mind that the chosen parameters imply a low component of stochastic labor income at the beginning of the working period. Increasing the importance of Y compared to \bar{Y} would give even more weight to direct and indirect labor hedging demand. A similar statement is true for risk aversion. In the numerical examples, a level of risk aversion of $\gamma = 4$ was chosen. A higher level of risk aversion would lower myopic and state variable hedging demand and thus, the relative importance of the two labor hedging demands would rise.

⁴²See, for example, Koo (1998).

⁴³See, for example, Liu and Loewenstein (2002).

⁴⁴See, for example, Xia (2001).

2.A Appendix

2.A.1 Solution of the Wachter Model

For the sake of completeness, this appendix contains the main results of part of the model that is identical to the Wachter model. For more details the reader is referred to Wachter (2002).

The solutions to the SODE (2.11) - (2.13) with initial conditions $c_2(0) = c_1(0) = c_0(0) = 0$ are given by

$$c_2(s) = \frac{2k_0(1 - e^{-\eta s})}{2\eta - (k_1 + \eta)(1 - e^{-\eta s})}$$

$$c_1(s) = \frac{4k_0k_3(1 - e^{-\eta s/2})^2}{\eta[2\eta - (k_1 + \eta)(1 - e^{-\eta s})]}$$

$$c_0(s) = \int_0^s k_5 + k_3c_1(s) + \frac{k_2}{2}c_1(s)^2 + k_4c_2(s) ds$$

where $\eta \equiv \sqrt{q}$ and $q = k_1^2 - 4k_0k_2$.

The negativity and the convergence of $c_2(s)$ can be derived analytically⁴⁵ or seen in the phase plane analysis of Figure 2.14. Because $k_2 > 0$, the parabola in equation (2.11) opens upward. Furthermore, because of the assumption $\gamma > 1$ it must be noted that

$$k_0 \equiv \frac{1 - \gamma}{\gamma} \frac{\lambda_1^2}{\sigma_s^2} < 0$$

Hence, there exists only one case with two real particular solutions. As can be recognized, $c_2(s)$ starts in the origin ($c_2(0) = 0$) and moves to the left.

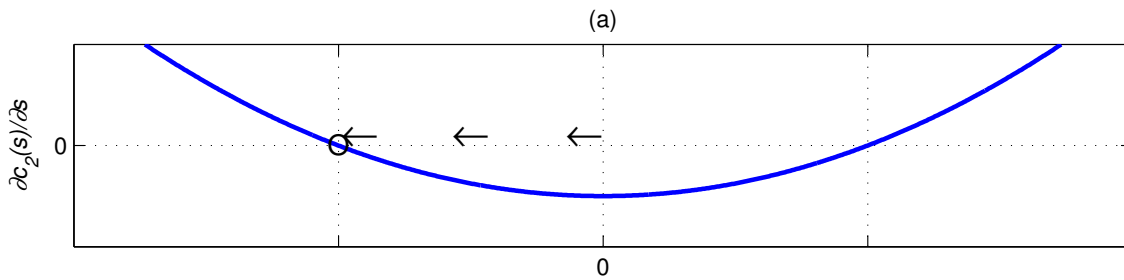


Figure 2.14: Phase Plane Analysis of $c_2(s)$

This Figure shows a phase plane analysis of the equation $\frac{\partial c_2(s)}{\partial s} = k_0 + k_1c_2(s) + k_2c_2(s)^2$ for $q > 0$. In this case, two real particular solutions exist and $c_2(s)$ converges to a stable solution marked by a circle.

In analogy to the phase plane analysis in Figure 2.1, the negativity of $c_1(s)$ follows from the negativity of the vertical axis intercept of equation (2.12). In fact, $k_3c_2(s) < 0$ because $k_3 > 0$ and the negativity of $c_2(s)$.

⁴⁵As performed by Wachter (2002, pp. 87-88).

2.A.2 Solution of the HJB-Equation for the Consumption Problem

The relevant partial derivatives of (2.9) are given by⁴⁶

$$\begin{aligned}
J_\tau &= e^{-\delta(T-\tau)} \left(\begin{aligned} &\frac{1}{1-\gamma} [\dots]^{\gamma-1} (\dots)^{1-\gamma} \int_0^\tau \left(\frac{\partial c_0(s)}{\partial s} + \frac{\partial c_1(s)}{\partial s} X + \frac{1}{2} \frac{\partial c_2(s)}{\partial s} X^2 \right) e^{C(X,s)} ds \\ &+ \frac{\delta}{1-\gamma} [\dots]^\gamma (\dots)^{1-\gamma} + \frac{\gamma}{1-\gamma} [\dots]^{\gamma-1} (\dots)^{1-\gamma} \\ &+ [\dots]^\gamma (\dots)^{-\gamma} \left(\frac{\partial k}{\partial \tau} Y - \frac{\partial R}{\partial \tau} \right) \end{aligned} \right) \\
J_A &= e^{-\delta(T-\tau)} [\dots]^\gamma (\dots)^{-\gamma}, \quad J_{AA} = -\gamma e^{-\delta(T-\tau)} [\dots]^\gamma (\dots)^{-\gamma-1} \\
J_Y &= e^{-\delta(T-\tau)} [\dots]^\gamma (\dots)^{-\gamma} k, \quad J_{YY} = -\gamma e^{-\delta(T-\tau)} [\dots]^\gamma (\dots)^{-\gamma-1} k^2 \\
J_X &= e^{-\delta(T-\tau)} \left(\begin{aligned} &\frac{1}{1-\gamma} [\dots]^{\gamma-1} (\dots)^{1-\gamma} \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \\ &+ [\dots]^\gamma (\dots)^{-\gamma} \frac{\partial k}{\partial X} Y \end{aligned} \right) \\
J_{XX} &= e^{-\delta(T-\tau)} \left(\begin{aligned} &-\frac{1}{\gamma} [\dots]^{\gamma-2} (\dots)^{1-\gamma} \left[\int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \right]^2 \\ &+ \frac{1}{1-\gamma} [\dots]^{\gamma-1} (\dots)^{1-\gamma} \int_0^\tau \frac{1}{\gamma} \left(\begin{aligned} &\gamma c_2(s) + c_1^2(s) \\ &+ 2c_1(s) c_2(s) X \\ &+ c_2(s)^2 X^2 \end{aligned} \right) e^{C(X,s)} ds \\ &+ 2 [\dots]^{\gamma-1} (\dots)^{-\gamma} \frac{\partial k}{\partial X} Y \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \\ &- \gamma [\dots]^\gamma (\dots)^{-\gamma-1} \left(\frac{\partial k}{\partial X} Y \right)^2 + [\dots]^\gamma (\dots)^{-\gamma} \frac{\partial^2 k}{\partial X^2} Y \end{aligned} \right) \\
J_{AX} &= e^{-\delta(T-\tau)} \left(\begin{aligned} &[\dots]^{\gamma-1} (\dots)^{-\gamma} \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \\ &- \gamma [\dots]^\gamma (\dots)^{-\gamma-1} \frac{\partial k}{\partial X} Y \end{aligned} \right) \\
J_{AY} &= -\gamma e^{-\delta(T-\tau)} [\dots]^\gamma (\dots)^{-\gamma-1} k \\
J_{XY} &= e^{-\delta(T-\tau)} \left(\begin{aligned} &[\dots]^{\gamma-1} (\dots)^{-\gamma} k \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \\ &- \gamma [\dots]^\gamma (\dots)^{-\gamma-1} k \frac{\partial k}{\partial X} Y + [\dots]^\gamma (\dots)^{-\gamma} \frac{\partial k}{\partial X} \end{aligned} \right)
\end{aligned}$$

where for the sake of brevity we define

$$\begin{aligned}
[\dots] &\equiv \left[\int_0^\tau e^{\frac{1}{\gamma} (c_0(s) + c_1(s) X + \frac{1}{2} c_2(s) X^2)} ds \right] \\
(\dots) &\equiv (A + k(\tau, X) Y - R(\tau))
\end{aligned}$$

and

$$C(X, \tau) \equiv \frac{1}{\gamma} \left(c_0(\tau) + c_1(\tau) X + \frac{1}{2} c_2(\tau) X^2 \right)$$

It should be noted that for J_τ , the following rule was applied:

$$f(a, b) = \int_b^a g(x) dx = G(a) - G(b)$$

\Rightarrow

$$\frac{\partial f(a, b)}{\partial a} = \frac{\partial G(a)}{\partial a} = g(a) - g(b) + g(b) = \int_b^a \frac{\partial g(x)}{\partial x} dx + g(b)$$

Moreover, only the terminal conditions $c_0(0) = c_1(0) = c_2(0) = 0$ ensure that J_τ contains $\frac{\gamma}{1-\gamma} [\dots]^{\gamma-1} (\dots)^{1-\gamma}$ and this term is inevitable to find a solution for the HJB⁴⁷.

⁴⁶More details with respect to the derivation of J_τ can be found at the bottom of the page.

⁴⁷See also Wachter (2010, p. 195).

Plugging in the relevant partial derivatives into the FOCs (2.6) and (2.7) leads to

$$c_t^* = \frac{(\dots)}{\int_0^\tau e^{C(X,s)} ds} + \bar{c} \quad (2.34)$$

and

$$\begin{aligned} A\pi_t^* &= \frac{1}{\gamma} \frac{\lambda_1}{\sigma_s^2} X (\dots) + \frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \frac{\int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} (\dots) \\ &\quad - \frac{\rho_{sx}\sigma_x}{\sigma_s} \frac{\partial k}{\partial X} Y - \frac{\rho_{sy}\sigma_y}{\sigma_s} kY \end{aligned} \quad (2.35)$$

The solution of the HJB equation is tedious but leads to simple and interpretable results⁴⁸. Plugging in the relevant partial derivatives, (2.34) and (2.35) into the HJB and multiplying by $e^{\delta(T-\tau)}$ yields

$$\begin{aligned} 0 &= - \left(\frac{1}{1-\gamma} [\dots]^{\gamma-1} (\dots)^{1-\gamma} \int_0^\tau \left(\frac{\partial c_0(s)}{\partial s} + \frac{\partial c_1(s)}{\partial s} X + \frac{1}{2} \frac{\partial c_2(s)}{\partial s} X^2 \right) e^{C(X,s)} ds \right. \\ &\quad \left. + \frac{\delta}{1-\gamma} [\dots]^\gamma (\dots)^{1-\gamma} + \frac{\gamma}{1-\gamma} [\dots]^{\gamma-1} (\dots)^{1-\gamma} + [\dots]^\gamma (\dots)^{-\gamma} \left(\frac{\partial k}{\partial \tau} Y - \frac{\partial R}{\partial \tau} \right) \right) \\ &\quad - [\dots]^\gamma (\dots)^{-\gamma} \bar{c} + \frac{\gamma}{1-\gamma} [\dots]^{\gamma-1} (\dots)^{1-\gamma} + [\dots]^\gamma (\dots)^{-\gamma} A r_0 \\ &\quad + [\dots]^\gamma (\dots)^{-\gamma} \bar{Y} + [\dots]^\gamma (\dots)^{-\gamma} Y \\ &\quad - \left(\frac{1}{1-\gamma} [\dots]^{\gamma-1} (\dots)^{1-\gamma} \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds + [\dots]^\gamma (\dots)^{-\gamma} \frac{\partial k}{\partial X} Y \right) \kappa_x (X - \bar{X}) \\ &\quad + [\dots]^\gamma (\dots)^{-\gamma} kY (y_0 + y_1 X) \\ &\quad + \frac{1}{2} [\dots]^\gamma (\dots)^{-\gamma} \lambda_1 X \left(\frac{1}{\gamma} (\dots) \frac{\lambda_1}{\sigma_s^2} X + \frac{1}{\gamma} (\dots) \frac{\int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} \frac{\rho_{sx}\sigma_x}{\sigma_s} \right. \\ &\quad \left. - \frac{\partial k}{\partial X} Y \frac{\rho_{sx}\sigma_x}{\sigma_s} - kY \frac{\rho_{sy}\sigma_y}{\sigma_s} \right) \\ &\quad + \frac{1}{2} \rho_{sx}\sigma_x\sigma_s \left[\begin{aligned} &\left([\dots]^{\gamma-1} (\dots)^{-\gamma} \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \right) \cdot \\ &-\gamma [\dots]^\gamma (\dots)^{-\gamma-1} \frac{\partial k}{\partial X} Y \\ &\left(\frac{1}{\gamma} (\dots) \frac{\lambda_1}{\sigma_s^2} X + \frac{1}{\gamma} (\dots) \frac{\int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} \frac{\rho_{sx}\sigma_x}{\sigma_s} \right) \\ &-\frac{\partial k}{\partial X} Y \frac{\rho_{sx}\sigma_x}{\sigma_s} - kY \frac{\rho_{sy}\sigma_y}{\sigma_s} \end{aligned} \right] \\ &\quad - \frac{1}{2} \gamma \rho_{sy}\sigma_y\sigma_s [\dots]^\gamma (\dots)^{-\gamma-1} kY \left(\frac{1}{\gamma} (\dots) \frac{\lambda_1}{\sigma_s^2} X + \frac{1}{\gamma} (\dots) \frac{\int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} \frac{\rho_{sx}\sigma_x}{\sigma_s} \right. \\ &\quad \left. - \frac{\partial k}{\partial X} Y \frac{\rho_{sx}\sigma_x}{\sigma_s} - kY \frac{\rho_{sy}\sigma_y}{\sigma_s} \right) \\ &\quad + \rho_{xy}\sigma_x\sigma_y Y \left([\dots]^{\gamma-1} (\dots)^{-\gamma} k \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \right. \\ &\quad \left. - \gamma [\dots]^\gamma (\dots)^{-\gamma-1} k \frac{\partial k}{\partial X} Y + [\dots]^\gamma (\dots)^{-\gamma} \frac{\partial k}{\partial X} Y \right) \\ &\quad - \frac{1}{2} \gamma [\dots]^\gamma (\dots)^{-\gamma-1} k^2 Y^2 \sigma_y^2 \\ &\quad + \frac{1}{2} \sigma_x^2 \left(\begin{aligned} &-\frac{1}{\gamma} [\dots]^{\gamma-2} (\dots)^{1-\gamma} \left[\int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \right]^2 \\ &+ \frac{1}{1-\gamma} [\dots]^{\gamma-1} (\dots)^{1-\gamma} \int_0^\tau \frac{1}{\gamma} \begin{pmatrix} \gamma c_2(s) + c_1^2(s) \\ + 2c_1(s) c_2(s) X \\ + c_2(s)^2 X^2 \end{pmatrix} e^{C(X,s)} ds \\ &+ 2 [\dots]^{\gamma-1} (\dots)^{-\gamma} \frac{\partial k}{\partial X} Y \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \\ &- \gamma [\dots]^\gamma (\dots)^{-\gamma-1} \left(\frac{\partial k}{\partial X} Y \right)^2 + [\dots]^\gamma (\dots)^{-\gamma} \frac{\partial^2 k}{\partial X^2} Y \end{aligned} \right) \end{aligned}$$

⁴⁸For a textbook treatment of stochastic control, the reader is referred to Øksendal (2003) Chapter 11.

Multiplying by $[\dots]^{-(\gamma-1)} (\dots)^\gamma$ gives

$$\begin{aligned}
0 = & -\frac{\delta}{1-\gamma} (\dots) [\dots] - \frac{1}{1-\gamma} (\dots) \int_0^\tau \left(\frac{\partial c_0(s)}{\partial s} + \frac{\partial c_1(s)}{\partial s} X + \frac{1}{2} \frac{\partial c_2(s)}{\partial s} X^2 \right) e^{C(X,s)} ds \\
& - \left(\frac{\partial k}{\partial \tau} Y - \frac{\partial R}{\partial \tau} \right) [\dots] - \bar{c} [\dots] + A r_0 [\dots] + \bar{Y} [\dots] + Y [\dots] + (y_0 + y_1 X) k Y [\dots] \\
& - \frac{1}{1-\gamma} \kappa_x (X - \bar{X}) (\dots) \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds - \kappa_x (X - \bar{X}) \frac{\partial k}{\partial X} Y [\dots] \\
& + \frac{1}{2} \frac{1}{\gamma} \frac{\lambda_1^2}{\sigma_s^2} X^2 (\dots) [\dots] + \frac{1}{2} \frac{1}{\gamma} \frac{\rho_{sx} \sigma_x}{\sigma_s} \lambda_1 X (\dots) \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \\
& - \frac{1}{2} \frac{\rho_{sx} \sigma_x}{\sigma_s} \lambda_1 X \frac{\partial k}{\partial X} Y [\dots] - \frac{1}{2} \frac{\rho_{sy} \sigma_y}{\sigma_s} \lambda_1 X k Y [\dots] \\
& + \frac{1}{2} \frac{1}{\gamma} \frac{\rho_{sx} \sigma_x}{\sigma_s} \lambda_1 X (\dots) \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \\
& + \underbrace{\frac{1}{2} \frac{1}{\gamma} \rho_{sx}^2 \sigma_x^2 (\dots) [\dots]^{-1} \left[\int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \right]^2}_{i)} \\
& - \frac{1}{2} \rho_{sx}^2 \sigma_x^2 \frac{\partial k}{\partial X} Y \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds - \frac{1}{2} \rho_{sx} \sigma_x \rho_{sy} \sigma_y k Y \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \\
& - \frac{1}{2} \frac{\rho_{sx} \sigma_x}{\sigma_s} \lambda_1 X \frac{\partial k}{\partial X} Y [\dots] - \frac{1}{2} \rho_{sx}^2 \sigma_x^2 \frac{\partial k}{\partial X} Y \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \\
& + \underbrace{\frac{1}{2} \gamma \rho_{sx}^2 \sigma_x^2 (\dots)^{-1} \left(\frac{\partial k}{\partial X} Y \right)^2 [\dots]}_{ii)} + \underbrace{\frac{1}{2} \gamma \rho_{sx} \sigma_x \rho_{sy} \sigma_y (\dots)^{-1} k Y \frac{\partial k}{\partial X} Y [\dots]}_{iii)} \\
& - \frac{1}{2} \frac{\rho_{sy} \sigma_y}{\sigma_s} \lambda_1 X k Y [\dots] - \frac{1}{2} \rho_{sx} \sigma_x \rho_{sy} \sigma_y k Y \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \\
& + \underbrace{\frac{1}{2} \gamma \rho_{sx} \sigma_x \rho_{sy} \sigma_y (\dots)^{-1} k Y \frac{\partial k}{\partial X} Y [\dots]}_{iii)} + \underbrace{\frac{1}{2} \gamma \rho_{sy}^2 \sigma_y^2 (\dots)^{-1} k^2 Y^2 [\dots]}_{iv)} \\
& + \rho_{xy} \sigma_x \sigma_y k Y \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds - \underbrace{\gamma \rho_{xy} \sigma_x \sigma_y (\dots)^{-1} k Y \frac{\partial k}{\partial X} Y [\dots]}_{iii)} \\
& + \rho_{xy} \sigma_x \sigma_y \frac{\partial k}{\partial X} Y [\dots] - \underbrace{\frac{1}{2} \gamma \sigma_y^2 (\dots)^{-1} k^2 Y^2 [\dots]}_{iv)} \\
& - \underbrace{\frac{1}{2} \frac{1}{\gamma} \sigma_x^2 (\dots) [\dots]^{-1} \left[\int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \right]^2}_{i)} \\
& + \frac{1}{2} \frac{1}{1-\gamma} \sigma_x^2 (\dots) \int_0^\tau \frac{1}{\gamma} (\gamma c_2(s) + c_1^2(s) + 2c_1(s) c_2(s) X + c_2(s)^2 X^2) e^{C(X,s)} ds \\
& + \sigma_x^2 \frac{\partial k}{\partial X} Y \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \\
& - \underbrace{\frac{1}{2} \gamma \sigma_x^2 (\dots)^{-1} \left(\frac{\partial k}{\partial X} Y \right)^2 [\dots]}_{ii)} + \frac{1}{2} \sigma_x^2 \frac{\partial^2 k}{\partial X^2} Y [\dots] \tag{2.36}
\end{aligned}$$

To our knowledge, closed-form solutions for this general PDE are not available⁴⁹. The highlighted terms make it impossible to separate the equation into a system of ODEs. However, the highlighted terms *i)* and *ii)* vanish under the assumption of $\rho_{sx} \in \{-1, 1\}$, which is the assumption in Wachter (2002). Furthermore, if $\rho_{sy} \in \{-1, 1\}$ then $\rho_{xy} = \rho_{sx} \rho_{sy} \in \{-1, 1\}$ or if $\sigma_y = 0$, the terms indicated by *iii)* and *iv)* vanish.

⁴⁹See Huang and Milevsky (2008), Huang et al. (2008), Munk and Sørensen (2010).

Without the terms highlighted by *i*), *ii*), *iii*) and *iv*), the HJB simplifies to

$$\begin{aligned}
0 = & -\frac{\delta}{1-\gamma}(\dots)[\dots] - \frac{1}{1-\gamma}(\dots) \int_0^\tau \left(\frac{\partial c_0(s)}{\partial s} + \frac{\partial c_1(s)}{\partial s} X + \frac{1}{2} \frac{\partial c_2(s)}{\partial s} X^2 \right) e^{C(X,s)} ds \\
& - \left(\frac{\partial k}{\partial \tau} Y - \frac{\partial R}{\partial \tau} \right) [\dots] - \bar{c}[\dots] + r_0(\dots)[\dots] - r_0(kY - R)[\dots] \\
& + \bar{Y}[\dots] + Y[\dots] + (y_0 + y_1 X) kY[\dots] \\
& - \frac{1}{1-\gamma} \kappa_x (X - \bar{X})(\dots) \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds - \kappa_x (X - \bar{X}) \frac{\partial k}{\partial X} Y[\dots] \\
& + \frac{1}{2} \frac{1}{\gamma} \frac{\lambda_1^2}{\sigma_s^2} X^2 (\dots)[\dots] + \frac{1}{2} \frac{1}{\gamma} \frac{\rho_{sx} \sigma_x}{\sigma_s} \lambda_1 X \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds (\dots) \\
& - \frac{1}{2} \frac{\rho_{sx} \sigma_x}{\sigma_s} \lambda_1 X \frac{\partial k}{\partial X} Y[\dots] - \frac{1}{2} \frac{\rho_{sy} \sigma_y}{\sigma_s} \lambda_1 X kY[\dots] \\
& + \frac{1}{2} \frac{1}{\gamma} \frac{\rho_{sx} \sigma_x}{\sigma_s} \lambda_1 X (\dots) \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \\
& - \frac{1}{2} \rho_{sx}^2 \sigma_x^2 \frac{\partial k}{\partial X} Y \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \\
& - \frac{1}{2} \rho_{sy} \sigma_y \rho_{sx} \sigma_x kY \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \\
& - \frac{1}{2} \frac{\rho_{sx} \sigma_x}{\sigma_s} \lambda_1 X \frac{\partial k}{\partial X} Y[\dots] - \frac{1}{2} \rho_{sx}^2 \sigma_x^2 \frac{\partial k}{\partial X} Y \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \\
& - \frac{1}{2} \frac{\rho_{sy} \sigma_y}{\sigma_s} \lambda_1 X kY[\dots] - \frac{1}{2} \rho_{sx} \sigma_x \rho_{sy} \sigma_y kY \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds \\
& + \rho_{xy} \sigma_x \sigma_y kY \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds + \rho_{xy} \sigma_x \sigma_y \frac{\partial k}{\partial X} Y[\dots] \\
& + \frac{1}{2} \frac{1}{1-\gamma} \sigma_x^2 (\dots) \int_0^\tau \frac{1}{\gamma} (\gamma c_2(s) + c_1^2(s) + 2c_1(s) c_2(s) X + c_2(s)^2 X^2) e^{C(X,s)} ds \\
& + \sigma_x^2 \frac{\partial k}{\partial X} Y \int_0^\tau (c_1(s) + c_2(s) X) e^{C(X,s)} ds + \frac{1}{2} \sigma_x^2 \frac{\partial^2 k}{\partial X^2} Y[\dots] \tag{2.37}
\end{aligned}$$

Moreover, it should be noticed that the trivial expansion

$$(kY - R) r_0[\dots] - (kY - R) r_0[\dots]$$

was made in the second line of (2.37). This equation can now be separated into a system of ODEs.

2.A.3 The Dynamics of Total Wealth

From the definition $\hat{A} \equiv A + k(X, t)Y - R(t)$, application of Ito's lemma yields the dynamics of total wealth

$$d\hat{A} = dA + \frac{\partial k}{\partial X} Y dX + \frac{1}{2} \frac{\partial^2 k}{\partial X^2} Y dX^2 + \frac{\partial k}{\partial t} Y dt + k dY + \frac{\partial k}{\partial X} dX dY - \frac{\partial R}{\partial t} dt$$

Plugging in (2.2) - (2.4) and the optimal policies (2.23) and (2.24) leads to

$$\begin{aligned} d\hat{A}^* = & \left[\frac{1}{\gamma} \frac{\lambda_1^2}{\sigma_s^2} X^2 \hat{A}^* + \frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 \frac{\int_0^\tau (c_1(s)X + c_2(s)X^2) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} \hat{A}^* \right. \\ & \left. - \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 X \frac{\partial k}{\partial X} Y - \frac{\rho_{sy}\sigma_y}{\sigma_s} \lambda_1 X k Y \right. \\ & \left. + r_0 \hat{A}^* - r_0 (kY - R) + \bar{Y} + Y - \frac{1}{\int_0^\tau e^{C(X,s)} ds} \hat{A}^* - \bar{c} \right] dt \\ & + \left(\frac{1}{\gamma} \frac{\lambda_1}{\sigma_s} X \hat{A}^* + \frac{1}{\gamma} \rho_{sx}\sigma_x \frac{\int_0^\tau (c_1(s) + c_2(s)X) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} \hat{A}^* \right) dW_s(t) \\ & - \frac{\partial k}{\partial X} Y \kappa_x (X - \bar{X}) dt + \frac{\partial k}{\partial X} Y \sigma_x dW_x(t) + \frac{1}{2} \sigma_x^2 \frac{\partial^2 k}{\partial X^2} Y dt - \frac{\partial k}{\partial \tau} Y dt \\ & + kY (y_0 + y_1 X) dt + kY \sigma_y dW_y(t) + \frac{\partial k}{\partial X} \rho_{xy} \sigma_x \sigma_y Y dt + \frac{\partial R}{\partial \tau} dt \end{aligned}$$

Arranging in proper order

$$\begin{aligned} d\hat{A}^* = & \left(r_0 + \frac{1}{\gamma} \frac{\lambda_1^2}{\sigma_s^2} X^2 - \frac{1}{\int_0^\tau e^{C(X,s)} ds} \right) \hat{A}^* dt \\ & + \left(\frac{1}{\gamma} \frac{\lambda_1}{\sigma_s} X + \frac{1}{\gamma} \rho_{sx}\sigma_x \frac{\int_0^\tau (c_1(s) + c_2(s)X) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} \right) \hat{A}^* dW_s(t) \\ & + \left[-\frac{\partial k}{\partial \tau} + 1 - r_0 k - \frac{\partial k}{\partial X} \kappa_x (X - \bar{X}) + k(y_0 + y_1 X) \right. \\ & \left. - \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 \frac{\partial k}{\partial X} X - \frac{\rho_{sy}\sigma_y}{\sigma_s} \lambda_1 k X + \frac{\partial k}{\partial X} \rho_{xy} \sigma_x \sigma_y + \frac{1}{2} \sigma_x^2 \frac{\partial^2 k}{\partial X^2} \right] Y dt \\ & + \left[\frac{\partial R}{\partial \tau} + \bar{Y} + r_0 R - \bar{c} \right] dt \\ & + [dW_y(t) - \rho_{sy} dW_s(t)] \sigma_y k Y + [dW_x(t) - \rho_{sx} dW_s(t)] \sigma_x \frac{\partial k}{\partial X} Y \end{aligned} \quad (2.38)$$

The last line is equal to zero due to the assumptions about perfect dependence (2.c.1) - (2.c.2) and locally riskfree labor income (2.c.3), i.e. $dW_x(t) = \rho_{sx} dW_s(t)$ and $dW_y(t) = \rho_{sy} dW_s(t)$ or $\sigma_y = 0$. Inspecting the parts in the square brackets one can identify (2.14) and (2.21) which are also equal to zero. The dynamics of (2.25) follow directly.

2.A.4 Solution of the HJB-Equation for the Terminal Wealth Problem

For the terminal wealth problem, a reasonable candidate for the value function is given by

$$J = \frac{e^{c_0(\tau)+c_1(\tau)X+\frac{1}{2}c_2(\tau)X^2} (A + k(\tau, X)Y - R(\tau))^{1-\gamma}}{1-\gamma}$$

where $\tau \equiv T - t$, $k(X, \tau)$ and $R(\tau)$ are in analogy to Appendix 2.A.2. The expected utility implies that

$$J(\tau = 0) = \frac{(A_T - \bar{A})^{1-\gamma}}{1-\gamma}$$

and hence, $c_0(0) = c_1(0) = c_2(0) = 0$ and $R(0) = \bar{A}$. The relevant partial derivatives are given by

$$\begin{aligned} J_\tau &= e^{C(X, \tau)} \left(\frac{1}{1-\gamma} (\dots)^{1-\gamma} \left(\frac{\partial c_0(\tau)}{\partial \tau} + \frac{\partial c_1(\tau)}{\partial \tau} X + \frac{1}{2} \frac{\partial c_2(\tau)}{\partial \tau} X^2 \right) + (\dots)^{-\gamma} \left(\frac{\partial k}{\partial \tau} Y - \frac{\partial R}{\partial \tau} \right) \right) \\ J_A &= e^{C(X, \tau)} (\dots)^{-\gamma}, \quad J_{AA} = -\gamma e^{C(X, \tau)} (\dots)^{-\gamma-1} \\ J_Y &= e^{C(X, \tau)} (\dots)^{-\gamma} k, \quad J_{YY} = -\gamma e^{C(X, \tau)} (\dots)^{-\gamma-1} k^2 \\ J_X &= e^{C(X, \tau)} \left(\frac{1}{1-\gamma} (c_1(\tau) + c_2(\tau)X) (\dots)^{1-\gamma} + (\dots)^{-\gamma} \frac{\partial k}{\partial X} Y \right) \\ J_{XX} &= e^{C(X, \tau)} \left(\frac{1}{1-\gamma} (\dots)^{1-\gamma} (c_2(\tau) + c_1^2(\tau) + 2c_1(\tau)c_2(\tau)X + c_2(\tau)^2 X^2) + 2(c_1(\tau) + c_2(\tau)X) (\dots)^{-\gamma} \frac{\partial k}{\partial X} Y - \gamma (\dots)^{-\gamma-1} \left(\frac{\partial k}{\partial X} Y \right)^2 + (\dots)^{-\gamma} \frac{\partial^2 k}{\partial X^2} Y \right) \\ J_{AX} &= e^{C(X, \tau)} \left((c_1(\tau) + c_2(\tau)X) (\dots)^{-\gamma} - \gamma (\dots)^{-\gamma-1} \frac{\partial k}{\partial X} Y \right) \\ J_{AY} &= -\gamma e^{C(X, \tau)} (\dots)^{-\gamma-1} k \\ J_{XY} &= e^{C(X, \tau)} \left((c_1(\tau) + c_2(\tau)X) (\dots)^{-\gamma} k - \gamma (\dots)^{-\gamma-1} k \frac{\partial k}{\partial X} Y + (\dots)^{-\gamma} \frac{\partial k}{\partial X} \right) \end{aligned}$$

where $C(X, \tau) \equiv c_0(\tau) + c_1(\tau)X + \frac{1}{2}c_2(\tau)X^2$. Plugging the relevant partial derivatives into (2.7) gives

$$\begin{aligned} A\pi_t^* &= \frac{1}{\gamma} \frac{\lambda_1}{\sigma_s^2} X \hat{A} + \frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} (c_1(\tau) + c_2(\tau)X) \hat{A} \\ &\quad - \frac{\rho_{sx}\sigma_x}{\sigma_s} \left(\int_0^\tau d_1(s) e^{d_0(s)+d_1(s)X} ds \right) Y - \frac{\rho_{sy}\sigma_y}{\sigma_s} kY \end{aligned}$$

The solution of the HJB-equation follows the same steps as in the consumption case. For this reason the derivation is shortened. Plugging in the relevant partial derivatives and the optimal policy in the HJB (2.28) and multiplying by $e^{-C(X,\tau)} (\dots)^\gamma$ yields

$$\begin{aligned}
0 = & -\frac{1}{1-\gamma} (\dots) \left(\frac{\partial c_0(\tau)}{\partial \tau} + \frac{\partial c_1(\tau)}{\partial \tau} X + \frac{1}{2} \frac{\partial c_2(\tau)}{\partial \tau} X^2 \right) - \left(\frac{\partial k}{\partial \tau} Y - \frac{\partial R}{\partial \tau} \right) \\
& + Ar_0 + \bar{Y} + Y + kY (y_0 + y_1 X) \\
& - \frac{1}{1-\gamma} \kappa_x (X - \bar{X}) (\dots) (c_1(\tau) + c_2(\tau) X) - \kappa_x (X - \bar{X}) \frac{\partial k}{\partial X} Y \\
& + \frac{1}{2} \frac{1}{\gamma} \frac{\lambda_1^2}{\sigma_s^2} X^2 (\dots) + \frac{1}{2} \frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 X (\dots) (c_1(\tau) + c_2(\tau) X) \\
& - \frac{1}{2} \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 X \frac{\partial k}{\partial X} Y - \frac{1}{2} \frac{\rho_{sy}\sigma_y}{\sigma_s} \lambda_1 X kY \\
& + \frac{1}{2} \frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 X (\dots) (c_1(\tau) + c_2(\tau) X) + \frac{1}{2} \frac{1}{\gamma} \rho_{sx}^2 \sigma_x^2 (\dots) (c_1(\tau) + c_2(\tau) X)^2 \\
& - \frac{1}{2} \rho_{sx}^2 \sigma_x^2 \frac{\partial k}{\partial X} Y ((c_1(\tau) + c_2(\tau) X)) - \frac{1}{2} \rho_{sx}\sigma_x \rho_{sy}\sigma_y kY ((c_1(\tau) + c_2(\tau) X)) \\
& - \frac{1}{2} \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 X \frac{\partial k}{\partial X} Y - \frac{1}{2} \rho_{sx}^2 \sigma_x^2 \frac{\partial k}{\partial X} Y (c_1(\tau) + c_2(\tau) X) \\
& + \underbrace{\frac{1}{2} \rho_{sx}^2 \sigma_x^2 (\dots)^{-1} \left(\frac{\partial k}{\partial X} Y \right)^2}_{ii)} + \underbrace{\frac{1}{2} \gamma \rho_{sx}\sigma_x \rho_{sy}\sigma_y (\dots)^{-1} kY \frac{\partial k}{\partial X} Y}_{iii)} \\
& - \frac{1}{2} \frac{\rho_{sy}\sigma_y}{\sigma_s} \lambda_1 X kY - \frac{1}{2} \rho_{sx}\sigma_x \rho_{sy}\sigma_y kY (c_1(\tau) + c_2(\tau) X) \\
& + \underbrace{\frac{1}{2} \gamma \rho_{sx}\sigma_x \rho_{sy}\sigma_y (\dots)^{-1} kY \frac{\partial k}{\partial X} Y}_{iii)} + \underbrace{\frac{1}{2} \gamma \rho_{sy}^2 \sigma_y^2 (\dots)^{-1} k^2 Y^2}_{iv)} \\
& + \underbrace{\rho_{xy}\sigma_x \sigma_y kY (c_1(\tau) + c_2(\tau) X) - \gamma \rho_{xy}\sigma_x \sigma_y (\dots)^{-1} kY \frac{\partial k}{\partial X} Y}_{iii)} \\
& + \underbrace{\rho_{xy}\sigma_x \sigma_y \frac{\partial k}{\partial X} Y - \frac{1}{2} \gamma (\dots)^{-1} k^2 Y^2 \sigma_y^2}_{iv)} \\
& + \frac{1}{2} \frac{1}{1-\gamma} \sigma_x^2 (\dots) (c_2(s) + c_1^2(\tau) + 2c_1(\tau) c_2(\tau) X + c_2(\tau)^2 X^2) \\
& + \sigma_x^2 \frac{\partial k}{\partial X} Y (c_1(\tau) + c_2(\tau) X) \\
& - \underbrace{\frac{1}{2} \gamma \sigma_x^2 (\dots)^{-1} \left(\frac{\partial k}{\partial X} Y \right)^2}_{ii)} + \frac{1}{2} \sigma_x^2 \frac{\partial^2 k}{\partial X^2} Y
\end{aligned}$$

Kim and Omberg (1996) show that in the case without labor income, the PDE can be solved in closed-form for all values of ρ_{sx} . This is not the case in the presence of labor income. On one hand, it can be noticed that terms highlighted by *i*) in (2.36) do not cause any problems in this equation. On the other hand, the assumption $\rho_{sx} \in \{-1, 1\}$ is needed in order to get rid of terms highlighted by *ii*). Hence, we impose the same assumptions as in the case of consumption.

Without the terms highlighted by *ii*), *iii*) and *iv*), the HJB simplifies to

$$\begin{aligned}
0 = & -\frac{1}{1-\gamma} (\dots) \left(\frac{\partial c_0(\tau)}{\partial \tau} + \frac{\partial c_1(\tau)}{\partial \tau} X + \frac{1}{2} \frac{\partial c_2(\tau)}{\partial \tau} X^2 \right) \\
& - \left(\frac{\partial k}{\partial \tau} Y - \frac{\partial R}{\partial \tau} \right) + r_0 (\dots) - r_0 (kY - R) \\
& + \bar{Y} + Y + (y_0 + y_1 X) kY \\
& - \frac{1}{1-\gamma} \kappa_x (X - \bar{X}) (\dots) (c_1(\tau) + c_2(\tau) X) - \kappa_x (X - \bar{X}) \frac{\partial k}{\partial X} Y \\
& + \frac{1}{2} \frac{1}{\gamma} \frac{\lambda_1^2}{\sigma_s^2} X^2 (\dots) + \frac{1}{2} \frac{1}{\gamma} \frac{\rho_{sx} \sigma_x}{\sigma_s} \lambda_1 X (\dots) (c_1(\tau) + c_2(\tau) X) \\
& - \frac{1}{2} \frac{\rho_{sx} \sigma_x}{\sigma_s} \lambda_1 X \frac{\partial k}{\partial X} Y - \frac{1}{2} \frac{\rho_{sy} \sigma_y}{\sigma_s} \lambda_1 X kY \\
& + \frac{1}{2} \frac{1}{\gamma} \frac{\rho_{sx} \sigma_x}{\sigma_s} \lambda_1 X (\dots) (c_1(\tau) + c_2(\tau) X) \\
& - \frac{1}{2} \rho_{sx}^2 \sigma_x^2 \frac{\partial k}{\partial X} Y (c_1(\tau) + c_2(\tau) X) - \frac{1}{2} \rho_{sx} \sigma_x \rho_{sy} \sigma_y kY (c_1(\tau) + c_2(\tau) X) \\
& - \frac{1}{2} \frac{\rho_{sx} \sigma_x}{\sigma_s} \lambda_1 X \frac{\partial k}{\partial X} Y - \frac{1}{2} \rho_{sx}^2 \sigma_x^2 \frac{\partial k}{\partial X} Y (c_1(\tau) + c_2(\tau) X) \\
& - \frac{1}{2} \frac{\rho_{sy} \sigma_y}{\sigma_s} \lambda_1 X kY + -\frac{1}{2} \rho_{sx} \sigma_x \rho_{sy} \sigma_y kY (c_1(s) + c_2(s) X) \\
& + \rho_{xy} \sigma_x \sigma_y kY (c_1(\tau) + c_2(\tau) X) + \rho_{xy} \sigma_x \sigma_y \frac{\partial k}{\partial X} Y \\
& + \frac{1}{2} \frac{1}{1-\gamma} \sigma_x^2 (\dots) c_2(s) + \sigma_x^2 \frac{\partial k}{\partial X} Y (c_1(\tau) + c_2(\tau) X) + \frac{1}{2} \sigma_x^2 \frac{\partial^2 k}{\partial X^2} Y \\
& + \frac{1}{2} \frac{1}{1-\gamma} \frac{1}{\gamma} \sigma_x^2 (\dots) (c_1^2(s) + 2c_1(s) c_2(s) X + c_2(s)^2 X^2)
\end{aligned} \tag{2.39}$$

Moreover, it should be noticed that the trivial expansion

$$(kY - R) r_0 - (kY - R) r_0$$

was made in the second line of (2.39).

Furthermore,

$$\begin{aligned}
& \frac{1}{2} \left(\frac{1}{1-\gamma} + \frac{1}{\gamma} \rho_{sx}^2 \right) \sigma_x^2 (\dots) \left(c_1^2(s) + 2c_1(s) c_2(s) X + c_2(s)^2 X^2 \right) \\
= & \frac{1}{2} \frac{1}{\gamma(1-\gamma)} \sigma_x^2 (\dots) \left(c_1^2(s) + 2c_1(s) c_2(s) X + c_2(s)^2 X^2 \right)
\end{aligned}$$

because $\rho_{sx}^2 = 1$ by assumption.

Comparing (2.39) with (2.37) shows that by separating the equations, one gets the same systems of differential equations⁵⁰, except that $\delta = 0$. As a consequence, it can be referred to the discussion of the consumption case. Besides, it should be noticed that assuming $\delta \neq 0$ would lead to a change in $c_0(s)$ only. Hence, optimal risky investment and the valuation of the reserves would not be affected.

The equation that determines the value of the net reserves is given by

$$0 = \int_0^\tau e^{C(X,s)} ds \left\{ \frac{\partial R}{\partial \tau} + \bar{Y} + r_0 R \right\} \tag{2.40}$$

⁵⁰This is known from Wachter (2002) and Liu (2007) in the context without labor income.

with initial condition $R(0) = \bar{A}$. It can be verified that (2.29) is the solution to (2.40) with $R(0) = \bar{A}$.

2.A.5 Invariant Affine Transformation

Invariant affine transformations (IAT) are well-known from the term-structure literature⁵¹. Given (2.1) and (2.2)

$$\begin{aligned}\frac{dS_1(t)}{S_1(t)} &= (\lambda_1 X(t) + r_0) dt + \sigma_s dW_s(t) \\ dX(t) &= -\kappa_x (X(t) - \bar{X}) dt + \sigma_x dW_x(t)\end{aligned}$$

It can be noticed that the system is over-identified. To give an example, instead of (2.1) it could be assumed that the same risky asset follows

$$\frac{dS_1(t)}{S_1(t)} = \left(\tilde{\lambda}_1 \tilde{X}(t) + r_0 \right) dt + \sigma_s dW_s(t)$$

where $\tilde{\lambda}_1 \neq \lambda_1$. Now,

$$\tilde{\lambda}_1 \tilde{X}(t) + r_0 = \lambda_1 X(t) + r_0 \Leftrightarrow \tilde{X}(t) = \frac{\lambda_1}{\tilde{\lambda}_1} X(t)$$

This leads to $d\tilde{X}(t) = \frac{\lambda_1}{\tilde{\lambda}_1} dX(t)$

$$\begin{aligned}d\tilde{X}(t) &= -\frac{\lambda_1}{\tilde{\lambda}_1} \kappa_x (X(t) - \bar{X}) dt + \frac{\lambda_1}{\tilde{\lambda}_1} \sigma_x dW_x(t) \\ &= -\kappa_x \left(\frac{\lambda_1}{\tilde{\lambda}_1} X(t) - \frac{\lambda_1}{\tilde{\lambda}_1} \bar{X} \right) dt + \frac{\lambda_1}{\tilde{\lambda}_1} \sigma_x dW_x(t) \\ &= -\kappa_x \left(\tilde{X}(t) - \bar{\tilde{X}} \right) dt + \tilde{\sigma}_x dW_x(t)\end{aligned}$$

where $\bar{\tilde{X}} \equiv \frac{\lambda_1}{\tilde{\lambda}_1} \bar{X}$ and $\tilde{\sigma}_x \equiv \frac{\lambda_1}{\tilde{\lambda}_1} \sigma_x$. In other words, changes in the sensitivity parameter λ_1 can be compensated by appropriate changes in (2.2). For the system (2.1) - (2.2), invariant affine transformation have the following form

$$T_A \mathbf{X}(t) = p \mathbf{X}(t), T_A \Psi \equiv (p^{-1} \lambda_1, p^{-1} p \kappa_x = \kappa_x, p \bar{X}, p \sigma_x)$$

where $p \neq 0$.

2.A.6 Valuation of the Labor Income Stream

From

$$k(X, \tau) = \int_0^\tau e^{d_0(s) + d_1(s)X} ds$$

it is clear that the $d_0(s) + d_1(s)X$ is crucial for the valuation of the income stream. It should be kept in mind that $d_1(s)$ and $d_0(s)$ are given by equation (2.19) and (2.20) respectively. For

⁵¹See Dai and Singleton (2000).

the sake of comparability we focus on $X = \bar{X}$

$$\begin{aligned}
& d_0(s) + d_1(s) \bar{X} \\
&= \left(l_2 - l_3 \frac{l_0}{l_1} + l_4 \frac{l_0^2}{l_1^2} \right) s + \left(l_3 \frac{l_0}{l_1^2} - 2l_4 \frac{l_0^2}{l_1^3} \right) (e^{l_1 s} - 1) \\
&\quad + \frac{1}{2} l_4 \frac{l_0^2}{l_1^3} (e^{2l_1 s} - 1) + \frac{l_0}{l_1} (e^{l_1 s} - 1) \bar{X} \\
&= (\bar{y} - r_0) s - y_1 \bar{X} s + \frac{\rho_{sy} \sigma_y}{\sigma_s} \lambda_1 \bar{X} s - \frac{\rho_{sy} \sigma_y}{\sigma_s} \lambda_1 \bar{X} s + l_3 \frac{l_0}{l_1} \left[-s + \frac{1}{l_1} (e^{l_1 s} - 1) \right] \\
&\quad + l_4 \frac{l_0^2}{l_1^2} \left[s - 2 \frac{1}{l_1} (e^{l_1 s} - 1) + \frac{1}{2l_1} (e^{2l_1 s} - 1) \right] + l_0 \frac{1}{l_1} (e^{l_1 s} - 1) \bar{X} \\
&= (\bar{y} - r_0) s + \left\{ \begin{aligned} & l_0 \left[\frac{1}{l_1} (e^{l_1 s} - 1) - s \right] \left(\frac{l_3}{l_1} + \bar{X} \right) \\ & + l_4 \frac{l_0^2}{l_1^2} \left[s - 2 \frac{1}{l_1} (e^{l_1 s} - 1) + \frac{1}{2l_1} (e^{2l_1 s} - 1) \right] \\ & - \frac{\rho_{sy} \sigma_y}{\sigma_s} \lambda_1 \bar{X} s \end{aligned} \right\}
\end{aligned}$$

for⁵² $l_1 \neq 0$.

It should be noticed that after the second equals sign the relation $y_0 = \bar{y} - y_1 \bar{X}$ was used and the trivial expansion $\frac{\rho_{sy} \sigma_y}{\sigma_s} \lambda_1 \bar{X} s - \frac{\rho_{sy} \sigma_y}{\sigma_s} \lambda_1 \bar{X} s = 0$ was made. Furthermore, after the third equals sign $l_0 = y_1 - \frac{\rho_{sy} \sigma_y}{\sigma_s} \lambda_1$ was used.

It should be kept in mind that $\lambda_1 > 0$ and $\bar{X} \geq 0$. The first part of the last line, $(\bar{y} - r_0) s$ corresponds to the value of the income stream under a constant growth rate ($y_1 = 0$). Hence, the term in the brackets determines whether the income stream is valued higher or lower than the constant counterpart.

Under locally riskfree labor income, the last term in the brackets vanishes. Under risky labor income, a positive correlation between the risky asset and labor income leads to a lower valuation. The second term in the brackets is positive for⁵³ $s > 0$. Under locally riskfree labor income, the definition $l_4 = \frac{1}{2} \sigma_x^2$ reveals that state variable volatility has an unambiguously positive effect on the valuation of the income stream. Nevertheless, for the parameter values as chosen in Table 2.4, the second term is small in magnitude. Hence, the term

$$l_0 \left[\frac{1}{l_1} (e^{l_1 s} - 1) - s \right] \left(\frac{l_3}{l_1} + \bar{X} \right)$$

becomes the key for the valuation of the income stream at $X = \bar{X}$ under time-varying income growth.

⁵²For the case $l_1 = 0$ the solution is given by $d_0(s) + d_1(s) \bar{X} = (\bar{y} - r_0) s + y_1 \bar{X} s + \frac{1}{2} l_0 l_3 s^2 + \frac{1}{3} l_4 l_0^2 s^3 - \frac{\rho_{sy} \sigma_y}{\sigma_s} \lambda_1 \bar{X} s$.

⁵³This follows from the fact that at $s = 0$ the term is zero and the first derivative with respect to s is given by $(e^{l_1 s} - 1)^2 > 0, s > 0$.

Chapter 3

Portfolio and Consumption Decisions under Mean-Reverting Returns and Labor Income Growth and Stochastic Labor Income Volatility

The model presented in this chapter is an extension of the basic model of Chapter 2. The chapter is written so that the main implications can be understood without having read Chapter 2. Nevertheless, the author recommends reading the aforementioned chapter first.

The main motivation for this chapter is that Lynch and Tan (2009) report that adding stochastic volatility to labor income has an even more pronounced effect on risky investment as a time-varying income drift¹. Specifically, they show that adding stochastic labor income volatility reduces risky investment further. In their numerical exercise they assume that labor income volatility can take two states depending on the state variable - the dividend yield (dy). In particular, it is assumed that $\sigma_y(t) = \bar{\sigma}_y$, if $dy(t) \leq dy^*$ and $\sigma_y(t) = 1.75 \cdot \bar{\sigma}_y$, if $dy(t) > dy^*$ and where dy^* is a threshold defined by Lynch and Tan².

A strong impact of labor income volatility on optimal investment is particularly interesting since Chacko and Viceira (2005) report only modest hedging demand for stock market volatility.

The inclusion of stochastic volatility in the labor income process does not lead to severe difficulties. In fact, the separation of the HJB can be done by the same methods. However, the system of ordinary differential equations to value the future income stream becomes more extensive and more sophisticated methods have to be applied.

The remainder of this chapter is as follows. In Section 3.1, the basic model with preferences over intermediate consumption of Chapter 2 is extended to stochastic volatility. In Section 3.2, the long-horizon stability of the solution is discussed. Section 3.3 contains the results of the model

¹Stochastic volatility in economic time series is a widely accepted phenomena and has received much attention since the work of Engle (1982) and Bollerslev (1986).

²Lynch and Tan (2009, p. 24). The paper by Lynch and Tan has been accepted for future publication in the *Journal of Financial Economics*, <http://jfe.rochester.edu/forth.htm> (10th January 2011).

for numerically realistic parameter values. The final section concludes. Mathematical derivations as the solution of the system of ordinary differential equation are given in Appendices 3.A.1 - 3.A.3.

3.1 Model

In order to get closed form solutions, the specification of Lynch and Tan cannot be implemented one-to-one but is adapted to the following form

$$\frac{dY(t)}{Y(t)} = (y_0 + y_1 X(t)) dt + (\sigma_{y0} + \sigma_{y1} X(t)) dW_y(t) \quad (3.1)$$

where $\sigma_{y1} \geq 0$. It should be noticed that with this assumption $\sigma_{y0} + \sigma_{y1} X(t)$ can turn negative. In this case, the correlation between labor income and the risky asset changes sign. Nevertheless, as long as σ_{y1} is small the probability is low.

Similarly to Lynch and Tan (2009) the financial assets and the state variable are still given as in Chapter 2.

While the separation of the HJB by A as described in Section 2.1.1 of Chapter 2 is not affected by the changes in the volatility of labor income, it can be verified that equation (2.14) of Chapter 2 changes to

$$0 = \int_0^\tau e^{C(X,s)} ds \left\{ \begin{aligned} & -\frac{\partial k}{\partial \tau} - r_0 k + 1 + k(y_0 + y_1 X) - \kappa_x X \frac{\partial k}{\partial X} + \kappa_x \bar{X} \frac{\partial k}{\partial \bar{X}} \\ & -\frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 X \frac{\partial k}{\partial X} - \frac{\rho_{sy}}{\sigma_s} (\sigma_{y0} + \sigma_{y1} X) \lambda_1 X k \\ & + \rho_{xy} \sigma_x (\sigma_{y0} + \sigma_{y1} X) \frac{\partial k}{\partial X} + \frac{1}{2} \sigma_x^2 \frac{\partial^2 k}{\partial X^2} \end{aligned} \right\} \quad (3.2)$$

It should be noticed that the term in brackets can also be derived by the Martingale method. Appendix 3.A.1 shows the rather compact derivation.

As $\int_0^\tau e^{C(X,s)} ds > 0$, (3.2) is zero if the part in the brackets is zero. A function of the form³

$$k(X, \tau) = \int_0^\tau e^{d_0(s) + \hat{d}_1(s)X + \frac{1}{2}d_2(s)X^2} ds$$

will solve the equation (3.2) with $d_0(0) = \hat{d}_1(0) = d_2(0) = 0$. The relevant partial derivatives are as follows

$$\begin{aligned} k_\tau &= \int_0^\tau \left(\frac{\partial d_0(s)}{\partial s} + \frac{\partial \hat{d}_1(s)}{\partial s} X + \frac{1}{2} \frac{\partial d_2(s)}{\partial s} X^2 \right) e^{D(X,s)} ds + 1 \\ k_X &= \int_0^\tau \left(\hat{d}_1(s) + d_2(s) X \right) e^{D(X,s)} ds \\ k_{XX} &= \int_0^\tau \left(\hat{d}_1(s)^2 + d_2(s) + 2\hat{d}_1(s) d_2(s) X + d_2^2(s) X^2 \right) e^{D(X,s)} ds \end{aligned}$$

where $D(X, s) \equiv d_0(s) + \hat{d}_1(s)X + \frac{1}{2}d_2(s)X^2$. Plugging in the partial derivatives into (3.2)

³The notational change from $d_1(s)$ to $\hat{d}_1(s)$ will become clear below.

leads to

$$0 = \int_0^\tau e^{D(X,s)} \left\{ \begin{aligned} & - \left(\frac{\partial d_0(s)}{\partial s} + \frac{\partial \hat{d}_1(s)}{\partial s} X + \frac{1}{2} \frac{\partial d_2(s)}{\partial s} X^2 \right) - r_0 + (y_0 + y_1 X) \\ & - \kappa_x X \left(\hat{d}_1(s) + d_2(s) X \right) + \kappa_x \bar{X} \left(\hat{d}_1(s) + d_2(s) X \right) \\ & - \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 X \left(\hat{d}_1(s) + d_2(s) X \right) - \frac{\rho_{sy}}{\sigma_s} (\sigma_{y0} + \sigma_{y1} X) \lambda_1 X \\ & + \rho_{xy}\sigma_x (\sigma_{y0} + \sigma_{y1} X) \left(\hat{d}_1(s) + d_2(s) X \right) \\ & + \frac{1}{2} \sigma_x^2 \left(\hat{d}_1(s)^2 + d_2(s) + 2\hat{d}_1(s) d_2(s) X + d_2^2(s) X^2 \right) \end{aligned} \right\} ds$$

Matching coefficients on X^2 , X and the constant term leads to a system of three ordinary differential equations.

$$\frac{\partial d_2(s)}{\partial s} = l_0 + l_1 d_2(s) + l_2 d_2(s)^2 \quad (3.3)$$

$$\frac{\partial \hat{d}_1(s)}{\partial s} = l_6 + l_3 d_2(s) + \frac{l_1}{2} \hat{d}_1(s) + l_2 \hat{d}_1(s) d_2(s) \quad (3.4)$$

$$\frac{\partial d_0(s)}{\partial s} = l_7 + l_4 \hat{d}_1(s) + \frac{l_2}{2} \hat{d}_1(s)^2 + l_5 d_2(s) \quad (3.5)$$

where

$$\begin{aligned} l_0 &\equiv -2 \frac{\rho_{sy}\sigma_{y1}}{\sigma_s} \lambda_1, \quad l_1 \equiv 2 \left[-\kappa_x - \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 + \rho_{xy}\sigma_x \sigma_{y1} \right], \quad l_2 \equiv \sigma_x^2 \\ l_6 &\equiv y_1 - \frac{\rho_{sy}\sigma_{y0}}{\sigma_s} \lambda_1, \quad l_3 \equiv \kappa_x \bar{X} + \rho_{xy}\sigma_x \sigma_{y0} \\ l_7 &\equiv y_0 - r_0, \quad l_4 \equiv \kappa_x \bar{X} + \rho_{xy}\sigma_x \sigma_{y0}, \quad l_5 \equiv \frac{1}{2} \sigma_x^2 \end{aligned}$$

Remarks

- The first equation of the system of ordinary differential equations (SODE) (3.3) - (3.5) is a Riccati differential equation and has three solution forms depending on l_0 , l_1 and l_2 . As a consequence, three cases must be distinguished for (3.3). The solutions are given in Appendix 3.A.2 and the details of the derivation can be found in Appendix 3.A.3.
- The inclusion of stochastic volatility of this form always comes with a gap between the market price of the risky asset and that of labor income. To be more precise, assuming $\sigma_{y1} \neq 0$ will lead to $d_2(s) \neq 0$. As a consequence, there will be an effect of stochastic volatility on the optimal policies.
- The size of the effect depends on the stability of the Riccati differential equation. In instable cases, a strong hedging demand may arise for long horizons. A detailed discussion follows in the next section.

The results for the model with stochastic volatility are summarized in Proposition 3.1.

PROPOSITION 3.1 *Given the assumptions $\hat{A}(0) > 0$, $\rho_{sx} \in \{-1, 1\}$ and $\rho_{sy} \in \{-1, 1\}$ one obtains*

$$J = \frac{e^{-\delta(T-\tau)} \left[\int_0^\tau e^{\frac{1}{\gamma} (c_0(s) + c_1(s)X + \frac{1}{2} c_2(s)X^2)} ds \right]^\gamma (A + k(\tau, X)Y - R(\tau))^{1-\gamma}}{1 - \gamma}$$

with

$$k(\tau, X) = \int_0^\tau e^{d_0(s) + \hat{d}_1(s)X + \frac{1}{2}d_2(s)X^2} ds$$

where $d_0(s)$, $d_1(s)$ and $d_2(s)$ are the solution to the following system of ordinary differential equations

$$\begin{aligned} \frac{\partial d_2(s)}{\partial s} &= l_0 + l_1 d_2(s) + l_2 d_2(s)^2 \\ \frac{\partial d_1(s)}{\partial s} &= l_6 + l_3 d_2(s) + \frac{l_1}{2} \hat{d}_1(s) + l_2 d_1(s) d_2(s) \\ \frac{\partial d_0(s)}{\partial s} &= l_7 + l_4 \hat{d}_1(s) + \frac{l_2}{2} \sigma \hat{d}_1(s)^2 + l_5 d_2(s) \end{aligned}$$

with initial conditions $d_0(0) = 0$, $\hat{d}_1(0) = 0$ and $d_2(0) = 0$ and where

$$\begin{aligned} l_0 &\equiv -2 \frac{\rho_{sy}\sigma_{y1}}{\sigma_s} \lambda_1, \quad l_1 \equiv 2 \left[-\kappa_x - \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 + \rho_{xy}\sigma_x\sigma_{y1} \right], \quad l_2 \equiv \sigma_x^2 \\ l_6 &\equiv y_1 - \frac{\rho_{sy}\sigma_{y0}}{\sigma_s} \lambda_1, \quad l_3 \equiv \kappa_x \bar{X} + \rho_{xy}\sigma_x\sigma_{y0} \\ l_7 &\equiv y_0 - r_0, \quad l_4 \equiv \kappa_x \bar{X} + \rho_{xy}\sigma_x\sigma_{y0}, \quad l_5 \equiv \frac{1}{2} \sigma_x^2 \end{aligned}$$

The net reserves follow

$$R(\tau) = \frac{\bar{c} - \bar{Y}}{r_0} (1 - e^{-r_0\tau})$$

The solutions of $c_0(s)$, $c_1(s)$ and $c_2(s)$ are identical to Wachter (2002).

Optimal consumption and risky investment are given by

$$c_t^* = \frac{\hat{A}}{\int_0^\tau e^{C(X,s)} ds} + \bar{c} \quad (3.6)$$

$$\begin{aligned} A\pi_t^* &= \frac{1}{\gamma} \frac{\lambda_1}{\sigma_s^2} X \hat{A} + \frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \frac{\int_0^\tau (c_1(s) + c_2(s)X) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} \hat{A} \\ &\quad - \frac{\rho_{sx}\sigma_x}{\sigma_s} \left(\int_0^\tau (\hat{d}_1(s) + d_2(s)X) e^{D(X,s)} ds \right) Y \\ &\quad - \frac{\rho_{sy}}{\sigma_s} (\sigma_{y0} + \sigma_{y1}X) kY \end{aligned}$$

where $C(X, s) \equiv \frac{1}{\gamma} (c_0(s) + c_1(s)X + \frac{1}{2}c_2(s)X^2)$ and $D(X, s) \equiv d_0(s) + \hat{d}_1(s)X + \frac{1}{2}d_2(s)X^2$.

3.2 Long-Horizon Stability of the Solution

Figure 3.1 and 3.2 show a phase plane analysis for the Riccati differential equation (3.3). Since l_2 is unambiguously greater than zero, the parabola opens upward and six cases arise. Figure 3.1 shows the cases where (3.3) has two real particular solutions. If $l_0 < 0$ two real solutions with different signs exist. As can be seen in Panel (a), $d_2(s)$ converges to the negative solution. Given two real solutions, if $l_0 > 0$ and $l_1 < 0$ two positive solutions exist. From Panel (b) it can

be recognized that $d_2(s)$ converges to the smaller solution. Given two real solutions, if $l_0 > 0$ and $l_1 > 0$ two negative solutions exist. From Panel (c) it should be noticed that this setting is unstable as $d_2(s)$ grows without bound.

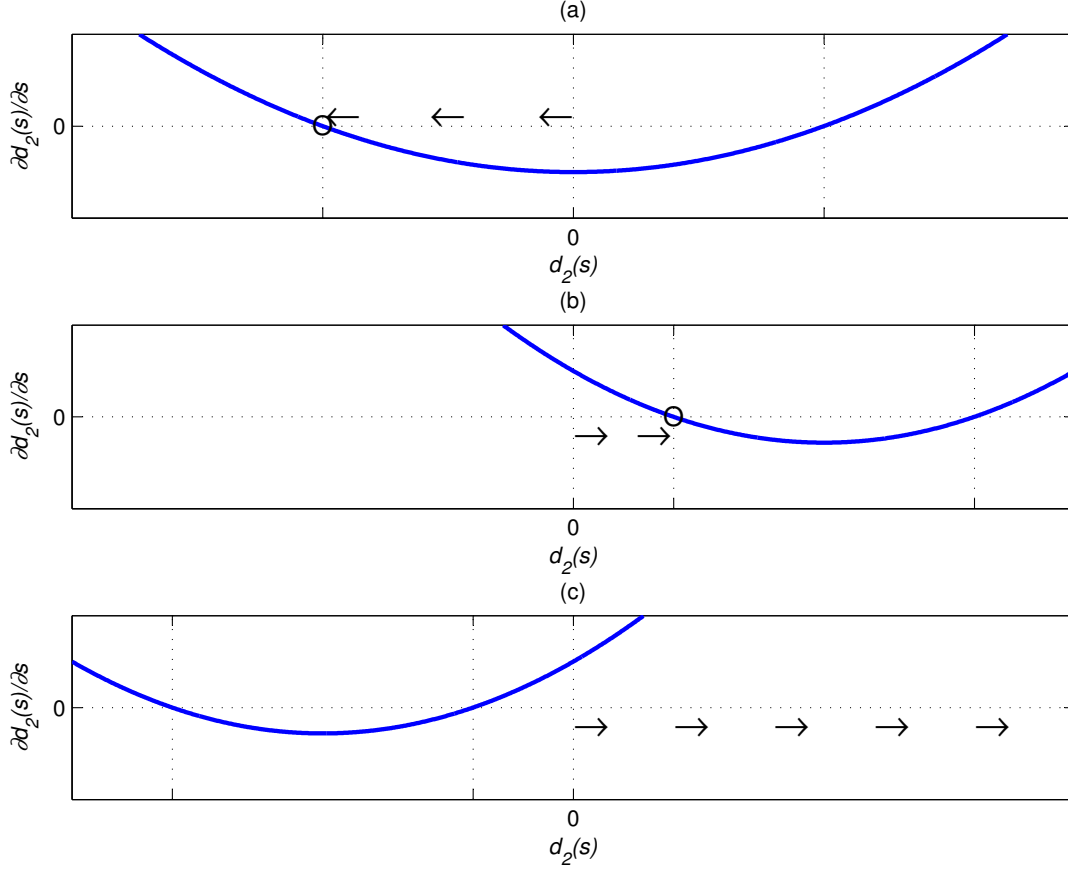


Figure 3.1: Phase Plane Analysis I

Panels (a), (b) and (c) show a phase plane analysis of the equation $\frac{\partial d_2(s)}{\partial s} = l_0 + l_1 d_2(s) + l_2 d_2(s)^2$ for $q_t > 0$. In all cases two real particular solutions exist. In Panel (a), $l_0 < 0$, in Panel (b), $l_0 > 0$ and $l_1 < 0$ and in Panel (c), $l_0 > 0$ and $l_1 > 0$. In Panels (a) and (b) $d_2(s)$ converges to a stable solution marked by a circle.

Panels (a) and (b) from Figure 3.2 show the case where only one real particular solution exists. The discussion is analogous to Panels (b) and (c) from Figure 3.1. In Panel (c) there are no real solutions and $d_2(s)$ grows without bound. In all cases in Figure 3.2 $l_0 > 0$. Table 3.1 summarizes the stability analysis.

As pointed out in Appendix 3.A.2, in the case of instability $d_2(s) > 0$, $\forall s > 0$ and approaches infinity at a finite horizon. The consequences are similar to Kim and Omberg (1996) who discovered this property for the solution of the state variable hedging part for utility functions with $\gamma < 1$. If the set of parameters leads to an unstable situation then the value of the future income stream becomes infinite and the optimal policies are not well-defined⁴.

⁴In the Kim and Omberg model, the cases with instability lead to an infinite utility. This is not the case in our model as $\gamma > 1$ by assumption and hence, the utility is bounded above by zero.

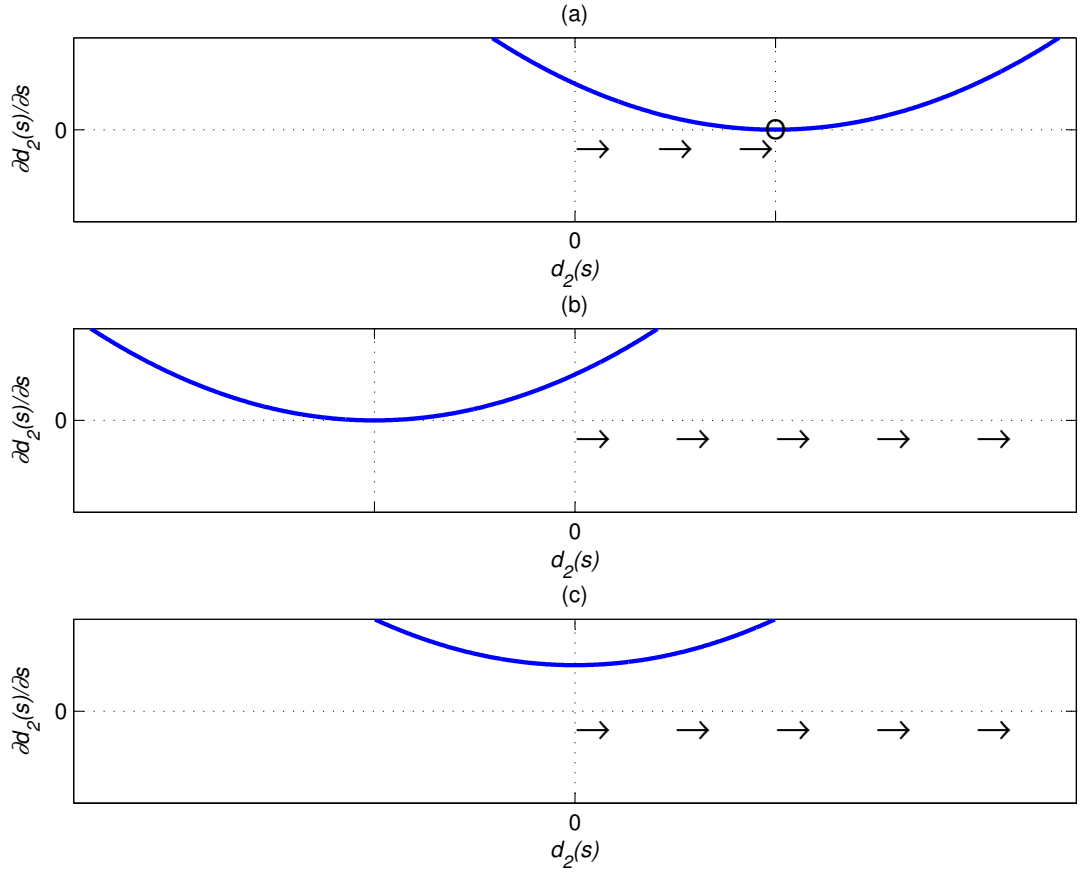


Figure 3.2: Phase Plane Analysis II

Panels (a), (b) and (c) show a phase plane analysis of the equation $\frac{\partial d_2(s)}{\partial s} = l_0 + l_1 d_2(s) + l_2 d_2(s)^2$. In Panels (a) and (b), one real particular solution exists ($q_l = 0$). Panel (c) shows the case without a particular solution ($q_l < 0$). In Panel (a), $l_1 < 0$ and in Panel (b), $l_1 > 0$. In Panel (a), $d_2(s)$ converges to a stable solution marked by a circle.

Number of Particular Solutions		Stable?
2	$q_l > 0, \quad l_0 < 0$	yes
	$q_l > 0, \quad l_0 > 0, \quad l_1 < 0$	yes
	$q_l > 0, \quad l_0 > 0, \quad l_1 > 0$	no
1	$q_l = 0, \quad l_1 < 0$	yes
	$q_l = 0, \quad l_1 > 0$	no
0	$q_l < 0$	no

Table 3.1: Stability Analysis

It should be noticed that the cases where an unstable solution results are limited. In particular, a (perfectly) positive correlation between the risky asset and labor income leads to $l_0 < 0$ and

thus a stable solution occurs. On the other hand, $l_0 > 0 \Leftrightarrow \rho_{sy} = -1$ implies that solutions that are not well-defined might exist. This point shows that a valuation of the future income stream that is so positive as to effect an infinite value is not arbitrary and depends on the economic environment. A negative correlation between the risky asset return and labor income is certainly a good thing as a natural diversification of financial and non-financial income exists.

Even for $\rho_{sy} = -1$, as long as λ_1 and σ_{y1} are not too high compared to σ_s , two particular solutions exist. In this case, the unstable solution asks for $l_1 > 0$. The definition

$$l_1 \equiv 2 \left[-\kappa_x - \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 + \rho_{xy}\sigma_x\sigma_{y1} \right]$$

reveals that l_1 consists of three components. The first part $-\kappa_x$ is unambiguously negative and therefore in favor of stability. Hence, as long as the persistency of the shocks is not too high (low κ_x), the solution tends to be stable. The other two parts contain correlation parameters. Since unstable solutions arise only for $\rho_{sy} = -1$ and $\lambda_1 > 0$, $\sigma_s > 0$, $\sigma_x > 0$, $\sigma_{y1} \geq 0$ only two cases can be distinguished.

In the first case

$$\rho_{sx} = -1 \Rightarrow \rho_{xy} = \rho_{sy}\rho_{sx} = 1.$$

In this case, $l_1 > 0$ is feasible and thus instability is possible.

In the second case

$$\rho_{sx} = 1 \Rightarrow \rho_{xy} = \rho_{sy}\rho_{sx} = -1$$

and $l_1 > 0$ is not feasible.

The reason why the individual prefers the first case to the second one is the following. $\rho_{sx} = -1$ implies not only a higher equity premium after a decline in the value of the risky asset, but higher labor income volatility too. As a consequence, the labor hedging portfolio must include more of the risky asset, which is desirable in states of high premium (the risks cancel out but the premium is high).

The ρ_{xy} must be interpreted in analogy to Section 2.4.3 of Chapter 2. As mentioned, for $X > 0$, $d_2(s) < 0$ and a negative correlation ρ_{xy} imply a positive drift in total wealth that must be accounted for.

The interpretations for the one particular solution case ($q_1 = 0$) is the same. In the case without a particular solution, the valuation is even unstable in case of $l_1 < 0$. In this case the diversification effect of the risky asset and labor income is so important that the counter effects are dominated.

Thus, it can be concluded that instable solutions arise only if labor income and the risky asset share very distinct dynamics. From an equilibrium perspective there should be a connection between labor and capital markets and therefore the rather extreme parameter set asked for instability are probably not in line with reality. This statement is similar to that in Kim and Omberg for their instable solutions⁵. The other explanation of Kim and Omberg is of course applicable as well. Namely, that the real world includes constraints and costs that are not part

⁵Kim and Omberg (1996, p. 151).

Hence, $d_2(s) < 0$ and this has, as shown in more detail below, a negative impact on risky asset holdings.

Moreover, the parameters estimated in Lynch and Tan suggest a positive l_1 . The sign of l_1 is more important than it may look on first sight for long-horizon investors. As can be seen from the phase plane analysis in Figure 3.3, an open upward parabola with $l_0 = 0$ has a particular solution equal to zero. Because of the initial condition $d_2(0) = 0$, the solution to the Riccati differential equation is given by

$$d_2(s) = 0, \forall s$$

The second particular solution is positive (negative) if $l_1 < 0$ ($l_1 > 0$), but not of interest. A difference arises as soon as $l_0 \neq 0$. For $l_1 < 0$, the stable solution of the Riccati equation is the one that was originally in the origin. Hence, even long-term behavior of $d_2(s)$ is smooth in l_0 . Loosely spoken, Panel (b) shows that as l_0 shifts away from zero, the stable solution is still close to the origin. This is not the case for $l_1 > 0$, Panel (a) exhibits that the solution from the differential equation converges to the negative particular solution. Hence, there is a jump in the long-term behavior of $d_2(s)$ around $l_0 = 0$ and extreme results may occur.

3.3 Illustration of the Results

Parameters are chosen similar to the model of Table 2.4 of Chapter 2, which is identical with the exception of the stochastic volatility part. Table 3.2 shows the parameters in detail.

The parameters for the labor income process (3.1) are chosen variably in order to show the effects clearly. For the sake of comparability, y_0 (σ_{y0}) and y_1 (σ_{y1}) are chosen so that the growth rate (volatility) at the long-run mean \bar{X} is constant. Specifically,

$$y_0 = \bar{y} - y_1 \bar{X}, \quad \sigma_{y0} = \bar{\sigma}_y - \sigma_{y1} \bar{X}$$

where \bar{y} ($\bar{\sigma}_y$) is the long-run growth rate (volatility) and given in Table 3.2.

Financial Market			
$r_0 = 0.0033$			
$\lambda_1 = 1$	$\sigma_s = 0.1579$		
$\kappa_x = 0.1755$	$\bar{X} = 0.0528$	$\sigma_x = 0.0089$	
$\rho_{sx} = -1$			
Individual			
$\gamma = 4$	$\delta = 0.06$		
$\bar{y} = 0.03$	$\bar{\sigma}_y = 0.04$	$\rho_{sy} = 1$	$\rho_{xy} = -1$
$A(0) = 50$	$Y(0) = 10$	$\bar{Y} = 40$	
$\bar{c} = 45$			

Table 3.2: Parameter Values

Compared to the model in the previous section, the inclusion of stochastic income volatility leads to a more difficult SODE. For the interpretation of the results it should be kept in mind that the sign of $d_2(s)$ is equal to the sign of l_0 . This has become clear from the phase plane analysis. Since it is focused on the case $\rho_{sy} = 1$

$$l_0 < 0 \Rightarrow d_2(s) < 0, \forall s$$

The assumption $l_0 < 0$ and the phase plane analysis revealed that for the illustrative examples only the results of case I of Appendix 3.A.3 are needed. Nevertheless, for future research the solution of the other cases can become important as well. Furthermore, for the interpretation of the results it is assumed that the critical parameter for stability is $l_1 < 0$. As can be seen from the parameters in Table 3.2, this is clearly the case for the numerical examples and the phase plane analysis above shows that this assumption seems to be in line with reality. Since l_2 is unambiguously greater than zero, in combination with the negative sign of $d_2(s)$, the (time-varying) slope parameter of (3.4) is

$$\frac{l_1}{2} + l_2 d_2(s) < 0$$

and hence the linear differential equation converges to a stable solution as well⁷.

Because of the high number of parameters, general results are hard to find. Nevertheless, the qualitative effects described in this section are valid for a wide range of realistic parameters.

In order to show the effect of stochastic labor income volatility clearly, the results consists of four scenarios. In the first case (green line with crosses), the individual has a constant labor income growth and constant volatility. In the second and the third cases, only stochastic volatility (green line with circles) or only varying labor income growth (blue line with crosses) is switched on. In the fourth case (blue line with circles), both channels are switched on. The cases without stochastic volatility are discussed extensively in the last chapter. For this reason only the most important points are stated.

For the sake of clarity, we introduce the following definitions for the components of risky investment

$$\begin{aligned}
 A\pi_t^* = & \underbrace{\frac{1}{\gamma} \frac{\lambda_1}{\sigma_s^2} X \hat{A}}_{\text{"myopic"}} + \underbrace{\frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \frac{\int_0^\tau (c_1(s) + c_2(s)X) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} \hat{A}}_{\text{"state variable hedging"}} \\
 & - \underbrace{\frac{\rho_{sx}\sigma_x}{\sigma_s} \left(\int_0^\tau (\hat{d}_1(s) + d_2(s)X) e^{D(X,s)} ds \right) Y}_{\text{"indirect labor hedging"}} \\
 & - \underbrace{\frac{\rho_{sy}(\sigma_{y0} + \sigma_{y1}X)}{\sigma_s} \left(\int_0^\tau e^{D(X,s)} ds \right) Y}_{\text{"direct labor hedging"}}
 \end{aligned} \tag{3.7}$$

Figure 3.4 shows total wealth dependent on the state variable. In Chapter 2 it was shown that without stochastic income volatility the *slope* of total wealth does not change sign over the

⁷For more details see phase plane analysis of a linear differential equation in Chapter 2.

range of X . In fact, without stochastic labor income volatility the linear differential equation (3.4) reduces to

$$\frac{\partial \hat{d}_1(s)}{\partial s} = l_6 + \frac{l_1}{2} \hat{d}_1(s)$$

and (3.8) reduces to

$$\frac{\partial \hat{A}}{\partial X} = \frac{\partial k}{\partial X} Y = \left(\int_0^\tau \hat{d}_1(s) e^{D(X,s)} ds \right) Y$$

It can be recognized that the sign of the slope does only depend on the sign of $\hat{d}_1(s)$ and is equal to the sign of⁸ l_6 .

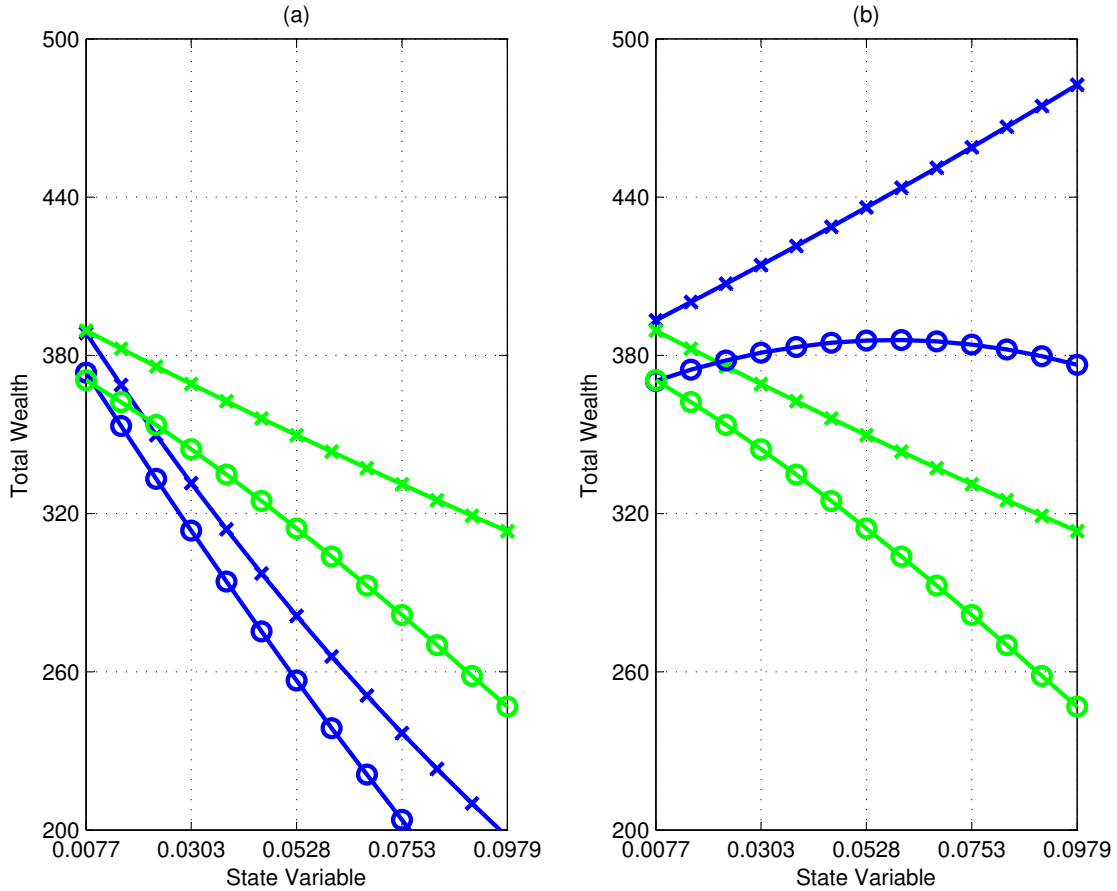


Figure 3.4: Total Wealth - Stochastic Labor Income Volatility

This Figure exhibits total wealth $\hat{A}(t)$ dependent on the state variable under stochastic labor income volatility. Parameters are chosen as in Table 3.2. In the panel to the left (right) the blue lines show the results for an individual with a negative (positive) sensitivity of labor income growth on $X(t)$ of $y_1 = -0.5$ ($y_1 = 0.5$). In both Panels the green lines show the case of constant labor income growth $y_1 = 0$. The lines with crosses (circles) display the results for an individual with constant (stochastic) labor income volatility. In the case of stochastic labor income volatility $\sigma_{y1} = 0.5$.

As Panel (b) shows, this is not the case in the presence of stochastic income volatility. In fact,

⁸For more details on the constant income volatility case see Chapter 2.

the slope varies with X and changes sign. From an analytical point of view

$$\frac{\partial \hat{A}}{\partial X} = \frac{\partial k}{\partial X} Y = \left(\int_0^\tau \left(\hat{d}_1(s) + d_2(s) X \right) e^{D(X,s)} ds \right) Y \quad (3.8)$$

By the positivity of the exponential function and Y , the sign is determined by $\hat{d}_1(s) + d_2(s) X$. As $d_2(s) < 0, \forall s$ and $\hat{d}_1(s)$ is bounded it can be stated that the slope of total wealth is positive for sufficiently small X and negative for sufficiently high X .

From

$$k(X, \tau) = \left(\int_0^\tau e^{d_0(s) + \hat{d}_1(s)X + \frac{1}{2}d_2(s)X^2} ds \right) Y$$

the negativity of $d_2(s)$ implies that $e^{\frac{1}{2}d_2(s)X^2}$ has an unambiguously negative effect on the *level* of total wealth at $X = \bar{X}$. However, a general statement cannot be made because of the impact of $\sigma_{y1} > 0$ on $\hat{d}_1(s)$ and the impact of $d_2(s)$ and $\hat{d}_1(s)$ on $d_0(s)$. Nevertheless, for the chosen parameters the total impact on the level of total wealth is negative.

Panels (a) and (b) of Figure 3.5 show total risky investment. It can be seen that at the long-run mean \bar{X} the inclusion of stochastic volatility leads to a decrease in the *level* of risky asset holdings. Moreover, the *slopes* become less steep and can even turn negative. A low sensitivity of optimal investment across states seems a desirable property because permanent portfolio shifts are avoided.

As can be seen from Panels (c) and (d) and the definition of state variable hedging demand in equation (3.7), the impact of X on state variable hedging demand is rather simple, as it is only affected by the changes in total wealth.

The changes in indirect labor hedging demand are far more interesting. While in the case $\sigma_{y1} = 0$ the *slope* of indirect hedging demand is unambiguously positive, this is not the case when stochastic labor volatility is included. The derivative of indirect labor hedging demand with respect to X is given by

$$\int_0^\tau \left(\hat{d}_1(s)^2 + d_2(s) + 2\hat{d}_1(s) d_2(s) X + d_2(s)^2 X^2 \right) e^{D(s,X)} ds$$

Without stochastic volatility the term in the parentheses reduces to $\hat{d}_1(s)^2$, which is positive. In the presence of stochastic volatility, the terms $\hat{d}_1(s)^2$ and $d_2(s)^2 X^2$ are also unambiguously positive. However, this is not the case for the other two terms. As shown above, $d_2(s) < 0$ and the term $2\hat{d}_1(s) d_2(s) X$ can have either sign. Hence, the absence of a clear relation is straightforward.

The change in *level* is mainly caused by the term $d_2(s) X$, which is unambiguously negative for $X > 0$. This can be clearly recognized in Panels (e) and (f) as the reduction in the difference between the solid and the dashed lines as X goes to zero.

The behavior of indirect labor hedging demand can be explained by the goal of intertemporal hedging. High income volatility in high states of X is clearly an undesirable feature. As a consequence, the individual takes a (short) position in the risky asset that delivers a high return before a negative state and vice versa.

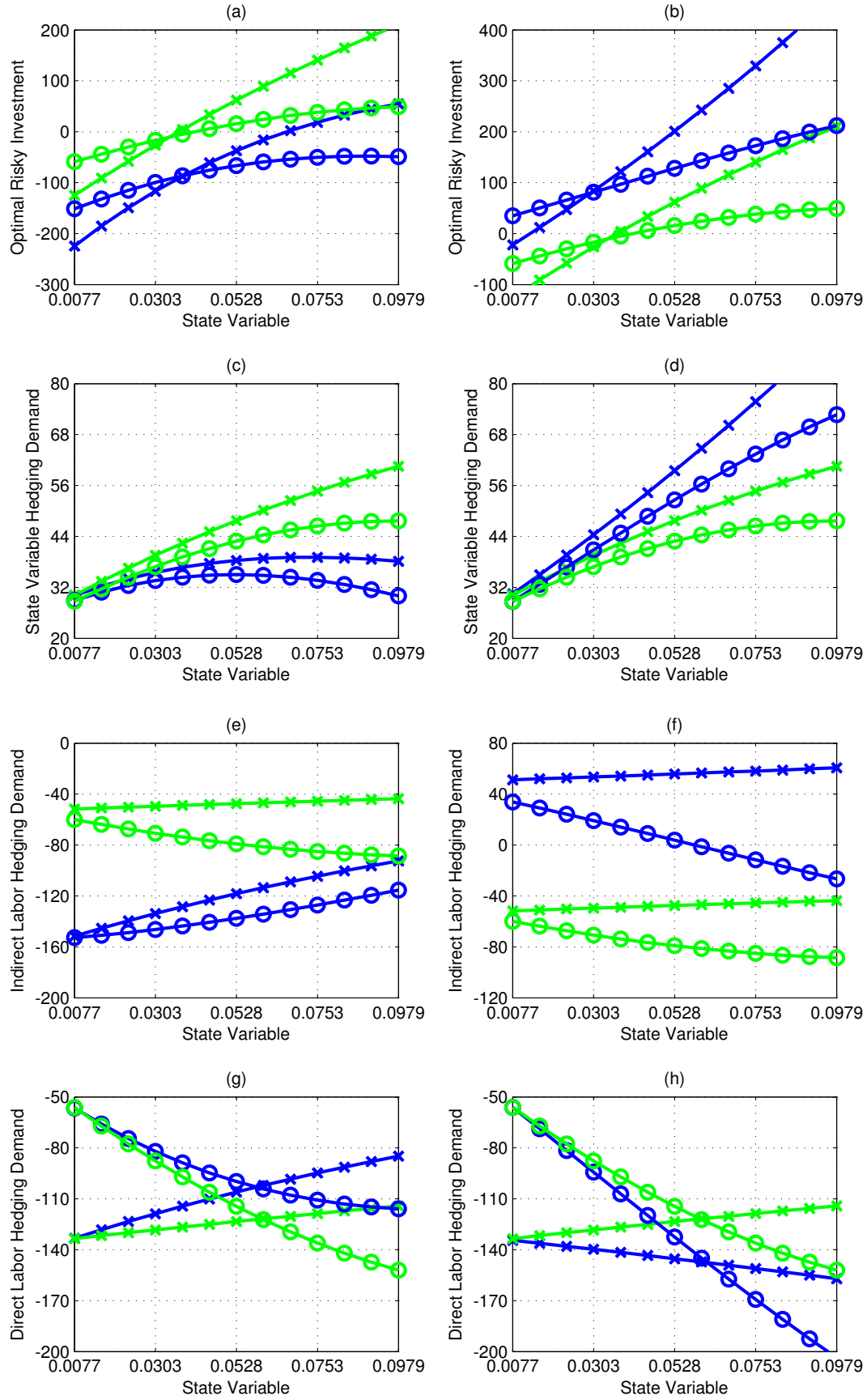


Figure 3.5: Optimal Risky Investment - Stochastic Labor Income Volatility

Figure 3.5 continued: Panels (a) and (b) show optimal total risky investment $A(t) \pi_t^*$ dependent on the state variable under stochastic labor income volatility. Panels (c) to (h) show the components of hedging demand as described in equation (3.2). Parameters are chosen as in Table 3.2. In the panels to the left (right) the blue lines show the results for an individual with a negative (positive) sensitivity of labor income growth on $X(t)$ of $y_1 = -0.5$ ($y_1 = 0.5$). In both panels the green line shows the case of constant labor income growth $y_1 = 0$. The lines with crosses (circles) display the results for an individual with constant (stochastic) labor income volatility. In the case of stochastic labor income volatility $\sigma_{y1} = 0.5$.

Direct labor hedging demand is displayed in Panels (g) and (h). From its definition it becomes clear that two effects are at work. On the one hand, the demand is directly affected by X through $\rho_{sy}(\sigma_{y0} + \sigma_{y1}X)/\sigma_s$. On the other hand, k changes with X . Moreover, it should be kept in mind that at $X = \bar{X}$, $\sigma_{y0} + \sigma_{y1}\bar{X}$ is the same for all cases. The derivative of direct labor hedging demand with respect to X is given by

$$-\frac{\rho_{sy}\sigma_{y1}}{\sigma_s}kY - \frac{\rho_{sy}(\sigma_{y0} + \sigma_{y1}X)}{\sigma_s} \left(\int_0^\tau \left(\hat{d}_1(s) + d_2(s)X \right) e^{D(s,X)} ds \right) Y \quad (3.9)$$

It should be noticed that the first term is unambiguously negative, but the second part is more complicated.

In the case of constant income volatility, the first term drops out and the second reduces to

$$-\frac{\rho_{sy}\sigma_{y0}}{\sigma_s} \left(\int_0^\tau \hat{d}_1(s) e^{d_0(s) + \hat{d}_1(s)X} ds \right) Y$$

and the sign of the *slope* of direct labor hedging demand is determined by the sign of $-\rho_{sy}\hat{d}_1(s)$.

For the slope of direct hedging demand in the case of stochastic income volatility, it should be noted that by the positivity of the exponential function and Y , the two terms $\sigma_{y0} + \sigma_{y1}X$ and $\hat{d}_1(s) + d_2(s)X$ become crucial. Indeed, if the terms have the same sign, the second term of (3.9) also becomes negative and the slope of direct labor hedging demand is clearly negative. For sufficiently high values of X ,

$$\sigma_{y0} + \sigma_{y1}X > 0 \quad \wedge \quad \hat{d}_1(s) + d_2(s)X < 0$$

The second relation follows from the phase plane analysis that showed $d_2(s) < 0$ and because $\hat{d}_1(s)$ is bounded. In analogy, for sufficiently small values of X ,

$$\sigma_{y0} + \sigma_{y1}X < 0 \quad \wedge \quad \hat{d}_1(s) + d_2(s)X > 0$$

For intermediate values of X , the two terms have the same sign. Thus, general statements are not possible. However, for the chosen parameter values around the steady state \bar{X} the first part of (3.9) dominates and the slope is negative.

At $X = \bar{X}$, the impact on the *level* of direct labor hedging demand stems exclusively from changes in k as $\sigma_{y0} + \sigma_{y1}\bar{X} = \bar{\sigma}_y$. Hence, the changes in the level of direct labor hedging demand are analogous to the changes in total wealth.

The stronger variation in direct labor income hedging demand is intuitive as high levels of X imply high labor income volatility and hence more pronounced positions in the risky asset are needed to hedge labor income.

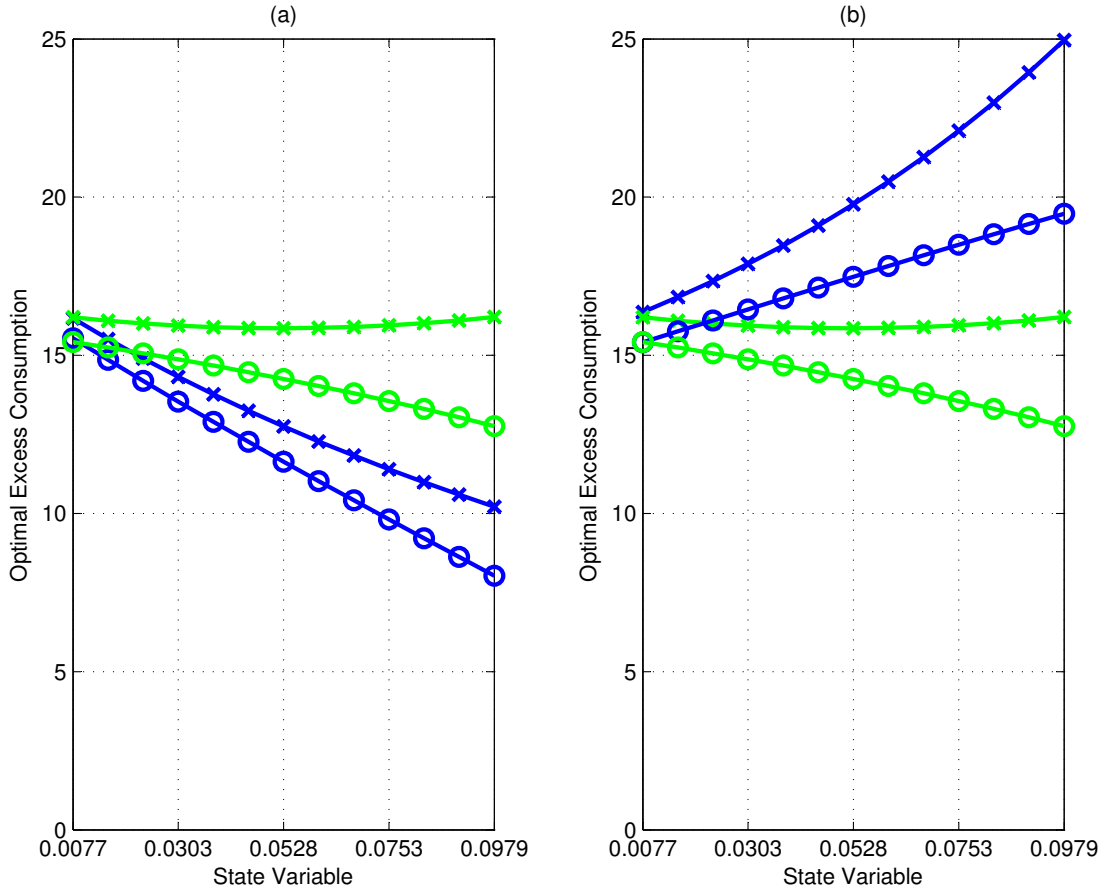


Figure 3.6: Optimal Consumption - Stochastic Labor Income Volatility

Panels (a) and (b) show optimal consumption exceeding the subsistence level $c_t^* - \bar{c}$ dependent on the state variable under risky labor income. Parameters are chosen as in Table 3.2. In the panel to the left (right) the blue lines show the results for an individual with a negative (positive) sensitivity of labor income growth on $X(t)$ of $y_1 = -0.5$ ($y_1 = 0.5$). In both panels the green lines show the case of constant labor income growth $y_1 = 0$. The lines with crosses (circles) display the results for an individual with constant (stochastic) labor income volatility. In the case of stochastic labor income volatility $\sigma_{y1} = 0.5$.

Figure 3.6 shows the results for optimal consumption exceeding the subsistence level of consumption. The results are rather easy to explain as the denominator of (3.6) is the same for all cases and hence the differences are caused exclusively by the differences in total wealth. A detailed discussion is omitted, but it should be noticed that stochastic labor income volatility leads to lower consumption in high states of X .

To show the impact of σ_{y1} more clearly, Figures 3.7 - 3.9 exhibit the blue cases with an additional value for $\sigma_{y1} = 0.08$. The analysis does not show any unexpected results. In general, the results go in the same direction and are stronger in magnitude. Nevertheless, the following points are worth mentioning.

Panels (a) and (b) of Figure 3.8 show that optimal risky investment can decrease as the state variable increases. Moreover, total risky investment can become rather insensitive to changes in the state variable. This is a desirable feature as it avoids strong shifts in risky investment across

states. As can be seen from Panels (g) and (h), these effects are mainly driven by changes in direct labor hedging demand, which becomes steeper.

Furthermore, the parameter values of the lines with squares in the panels to the left lead to both falling risky investment and consumption in states of high labor income volatility, even though expected returns of the risky asset rise with X .

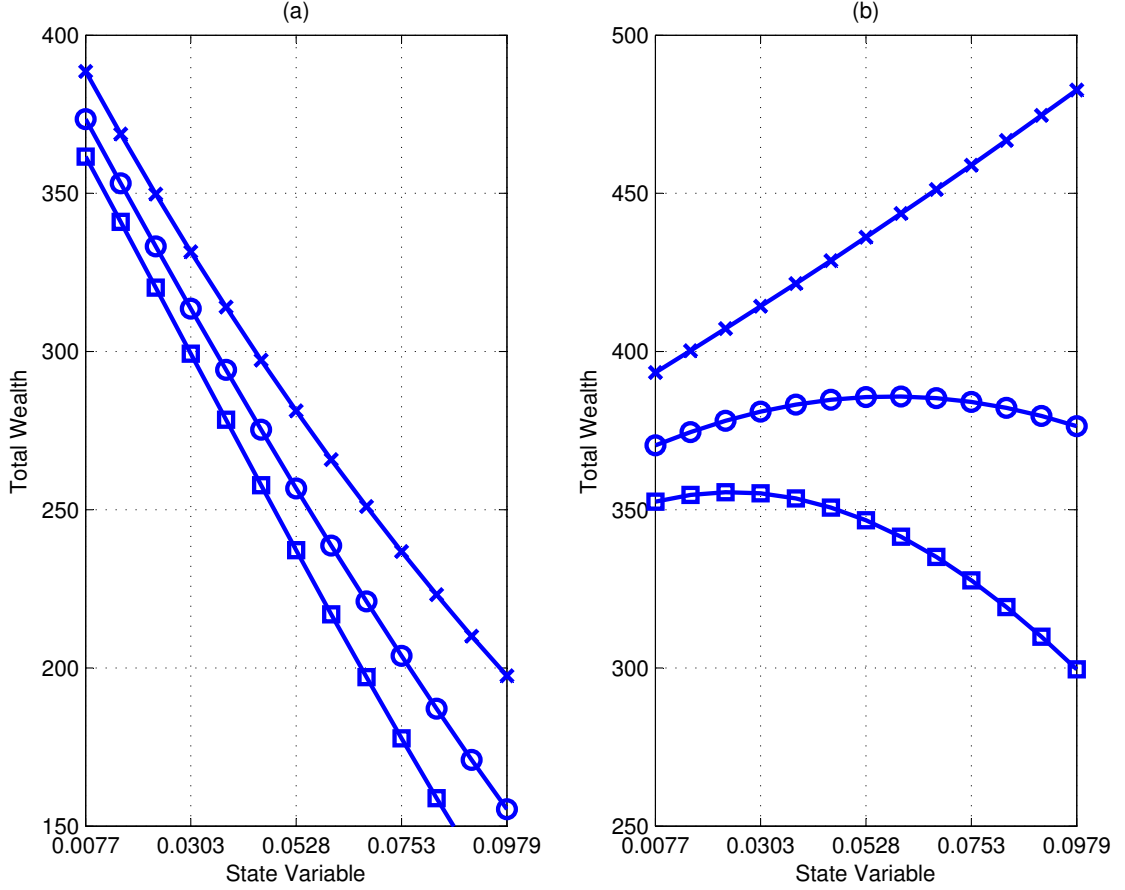


Figure 3.7: Total Wealth - Sensitivity

This Figure exhibits total wealth $\hat{A}(t)$ dependent on the state variable under stochastic labor income volatility. Parameters are chosen as in Table 3.2. In the panel to the left (right) the blue lines show the results for an individual with a negative (positive) sensitivity of labor income growth on $X(t)$ of $y_1 = -0.5$ ($y_1 = 0.5$). The lines with crosses (circles, squares) display the results for an individual with constant (stochastic) labor income volatility. In the case of stochastic labor income volatility $\sigma_{y1} = 0.5$ (circles) and $\sigma_{y1} = 1$ (squares).

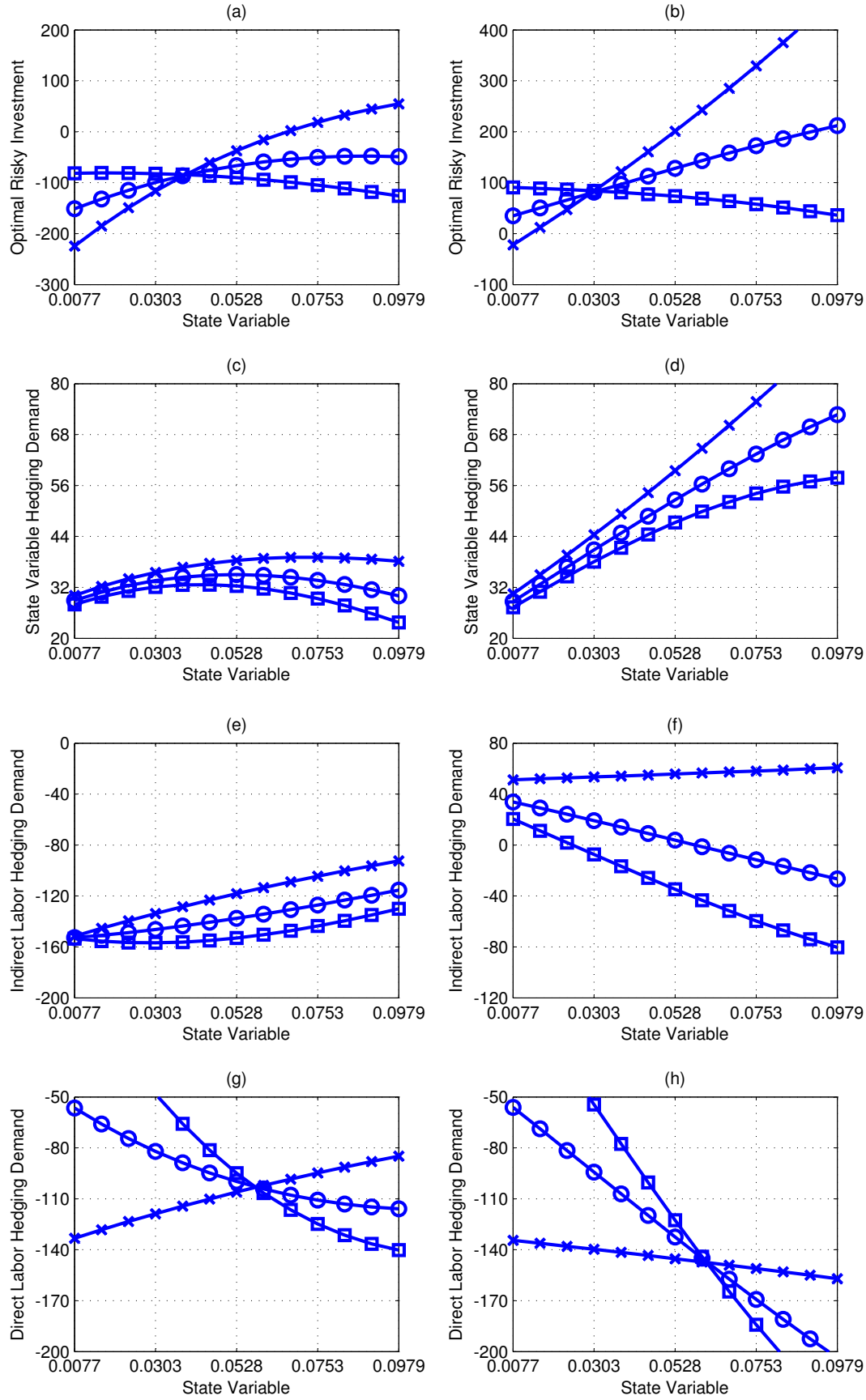


Figure 3.8: Optimal Risky Investment - Sensitivity

Figure 3.8 continued: Panels (a) and (b) show optimal total risky investment $A(t)\pi_t^*$ dependent on the state variable under stochastic labor income volatility. Panels (c) to (h) show the components of hedging demand as described in equation (3.2). Parameters are chosen as in Table 3.2. In the panels to the left (right) the blue lines show the results for an individual with a negative (positive) sensitivity of labor income growth on $X(t)$ of $y_1 = -0.5$ ($y_1 = 0.5$). The lines with crosses (circles, squares) display the results for an individual with constant (stochastic) labor income volatility. In the case of stochastic labor income volatility $\sigma_{y1} = 0.5$ (circles) and $\sigma_{y1} = 1$ (squares).

Finally, in Chapter 2 it was shown that for the cases with a positive sensitivity of labor income growth on the state variable ($y_1 > 0$), the optimal policies increase along X and seem unrealistically extreme. The inclusion of stochastic labor income volatility with high volatility in states of high income growth is an appropriate mean to achieve more realistic policies. In other words, a positive sensitivity of income growth can not be ruled out per se, but should be considered with a positive sensitivity of income volatility.

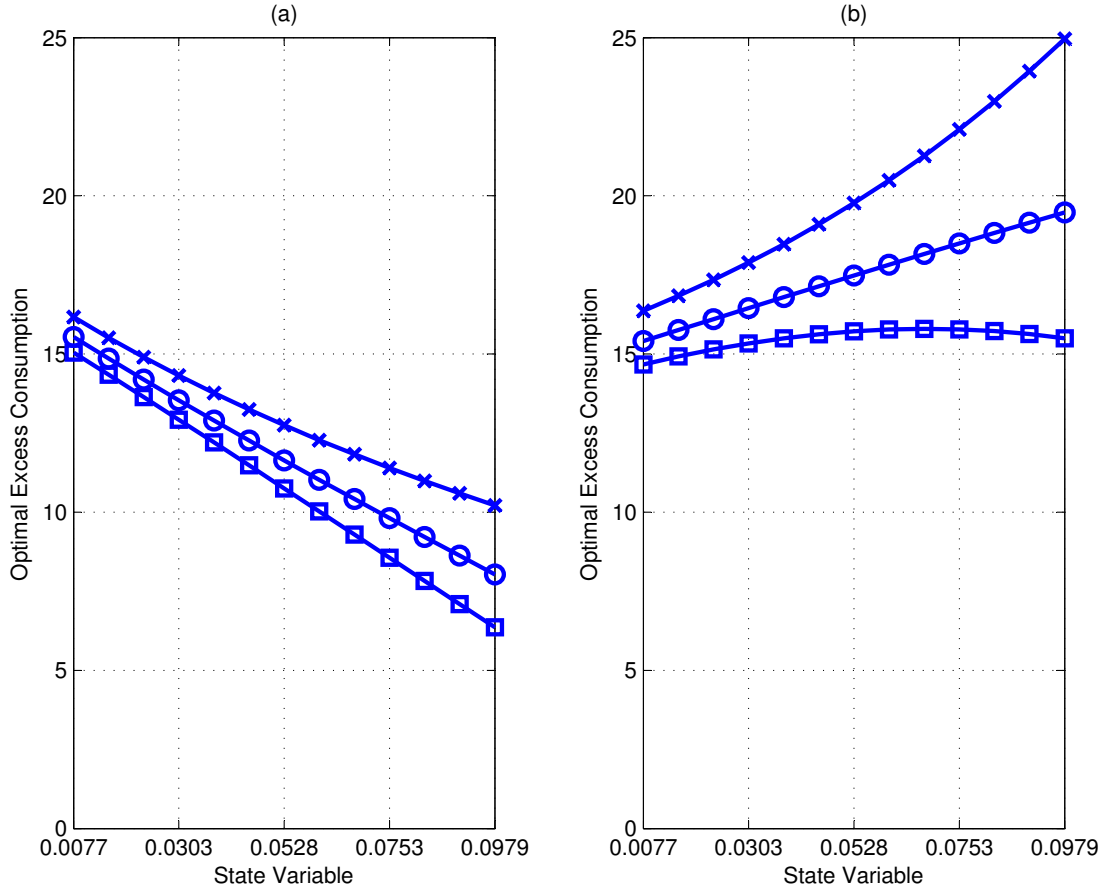


Figure 3.9: Optimal Consumption - Sensitivity

Panels (a) and (b) show optimal consumption exceeding the subsistence level $c_t^* - \bar{c}$ dependent on the state variable under risky labor income. Parameters are chosen as in Table 3.2. In the panel to the left (right) the blue lines show the results for an individual with a negative (positive) sensitivity of labor income growth on $X(t)$ of $y_1 = -0.5$ ($y_1 = 0.5$). The lines with crosses (circles, squares) display the results for an individual with constant (stochastic) labor income volatility. In the case of stochastic labor income volatility $\sigma_{y1} = 0.5$ (circles) and $\sigma_{y1} = 1$ (squares).

3.4 Conclusion

The inclusion of stochastic labor income volatility gives new insights. In addition to the results of the basic model presented in Chapter 2, the most important results are the following:

1. The inclusion of stochastic volatility in labor income always drives a wedge between the risky asset and labor income. Hence, the valuation of the labor income stream will always depend on state variable and indirect labor hedging demand is never zero.
2. If labor income and the risky asset are positively correlated, then the part of the hedging demand that is due to stochastic volatility converges to a stable solution even for an infinite horizon. Nevertheless, a highly persistent state variable can lead to $l_1 > 0$, which is in favor of extreme results.
3. The addition of stochastic labor income volatility allows for more interesting patterns of the two labor hedging demands. In fact, indirect and direct labor hedging demand can become non-monotone in the state variable.
4. Labor income volatility can generate risky investment that is rather insensitive to changes in the state variable. Furthermore, optimal policies which include both decreasing risky investment and consumption are possible even if expected returns of the risky asset are rising.

The results show that stochastic labor income volatility can have a considerable effect even if labor income volatility is low. Finally, it can be concluded that the major result of Lynch and Tan (2009) - stochastic labor income volatility leads to lower risky investment - could be qualitatively verified.

3.A Appendix

3.A.1 Valuation of the Labor Income Stream with the Martingale Approach

The risk-neutral valuation of the future labor income stream is given by⁹

$$G = E_0^Q \left[\int_0^\tau e^{-r_0 s} Y(s) ds \right]$$

Since it is assumed that there is only one shock that drives the economy, complete markets are implied. As stated in Pliska (1986) and He and Pearson (1991), the market price of risk is unique in complete markets. With a change in measure from the risk-neutral to the standard probability law, the value of the future income stream can be written as.

$$G = \frac{1}{\phi(0)} E_0 \left[\int_0^\tau \phi(s) Y(s) ds \right] \quad (3.10)$$

where

$$\frac{d\phi(t)}{\phi(t)} = -r_0 dt - \theta(t) dW_s(t) = -r_0 dt - \frac{\lambda_1}{\sigma_s} X(t) dW_s(t) \quad (3.11)$$

The process $\phi(t)$ can be interpreted as a system of Arrow-Debreu prices. Indeed, the value of $\phi(t)$ in each state gives the price per unit probability of a dollar in that state.

From (3.10) and (3.11) it can be recognized that the value of the labor income stream must be a function of X , Y and τ . Furthermore, from Cox et al. (1985) it is known that every asset must obey the following no-arbitrage condition

$$\begin{aligned} & Y + \frac{\partial G}{\partial t} - \frac{\partial G}{\partial X} \kappa_x (X - \bar{X}) + \frac{\partial G}{\partial Y} (y_0 + y_1 X) Y \\ & + \frac{1}{2} \left[\frac{\partial^2 G}{\partial X^2} \sigma_x^2 + \frac{\partial^2 G}{\partial Y^2} \sigma_y^2 Y^2 + 2 \frac{\partial^2 G}{\partial X \partial Y} \rho_{xy} \sigma_x \sigma_y \right] - r_0 G \\ & = \left[\frac{\partial G}{\partial X} \rho_{sx} \sigma_x + \frac{\partial G}{\partial Y} \rho_{sy} \sigma_y Y \right] \frac{\lambda_1}{\sigma_s} X \end{aligned} \quad (3.12)$$

where $\sigma_y \equiv \sigma_{y0} + \sigma_{y1} X(t)$. The first term Y is the instantaneous wage and is analogous to a dividend paying asset. The RHS shows the expected return every asset must deliver in order to comply with the premium of the tradable risky asset. Now, a function of the form

$$G(X, Y, \tau) = k(X, \tau) Y$$

implies the following partial derivatives

$$\begin{aligned} \frac{\partial G}{\partial t} &= -\frac{\partial G}{\partial \tau} = -\frac{\partial k}{\partial \tau} Y \\ \frac{\partial G}{\partial X} &= \frac{\partial k}{\partial X} Y, \quad \frac{\partial^2 G}{\partial X^2} = \frac{\partial^2 k}{\partial X^2} Y \\ \frac{\partial G}{\partial Y} &= k, \quad \frac{\partial^2 G}{\partial Y^2} = 0 \\ \frac{\partial^2 G}{\partial X \partial Y} &= \frac{\partial k}{\partial X} \end{aligned}$$

⁹For a textbook treatment of the martingale approach, the reader is referred to Korn (1997) Chapter 3.4. or Shreve (2004) Chapter 5.

Plugging in the partial derivatives into (3.12) and dividing by Y leads to

$$\begin{aligned} 0 = & 1 - \frac{\partial k}{\partial \tau} - \frac{\partial k}{\partial X} \kappa_x (X - \bar{X}) + k (y_0 + y_1 X) \\ & + \frac{1}{2} \frac{\partial^2 k}{\partial X^2} \sigma_x^2 + \frac{\partial k}{\partial X} \rho_{xy} \sigma_x (\sigma_{y0} + \sigma_{y1}) - r_0 k \\ & - \frac{\rho_{sx} \sigma_x}{\sigma_s} \lambda_1 X \frac{\partial k}{\partial X} - \frac{\rho_{sy} (\sigma_{y0} + \sigma_{y1} X)}{\sigma_s} \lambda_1 X k \end{aligned}$$

which is identical to the term in the brackets of (3.2).

3.A.2 A Special System of Ordinary Differential Equations - Solutions

The system of ordinary differential equation (SODE)

$$\frac{\partial d_2}{\partial s} = l_0 + l_1 d_2 + l_2 d_2^2 \quad (3.13)$$

$$\frac{\partial \hat{d}_1}{\partial s} = l_6 + l_3 d_2 + \frac{l_1}{2} \hat{d}_1 + l_2 d_2 \hat{d}_1 \quad (3.14)$$

$$\frac{\partial d_0}{\partial s} = l_7 + l_4 \hat{d}_1 + \frac{l_2}{2} \hat{d}_1^2 + l_5 d_2 \quad (3.15)$$

is related to the system discussed in Kim and Omberg (1996). In fact, (3.13) is identical to the first equation in the system of Kim and Omberg and hence their solutions apply as well. However, in the second equation there is an additional constant l_6 , which makes the solution of the system more complicated. Furthermore, there is an additional constant l_7 in the final equation. Initial conditions are $d_2(0) = \hat{d}_1(0) = d_0(0) = 0$.

In order to solve the second equation, the following form for the solution is attempted

$$\hat{d}_1(s) = d_1(s) + \tilde{d}_1(s)$$

Plugging in into (3.14) yields

$$\frac{\partial d_1(s)}{\partial s} + \frac{\partial \tilde{d}_1(s)}{\partial s} = l_6 + l_3 d_2 + \frac{l_1}{2} (d_1(s) + \tilde{d}_1(s)) + l_2 d_2 (d_1(s) + \tilde{d}_1(s))$$

This equation can be separated into

$$\frac{\partial d_1(s)}{\partial s} = l_3 d_2 + \frac{l_1}{2} d_1(s) + l_2 d_2 d_1(s) \quad (3.16)$$

$$\frac{\partial \tilde{d}_1(s)}{\partial s} = l_6 + \frac{l_1}{2} \tilde{d}_1(s) + l_2 d_2 \tilde{d}_1(s) \quad (3.17)$$

with $\tilde{d}_1(0) = d_1(0) = 0$. (3.16) is identical to the second equation in Kim and Omberg and thus the task is to find a solution to (3.17). Dealing with this equation demands solving a linear differential equation with time-varying coefficients and initial condition $\tilde{d}_1(0) = 0$.

The final equation (3.15) of the SODE can be solved by integration. Kim and Omberg show that closed-form solutions for $d_0(s)$ are available for their system, but are long and hard to interpret. Although closed-form solutions might be available even for this extended system, the

search for them is omitted. There are several good reason to refrain from this task¹⁰. Firstly, and most importantly, $d_0(s)$ is not needed in order to determine the sign of the corresponding hedging demand. Secondly, the closed-form solution will be extensive and its interpretation will be difficult. Finally, (3.15) can be solved easily numerically as long as $d_2(s)$ and $\hat{d}_1(s)$ are available in closed-form.

The first equation (3.13) is a Riccati differential equation and three cases can be distinguished¹¹. All derivations are given in Appendix 3.A.3.

Case I $q_l \equiv l_1^2 - 4l_0l_2 > 0$

In the first case $q_l > 0$, which means that (3.13) has two real particular solutions. In this case the solution for the SODE is given by¹².

$$\begin{aligned} d_2(s) &= \frac{2l_0(1 - e^{-\eta_l s})}{2\eta_l - (l_1 + \eta_l)(1 - e^{-\eta_l s})} \\ d_1(s) &= \frac{4l_0l_3(1 - e^{-\eta_l s/2})^2}{\eta_l [2\eta_l - (l_1 + \eta_l)(1 - e^{-\eta_l s})]} \\ \tilde{d}_1(s) &= \frac{2l_6}{2\eta_l - (l_1 + \eta_l)(1 - e^{-\eta_l s})} \left[2(1 - e^{-\eta_l s/2}) - \frac{l_1 + \eta_l}{\eta_l} (1 - e^{-\eta_l s/2})^2 \right] \\ d_0(s) &= \int_0^s l_7 + l_4 d_1(s) + \frac{l_2}{2} d_1(s)^2 + l_5 d_2(s) ds \end{aligned} \quad (3.18)$$

where $\eta_l \equiv \sqrt{q_l}$.

Since (3.13) is a quadratic equation and $q_l > 0$, two real solutions exist. $d_2^+(d_2^-)$ is the one associated with the positive (negative) root. The sign of the two solutions can be determined by the rule

$$d_2^+ d_2^- = \frac{l_0}{l_2}$$

Since $l_2 > 0$, the solutions have opposite signs if $l_0 < 0$, and equal sign otherwise. Figure 3.1 contains a phase plane analysis and makes clear that only one solution is a stable equilibrium. Moreover, because $d_2(0) = 0$, the solution converges to a finite number only if both particular solutions are positive or have opposite signs. As can be seen in Panel (c), if both particular solutions are negative, the solution will not converge and grows without bound. Kim and Omberg name this situation as “nirvana” solutions. Since $l_2 > 0$, this solution only occurs in cases where $l_0 > 0$ and $l_1 > 0$. More interestingly, the nirvana is obtained at a finite horizon, which is given by¹³

$$s_c = \frac{1}{\eta_l} \ln \left(\frac{l_1 + \eta_l}{l_1 - \eta_l} \right)$$

¹⁰We follow the example of Wachter (2002), who also resigned from this issue.

¹¹Kim and Omberg (1996) subdivide the second case into two cases called the “hyperbolic” and the “polynomial” solution. We omit the latter since in our model this case is not admissible. To be more precise, this case presumes $l_1 = l_2 = 0$ and as can be seen from the text $l_2 > 0$.

¹²The solution can be looked up in integral tables, see Gradshteyn and Ryzhik (2000, p. 78), or can be derived by general methods of solving Riccati differential equations as described in Zwillinger (1998, p. 392). Furthermore, this differential equation is well known from the term structure literature, see, for example, Ingersoll (1987, p. 397).

¹³ $d_2(s)$ approaches infinity when its denominator approaches zero.

Finally it should be noticed that

$$\lim_{s \rightarrow \infty} d_2(s) = \frac{2l_0}{\eta_l - l_1} = \frac{1}{\frac{-l_1 + \sqrt{l_1^2 - 4l_0l_2}}{2l_0} \frac{l_2}{l_2}} = \frac{1}{\frac{-l_1 + \sqrt{l_1^2 - 4l_0l_2}}{2l_2} \frac{l_2}{l_0}} = \frac{1}{d_2^+ \frac{l_2}{l_0}} = d_2^-$$

Hence, the stable solution is always the one associated with the negative root.

Case II $q_l \equiv l_1^2 - 4l_0l_2 = 0$

In the second case, the quadratic equation has only one solution, i.e. $q_l = 0$

$$\begin{aligned} d_2(s) &= -\frac{1}{l_2 \left(s - \frac{2}{l_1}\right)} - \frac{l_1}{2l_2} \\ d_1(s) &= -\frac{l_3}{l_1l_2 \left(s - \frac{2}{l_1}\right)} - \frac{l_1l_3 \left(s + \frac{2}{l_1}\right)}{4l_2} \\ \tilde{d}_1(s) &= 2l_6 \left[\frac{1}{4} \left(s + \frac{2}{l_1}\right) + \frac{1}{l_1^2 \left(s - \frac{2}{l_1}\right)} - \frac{1}{l_1 - \frac{2}{s}} \right] \\ d_0(s) &= \int_0^\tau l_7 + l_4d_1(s) + \frac{l_2}{2}d_1(s)^2 + l_5d_2(s) \, ds \end{aligned} \tag{3.19}$$

Panels (a) and (b) of Figure 3.2 display that the solution of $d_2(s)$ converges to a finite limit if $l_1 < 0$, and is a nirvana solution otherwise. In the nirvana case, the critical horizon is given by¹⁴

$$s_c = \frac{2}{l_1}$$

Case III $q_l \equiv l_1^2 - 4l_0l_2 < 0$

The third case covers $q_l < 0$, i.e. the quadratic equation does not have any real solutions. In this case, the solutions are given by

$$d_2(s) = \frac{\eta_l}{2l_2} \tan(\omega s + \varphi) - \frac{l_1}{2l_2}$$

where $\varpi \equiv \frac{\eta_l}{2}$ and $\varphi \equiv \arctan\left(\frac{l_1}{\eta_l}\right)$

$$\begin{aligned} d_1(s) &= -\frac{l_3}{l_2} [1 + \tan(\varphi) \tan(\omega s + \varphi) - \sec(\varphi) \sec(\omega s + \varphi)] \\ \tilde{d}_1(s) &= \sec(\omega s + \varphi) \cos(\varphi) \frac{2l_1l_6}{\eta_l^2} \left[-1 + \cos(\omega s + \varphi) + \frac{\eta_l}{l_1} \sin(\omega s + \varphi) \right] \\ d_0(s) &= \int_0^\tau l_7 + l_4d_1(s) + \frac{l_2}{2}d_1(s)^2 + l_5d_2(s) \, ds \end{aligned} \tag{3.20}$$

As can be seen from Panel (c) of Figure 3.2, there is no particular solution, the solution of $d_2(s)$ will not converge to a finite limit. Hence, this case results in nirvana solutions independent of l_1 and l_0 . The critical horizon is given by¹⁵

$$s_c = \frac{\pi}{\eta_l} - \frac{2}{\eta_l} \varphi$$

¹⁴As before, the result can be derived by setting the denominator of $c_2(s)$ to zero. Moreover, application of the rule of de l'Hospital yields that $\lim_{\eta_l \rightarrow 0^+} \frac{1}{\eta_l} \ln\left(\frac{l_1 + \eta_l}{l_1 - \eta_l}\right) = \frac{2}{l_1}$, indeed.

¹⁵For this case, it should be kept in mind that $\lim_{\psi \rightarrow \pi/2} \tan(\psi) = \infty$. Moreover, application of the rule of de l'Hospital to $s_c = \frac{2}{\eta_l} \left(\frac{\pi}{2} - \varphi\right)$ yields $\lim_{\eta_l \rightarrow 0^+} s_c = \frac{2}{l_1}$

where π is the mathematical constant and should not be confused with the risky investment policy.

3.A.3 A Special System of Ordinary Differential Equations - Derivations

The general formula for the solution of a linear differential equation with time-varying coefficients

$$\frac{\partial \tilde{d}_1(s)}{\partial s} + f_1(s) \tilde{d}_1(s) = f_0(s)$$

is given by¹⁶

$$\tilde{d}_1(s) = e^{-F} \int f_0(s) e^F ds + e^{-F} K$$

where

$$F \equiv \int f_1(s) ds$$

and K is the constant of integration. Starting from (3.17)

$$\frac{\partial \tilde{d}_1(s)}{\partial s} - \underbrace{\left(\frac{l_1}{2} + l_2 d_2(s) \right)}_{f_1(s)} \tilde{d}_1(s) = \underbrace{l_6}_{f_0(s)}$$

Obviously, only $f_1(s)$ is really a function of time through $d_2(s)$ and $f_0(s) = l_6$ is constant¹⁷. Hence, the main task is to find the primitive of $d_2(s)$.

Case I $q_l \equiv l_1^2 - 4l_0l_2 > 0$

In this case the primitive of $d_2(s)$ is given by¹⁸

$$\int d_2(s) ds = \frac{2l_0}{\eta_l - l_1} s + \frac{4l_0}{\eta_l^2 - l_1^2} \ln \left(\frac{2\eta_l - (l_1 + \eta_l)(1 - e^{-\eta_l s})}{2\eta_l} \right)$$

Hence,

$$\begin{aligned} F &= \int -\frac{l_1}{2} - l_2 d_2(s) ds \\ &= -\frac{l_1}{2} s - \frac{2l_0l_2}{\eta_l - l_1} s - \frac{4l_0l_2}{\eta_l^2 - l_1^2} \ln \left(\frac{2\eta_l - (l_1 + \eta_l)(1 - e^{-\eta_l s})}{2\eta_l} \right) \\ e^F &= e^{\eta_l s/2} \left(\frac{2\eta_l - (l_1 + \eta_l)(1 - e^{-\eta_l s})}{2\eta_l} \right) \end{aligned}$$

For the last step the definition of η_l should be kept in mind and thus $4l_0l_2 = l_1^2 - \eta_l^2$

$$\begin{aligned} \int f_0(s) e^F ds &= \int l_6 e^{\eta_l s/2} \left(\frac{2\eta_l - (l_1 + \eta_l)(1 - e^{-\eta_l s})}{2\eta_l} \right) ds \\ &= \frac{l_6}{\eta_l} \left(2e^{\eta_l s/2} - \frac{l_1 + \eta_l}{\eta_l} e^{\eta_l s/2} (1 - e^{-\eta_l s}) \right) \end{aligned}$$

¹⁶See, for example, Polyanin and Zaitsev (1995, p. 1).

¹⁷Since (3.16) contains time variation in $f_1(s)$ and $f_0(s)$, it could be expected that (3.17) is easier to solve and leads to simpler solutions. This is not the case, because in the solution of (3.16), the integral $\int f_0(s) e^F ds$ allows the cancelling out of some of the time-varying parts. However, this is not the case if $f_0(s)$ is simply the constant k_6 .

¹⁸All solutions are verified by the use of Mathematica (Version 7.0.1.0).

$$\tilde{d}_1(s) = \frac{2}{2\eta_l - (l_1 + \eta_l)(1 - e^{-\eta_l s})} \left[l_6 \left(2 - \frac{l_1 + \eta_l}{\eta_l} (1 - e^{-\eta_l s}) \right) + 2\eta_l e^{-\eta_l s/2} K \right] \quad (3.21)$$

The initial condition $\tilde{d}_1(0) = 0$ leads to

$$K = l_6 \frac{l_1 + \eta_l}{\eta_l^2}$$

Plugging in K into (3.21) leads to

$$\tilde{d}_1(s) = \frac{2l_6}{2\eta_l - (l_1 + \eta_l)(1 - e^{-\eta_l s})} \left[2 \left(1 - e^{-\eta_l s/2} \right) - \frac{l_1 + \eta_l}{\eta_l} \left(1 - e^{-\eta_l s/2} \right)^2 \right] \quad (3.22)$$

Case II $q_l \equiv l_1^2 - 4l_0l_2 = 0$

In this case the primitive of $d_2(s)$ is given by

$$\int d_2(s) ds = -\frac{l_1}{2l_2}s + \frac{1}{l_2} \ln \left(\frac{|l_1(s - \frac{2}{l_1})|}{2} \right)$$

Thus,

$$F = \int -\frac{l_1}{2} - l_2 d_2(s) ds$$

$$e^F = \frac{|l_1(s - \frac{2}{l_1})|}{2}$$

Furthermore,

$$\begin{aligned} \int f_0(s) e^F ds &= \int l_6 \frac{|l_1(s - \frac{2}{l_1})|}{2} ds \\ &= \frac{1}{4} l_6 l_1 s^2 - l_6 s + K \\ \tilde{d}_1(s) &= \frac{1}{\frac{1}{2} l_1 (s - \frac{2}{l_1})} \left[\frac{1}{4} l_6 l_1 s^2 - l_6 s + K \right] \end{aligned} \quad (3.23)$$

The initial condition $\tilde{d}_1(0) = 0$ leads to

$$K = 0$$

Plugging in K into (3.23) leads to

$$\begin{aligned} \tilde{d}_1(s) &= \frac{1}{\frac{1}{2} l_1 (s - \frac{2}{l_1})} \left[\frac{1}{4} l_6 l_1 s^2 - l_6 s \right] \\ &\quad \frac{l_6}{\frac{1}{2} l_1 (s - \frac{2}{l_1})} \left[\frac{1}{4} l_1 s^2 - \frac{4}{l_1^2} + \frac{4}{l_1^2} - s \right] \\ &= 2l_6 \left[\frac{1}{4} \left(s + \frac{2}{l_1} \right) + \frac{1}{l_1^2 (s - \frac{2}{l_1})} - \frac{1}{l_1 - \frac{2}{s}} \right] \end{aligned} \quad (3.24)$$

Case III $q_l \equiv l_1^2 - 4l_0l_2 < 0$

In this case the primitive of $d_2(s)$ is given by

$$\int d_2(s) ds = -\frac{l_1}{2l_2}s - \frac{1}{l_2} \ln [\cos(\omega s + \varphi)]$$

where ω and φ are defined as in the main text. Thus,

$$F = \int -\frac{l_1}{2} - l_2 d_2(s) ds$$

$$e^F = \cos(\omega s + \varphi)$$

Moreover,

$$\begin{aligned} \int f_0(s) e^F ds &= \int l_3 \cos(\omega s + \varphi) ds \\ &= \frac{2l_3}{\eta_l^2} \cos(\varphi) [l_1 \cos(\omega s + \varphi) + \eta_l \sin(\omega s + \varphi)] + K \\ \tilde{d}_1(s) &= \sec(\omega s + \varphi) \left[\frac{2l_3}{\eta_l^2} \cos(\varphi) [l_1 \cos(\omega s + \varphi) + \eta_l \sin(\omega s + \varphi)] + K \right] \end{aligned} \quad (3.25)$$

The initial condition $\tilde{d}_1(0) = 0$ leads to

$$K = -\frac{2l_1l_3}{\eta_l^2} \cos(\varphi)$$

Plugging in K into (3.25) leads to

$$\tilde{d}_1(s) = \sec(\omega s + \varphi) \cos(\varphi) \frac{2l_1l_3}{\eta_l^2} \left[-1 + \cos(\omega s + \varphi) + \frac{\eta_l}{l_1} \sin(\omega s + \varphi) \right] \quad (3.26)$$

Chapter 4

Portfolio and Consumption Decisions with Labor Income and a Volatility Premium

In this chapter we present a model that is related to the preceding chapters. However, there are crucial differences. In particular, it is assumed that the risky asset and labor income are exposed to stochastic volatility. Furthermore, the market price of risk is assumed to rise with the square root of volatility. Without labor income this setting is a special case of Liu (2007) and a closely related model was analyzed by Chacko and Viceira (2005). The most striking result of these papers is that stochastic volatility leads to hedging demand that is small in magnitude. This stands in contrast to models that contain market premiums, which depend linearly on a state variable as presented by Kim and Omberg (1996) and Wachter (2002). However, we show that the presence of labor income can have a strong impact on the results and considerable hedging demand can arise.

Moreover, in a second step we extend the basic model by integrating it into a simple life-cycle model. This model includes a phase of retirement without any non-financial income. As a consequence the individual has to adapt her saving behavior in order to afford the phase of retirement.

As in the models of Chapters 2 and 3 it must be assumed that the state variable and the risky asset are perfectly correlated. While this was not too problematic for the dividend yield, the assumption in this model is a drawback. Nevertheless, in several studies it is shown that the results of the special cases are close to numerically solved general cases¹. Hence, we expect that the main aspects of theoretical insights of this chapter are valid even for less restrictive models. Finally, as before one has to assume that labor income is either locally riskfree or perfectly correlated with the risky asset (i.e. markets are complete).

From a technical point of view, the solution can be derived by the same methods as in Chap-

¹See Cocco et al. (2005), Huang and Milevsky (2008), Huang et al. (2008), Bick et al. (2009) and Dybvig and Liu (2010).

ter 2. In particular, the HJB can be separated into systems of ordinary differential equations (SODE). Nevertheless, the SODE are different and other solutions apply. As a consequence, the separation of the HJB is not described in detail and the reader is referred to the appendix of Chapter 2. The solutions of the new SODE are shown in depth. Moreover, this chapter is written so that the implications can be understood without reading Chapter 2 first.

The remainder of this chapter is as follows. In Section 4.1, the basic model with preferences over intermediate consumption is introduced. In Section 4.2, the model is adapted to preferences over terminal wealth. Section 4.3 discusses the long-horizon stability of the solution. Section 4.4 contains the results of the basic model for numerically realistic parameter values. In Section 4.5, the basic model with preferences over intermediate consumption is integrated in a simple life-cycle model. The final Section concludes. Mathematical derivations are given in Appendices 4.A.1 - 4.A.3.

4.1 Model with Utility over Consumption

For the sake of simplicity, it is assumed that in the basic model the individual works during the entire planning horizon $[t, T]$. Thus, the conditional expected utility over the remaining horizon for an individual at t is

$$E_t \left[\int_t^T \frac{e^{-\delta s}}{1-\gamma} (c(s) - \bar{c})^{1-\gamma} ds \right], \quad \gamma > 1$$

where $\bar{c} > 0$ is the subsistence level of consumption, $\delta \geq 0$ is the time discount parameter and $\tau \equiv T - t$ is the fixed and certain time horizon.

It is assumed that the risky asset and labor income are exposed to stochastic volatility and receive a premium proportional to this exposure². In particular,

$$\frac{dS_1(t)}{S_1(t)} = (r_0 + \lambda_1 X(t)) dt + \sigma_s \sqrt{X(t)} dW_s(t)$$

where $\lambda_1 > 0$ and $\sigma_s > 0$. The riskless rate is assumed to be constant and the dynamics of a riskless bond are given by³

$$\frac{dS_0(t)}{S_0(t)} = r_0 dt$$

It should be noted that in this framework, the market price of risk is not linear in $X(t)$, but given by

$$\theta(t) \equiv \frac{\lambda_1}{\sigma_s} \sqrt{X(t)} \quad (4.1)$$

In particular, the market price of risk grows only in the square root of the state variable.

The only state variable is stochastic volatility that follows⁴.

$$dX(t) = -\kappa_x (X(t) - \bar{X}) dt + \sigma_x \sqrt{X(t)} dW_x(t) \quad (4.2)$$

²A similar model for asset prices was already proposed by Merton (1980).

³A specification of the short rate of the form $dS_0(t)/S_0(t) = (r_0 + r_1 X(t)) dt$ could be chosen without severe problems. Properties of affine term structure models are well-known, see, for example, Dai and Singleton (2000). For the sake of simplicity and in order to show the effects of non-financial income clearly, this is omitted.

⁴This process is well-known from the term structure literature, for example Cox et al. (1985) and from the option pricing literature, for example Heston (1993).

where $\kappa_x > 0$, $\bar{X} > 0$ and $\sigma_x > 0$. As stated above, the specification of stochastic volatility and a volatility premium in the financial market is a special case of Liu (2007)⁵.

Furthermore, it is assumed that the wage consists of two parts. In particular,

$$\hat{Y}(t) = \bar{Y} + Y(t)$$

where $\bar{Y} \geq 0$ is a constant and thus without risk. $Y(t)$ is risky and follows

$$\frac{dY(t)}{Y(t)} = [y_0 + y_1 X(t)] dt + \sigma_y \sqrt{X(t)} dW_y(t) \quad (4.3)$$

where $\sigma_y \geq 0$. Since $Y(t)$ cannot become negative, this part can be interpreted as a minimum wage that is guaranteed by a third party.

This leads to the following dynamics of the investors financial wealth

$$\begin{aligned} dA(t) = & \left[(\pi(t) \lambda_1 X(t) + r_0) A(t) + \hat{Y}(t) - c(t) \right] dt \\ & + \pi(t) A(t) \sigma_s \sqrt{X(t)} dW_s(t) \end{aligned} \quad (4.4)$$

The HJB is given by

$$\begin{aligned} 0 = & J_t + \sup_c \left[e^{-\delta t} \frac{(c_t - \bar{c})^{1-\gamma}}{1-\gamma} - J_A c_t \right] \\ & + \sup_\pi \left[J_A \pi(t) A(t) \lambda_1 X(t) + \frac{1}{2} J_{AA} \pi(t)^2 A(t)^2 \sigma_s^2 X(t) \right. \\ & \left. + J_{AX} \pi(t) A(t) \rho_{sx} \sigma_s \sigma_x X(t) + J_{AY} \pi(t) A(t) Y(t) \rho_{sy} \sigma_s \sigma_y X(t) \right] \\ & + J_A [A(t) r_0 + Y(t) + \bar{Y}] - J_X \kappa_x (X(t) - \bar{X}) + J_Y Y(t) [y_0 + y_1 X(t)] \\ & + \frac{1}{2} J_{XX} \sigma_x^2 X(t) + \frac{1}{2} J_{YY} Y(t)^2 \sigma_y^2 X(t) + J_{XY} Y(t) \rho_{xy} \sigma_x \sigma_y X(t) \end{aligned} \quad (4.5)$$

Hence, the following first order conditions (FOCs) result

$$c_t^* = \left(e^{\delta t} J_A \right)^{-\frac{1}{\gamma}} + \bar{c} \quad (4.6)$$

and

$$\pi_t^* = -\frac{J_A}{J_{AA} A(t) \sigma_s^2} \lambda_1 - \frac{J_{AX}}{J_{AA} A(t)} \frac{\rho_{sx} \sigma_x}{\sigma_s} - \frac{J_{AY}}{J_{AA} A(t)} \frac{\rho_{sy} \sigma_y}{\sigma_s} Y(t) \quad (4.7)$$

Plugging in the FOCs (4.6) and (4.7) into the HJB equation (4.5) yields

$$\begin{aligned} 0 = & J_t + \frac{\gamma}{1-\gamma} e^{-\frac{\delta}{\gamma} t} J_A^{1-\frac{1}{\gamma}} - J_A \bar{c} + J_A A(t) r_0 + J_A Y(t) + J_A \bar{Y} \\ & - J_X \kappa_x (X(t) - \bar{X}) + J_Y Y(t) [y_0 + y_1 X(t)] \\ & + \frac{1}{2} J_{AA} A(t) \pi_t^* \lambda_1 X(t) + \frac{1}{2} J_{AX} A(t) \pi_t^* \rho_{sx} \sigma_x \sigma_s X(t) \\ & + \frac{1}{2} J_{AY} A(t) Y(t) \pi_t^* \rho_{sy} \sigma_s \sigma_y X(t) \\ & + J_{XY} Y(t) \rho_{xy} \sigma_x \sigma_y X(t) + \frac{1}{2} J_{YY} Y(t)^2 \sigma_y^2 X(t) + \frac{1}{2} J_{XX} \sigma_x^2 X(t) \end{aligned}$$

⁵Chacko and Viceira (2005) use a slightly different specification of the volatility process. Liu presents this case as an example for his general model. See Liu (2007, p. 28-31).

One guesses a value function of the following form

$$J = \frac{e^{-\delta(T-\tau)} \left[\int_0^\tau e^{\frac{1}{\gamma}(c_0(s)+c_1(s)X)} ds \right]^\gamma (A + k(X, \tau)Y - R(\tau))^{1-\gamma}}{1 - \gamma} \quad (4.8)$$

where $\tau \equiv T - t$, $k(X, \tau)$ is a function of the state variable and the time horizon, and $R(\tau)$ is a function of the time horizon. Both functions will be parameterized below⁶. The solution of the HJB is shown in Appendix 4.A.1.

As the focus is on closed-form solutions, some assumptions have to be implemented. In fact, similarly to Chapter 2 it must be assumed that

$$\rho_{sx} \in \{-1, 1\} \quad (4.c.1)$$

and either

$$\rho_{sy} \in \{-1, 1\} \Rightarrow \rho_{xy} = \rho_{sx}\rho_{sy} \in \{-1, 1\} \quad (4.c.2)$$

or

$$\sigma_y = 0 \quad (4.c.3)$$

Admittedly, these assumptions do not match reality one-to-one. Nevertheless, several papers have shown that the results of the exactly solvable special cases are qualitatively similar to the cases with non-perfect correlation⁷. Hence, we expect that the qualitative results hold for more general cases. Furthermore, these assumptions come with an advantage besides the interpretability of the closed-form solutions. In fact, in the case of $\rho_{sy} \notin \{-1, 1\}$ and $\sigma_y > 0$, current financial wealth must be higher than the reserves for the future subsistence consumption⁸. This would be an unrealistic assumption, especially for young individuals who generally have a low financial wealth.

In analogy to Chapter 2, equation (4.31) from Appendix 4.A.1 can be separated into ordinary differential equations.

4.1.1 Separating the HJB by A

Given the ordinary differential equation (4.31) from Appendix 4.A.1 and separating by A gives the following equation

$$0 = \int_0^\tau e^{C(X,s)} \left\{ \begin{aligned} & -\frac{\delta}{1-\gamma} - \frac{1}{1-\gamma} \left(\frac{\partial c_0(s)}{\partial s} + \frac{\partial c_1(s)}{\partial s} X \right) + r_0 \\ & -\frac{1}{1-\gamma} \kappa_x c_1(s) X + \frac{1}{1-\gamma} \kappa_x \bar{X} c_1(s) \\ & + \frac{1}{2} \frac{1}{\gamma} \frac{\lambda_1^2}{\sigma_s^2} X + \frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 c_1(s) X + \frac{1}{2} \frac{1}{1-\gamma} \frac{1}{\gamma} \sigma_x^2 c_1^2(s) X \end{aligned} \right\} ds \quad (4.9)$$

⁶For the sake of brevity, the time subscripts for financial wealth, non-financial income and the state variable are omitted.

⁷See Cocco et al. (2005), Huang and Milevsky (2008), Huang et al. (2008), Bick et al. (2009) and Dybvig and Liu (2010).

⁸In Koo (1998) and Munk (2000) it is shown for an individual with power utility over consumption that under non-perfect correlation between the financial asset and labor income, total wealth and risky investment go to zero as the financial wealth approaches zero. This is intuitive as otherwise the individual risks to ending up with negative wealth and no income and hence, cannot afford a positive consumption level, which is clearly not optimal.

Matching coefficients on X and constants, one gets a system of two ordinary differential equations:

$$\begin{aligned}\frac{\partial c_1(s)}{\partial s} &= k_0 + k_1 c_1(s) + k_2 c_1(s)^2 \\ \frac{\partial c_0(s)}{\partial s} &= k_3 + k_4 c_1(s)\end{aligned}\tag{4.10}$$

where

$$\begin{aligned}k_0 &\equiv \frac{1}{2} \frac{1-\gamma}{\gamma} \frac{\lambda_1^2}{\sigma_s^2}, & k_1 &\equiv -\kappa_x + \frac{1-\gamma}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1, & k_2 &\equiv \frac{1}{2} \frac{1}{\gamma} \sigma_x^2 \\ k_3 &\equiv (1-\gamma) r_0 - \delta, & k_4 &\equiv \kappa_x \bar{X}\end{aligned}$$

and initial conditions $c_1(0) = c_0(0) = 0$. The solutions are given by

$$c_1(s) = \frac{2k_0(1 - e^{-\eta s})}{2\eta - (k_1 + \eta)(1 - e^{-\eta s})}\tag{4.11}$$

$$c_0(s) = k_3 s + \frac{2k_0 k_4}{\eta - k_1} s + \frac{4k_0 k_4}{\eta^2 - k_1^2} \ln \left(\frac{2\eta - (k_1 + \eta)(1 - e^{-\eta s})}{2\eta} \right)\tag{4.12}$$

where $q \equiv k_1^2 - 4k_0 k_2$ and $\eta \equiv \sqrt{|q|}$.

Remarks

- This is the solution from Liu (2007). It is easy to show that if $\gamma > 1 \Rightarrow k_0 < 0 \Rightarrow q > 0$ and thus, (4.10) has two real particular solutions. If $q \leq 0$, different solutions would apply⁹.
- Since $k_2 > 0 \Rightarrow k_1 + \eta < 2\eta$, $c_1(s)$ has the same sign as¹⁰ k_0 . Thus, the unambiguous negativity of k_0 leads to $c_1(s) < 0$ for $s > 0$.
- Given $\gamma > 1$, $c_1(s)$ converges to a finite number as $s \rightarrow \infty$. In other words, the solution of the Riccati differential equation is well-defined.

4.1.2 Separating the HJB by Y

Separating (4.31) by the Y parts yields

$$\begin{aligned}0 &= \int_0^\tau e^{C(X,s)} ds \left\{ \begin{aligned} &-\frac{\partial k}{\partial \tau} + 1 - \kappa_x X \frac{\partial k}{\partial X} + \kappa_x \bar{X} \frac{\partial k}{\partial \bar{X}} + k[y_0 + y_1 X(t)] \\ &-\frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 \frac{\partial k}{\partial X} X - \frac{\rho_{sy}\sigma_y}{\sigma_s} \lambda_1 k X \\ &+ \rho_{xy} \sigma_x \sigma_y \frac{\partial k}{\partial X} X + \frac{1}{2} \sigma_x^2 \frac{\partial^2 k}{\partial X^2} X - r_0 k \end{aligned} \right\} \\ &+ \int_0^\tau e^{C(X,s)} c_1(s) ds \left\{ \begin{aligned} &\sigma_x^2 \left[-\frac{1}{2} \rho_{sx}^2 - \frac{1}{2} \rho_{sx}^2 + 1 \right] \frac{\partial k}{\partial X} X \\ &\sigma_x \sigma_y \left[-\frac{1}{2} \rho_{sx} \rho_{sy} - \frac{1}{2} \rho_{sx} \rho_{sy} + \rho_{xy} \right] k X \end{aligned} \right\}\end{aligned}$$

Terms similar to A are directly set to zero by the means of (4.9).

⁹See next page or Kim and Omberg (1996).

¹⁰This can also be shown in a more demonstrative way by a phase plane analysis as presented in the next section.

With the assumptions about the correlations, the equation simplifies to

$$0 = \int_0^\tau e^{C(X,s)} ds \left\{ \begin{array}{l} -\frac{\partial k}{\partial \tau} + 1 - r_0 k - \kappa_x X \frac{\partial k}{\partial X} + \kappa_x \bar{X} \frac{\partial k}{\partial \bar{X}} \\ + k [y_0 + y_1 X(t)] - \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 \frac{\partial k}{\partial X} X - \frac{\rho_{sy}\sigma_y}{\sigma_s} \lambda_1 k X \\ + \rho_{xy}\sigma_x\sigma_y \frac{\partial k}{\partial X} X + \frac{1}{2}\sigma_x^2 \frac{\partial^2 k}{\partial X^2} X \end{array} \right\} \quad (4.13)$$

As $\int_0^\tau e^{C(X,s)} ds > 0$, (4.13) is zero if the part in the brackets is zero. A function of the form

$$k(X, \tau) = \int_0^\tau e^{d_0(s) + d_1(s)X} ds \quad (4.14)$$

with initial conditions¹¹ $d_1(0) = d_0(0) = 0$ will solve the equation (4.13). The relevant partial derivatives are as follows

$$\begin{aligned} k_\tau &= \int_0^\tau \left(\frac{\partial d_0(s)}{\partial s} + \frac{\partial d_1(s)}{\partial s} X \right) e^{d_0(s) + d_1(s)X} ds + 1 \\ k_X &= \int_0^\tau d_1(s) e^{d_0(s) + d_1(s)X} ds \\ k_{XX} &= \int_0^\tau d_1(s)^2 e^{d_0(s) + d_1(s)X} ds \end{aligned}$$

Plugging in the partial derivatives into (4.13) leads to

$$0 = \int_0^\tau e^{d_0(s) + d_1(s)X} \left\{ \begin{array}{l} -\left(\frac{\partial d_0(s)}{\partial s} + \frac{\partial d_1(s)}{\partial s} X \right) - r_0 - \kappa_x d_1(s) X + \kappa_x \bar{X} d_1(s) \\ + [y_0 + y_1 X(t)] - \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 d_1(s) X - \frac{\rho_{sy}\sigma_y}{\sigma_s} \lambda_1 X \\ + \rho_{xy}\sigma_x\sigma_y d_1(s) X + \frac{1}{2}\sigma_x^2 d_1(s)^2 X \end{array} \right\} ds$$

Matching coefficients on X and the constant term leads to a system of two ordinary differential equations.

$$\frac{\partial d_1(s)}{\partial s} = l_0 + l_1 d_1(s) + l_2 d_1(s)^2 \quad (4.15)$$

$$\frac{\partial d_0(s)}{\partial s} = l_3 + l_4 d_1(s) \quad (4.16)$$

where

$$\begin{aligned} l_0 &\equiv y_1 - \frac{\rho_{sy}\sigma_y}{\sigma_s} \lambda_1, & l_1 &\equiv -\kappa_x - \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 + \rho_{xy}\sigma_x\sigma_y, & l_2 &\equiv \frac{1}{2}\sigma_x^2 \\ l_3 &\equiv y_0 - r_0, & l_4 &\equiv \kappa_x \bar{X} \end{aligned}$$

The first equation (4.15) is a Riccati differential equation with constant coefficients, the second one (4.16) can be solved by integration.

¹¹These are the only initial conditions that solve equation (4.13) due to the second part of k_τ . Moreover, it can be shown that only these initial conditions ensure that the solutions converge to the solutions of the constant opportunity set model.

The general solutions with initial conditions $d_1(0) = d_0(0) = 0$ are given by

$$d_1(s) = \begin{cases} \frac{2l_0(1-e^{-\eta_l s})}{2\eta_l - (l_1 + \eta_l)(1-e^{-\eta_l s})}, & , q_l > 0 \\ -\frac{1}{l_2(s - \frac{2}{l_1})} - \frac{l_1}{2l_2}, & , q_l = 0 \\ \frac{\eta_l}{2l_2} \tan(\omega s + \varphi) - \frac{l_1}{2l_2}, & , q_l < 0 \end{cases} \quad (4.17)$$

$$d_0(s) = \begin{cases} l_3 s + \frac{2l_0 l_4}{\eta_l - l_1} s + \frac{4l_0 l_4}{\eta_l^2 - l_1^2} \ln\left(\frac{2\eta_l - (l_1 + \eta_l)(1-e^{-\eta_l s})}{2\eta_l}\right) & , q_l > 0 \\ l_3 s - \frac{l_1 l_4}{2l_2} s - \frac{l_4}{l_2} \ln\left(\frac{|l_1(s - \frac{2}{l_1})|}{2}\right) & , q_l = 0 \\ l_3 s - \frac{l_1 l_4}{2l_2} s + \frac{l_4}{l_2} [\cos(\omega s + \varphi) - \cos(\varphi)] & , q_l < 0 \end{cases} \quad (4.18)$$

where $\omega \equiv \frac{\eta_l}{2}$, $\varphi \equiv \arctan\left(\frac{l_1}{\eta_l}\right)$, $q_l \equiv l_1^2 - 4l_0 l_2$ and $\eta_l \equiv \sqrt{|q_l|}$. The derivations are given in Appendix 4.A.3.

Remarks

- From (4.14) it can be easily seen that $k(X, \tau) > 0$ for $\tau > 0$. This is intuitive as the risky part of labor income Y cannot become negative and hence the individual attaches a positive value to the future labor income stream.
- The system is identical to the system above in terms of structure. Again, since $l_2 > 0$, if $l_0 < 0 \Rightarrow q_l > 0$. While in the previous context this inequality is true for all coefficients of risk aversion γ greater than one (hence, the normal case is always valid), a similar statement is not possible. The sign of l_0 depends on the valuation of the income stream, which is similar to the CAPM. To be more specific, $l_0 = y_1 - \beta_{sy}\lambda_1$, where $\beta_{sy} \equiv \frac{\rho_{sy}\sigma_y}{\sigma_s}$. If $l_0 < 0$, labor income can be considered as an unfavorable asset since it does not deliver the premium of the hedging financial position. In this case, it will be shown that $d_1(s)$ becomes negative and is stable. In the case of labor income it is a favorable asset, $d_1(s)$ becomes positive and for stability an extensive discussion is needed. If labor income has a fair premium, $l_0 = 0 \Rightarrow d_1(s) = 0$, $\forall s$.
- In the case of locally risk free labor income, $\sigma_y = 0$, labor income is unambiguously a favorable (unfavorable) asset if $y_1 > 0$ ($y_1 < 0$).
- Risk aversion is not involved in the valuation of the income stream. This is intuitive as the assumption of complete markets allows for a perfect hedge of labor income risk.

4.1.3 Separating the HJB by the Constant Terms

Finally, separating (4.31) by the constant parts,

$$0 = \int_0^\tau e^{C(X,s)} ds \left\{ \frac{\partial R}{\partial \tau} + \bar{Y} - \bar{c} + r_0 R \right\} \quad (4.19)$$

Again, terms that are similar for A and R are directly neglected because they are equal to zero by the means of (4.9). By the same arguments as above, the equation is zero if the term in the brackets is zero. The equation in the brackets is a linear differential equation with constant coefficients and initial condition $R(\tau = 0) = 0$

$$R(\tau) = \frac{\bar{c} - \bar{Y}}{r_0} (1 - e^{-r_0 \tau}) \quad (4.20)$$

Since $\frac{\bar{c} - \bar{Y}}{r_0}$ is the value of perpetual bond that pays $\bar{c} - \bar{Y}$ as its coupon, it becomes clear that (4.20) can be interpreted as the reserves necessary to cover the subsistence level of consumption net of the minimum wage that is guaranteed.

Remarks

- From the derivation in Appendix 4.A.1 and (4.20) it can be noticed that the constant part of labor income enters in the same way as the (constant) subsistence level of consumption. For this reason, it can be stated that only the difference $\bar{c} - \bar{Y}$ really matters for the value of total wealth (and risky investment). Nevertheless, optimal consumption is directly affected by \bar{c} and hence, individuals with the same $\bar{c} - \bar{Y}$ but different \bar{c} hold an identical portfolio, have same excess consumption $c_t^* - \bar{c}$, but have different consumption levels.
- Given the solution of k and R , total wealth can be structured in a more interpretable form $\hat{A} \equiv A + H - N$. A is still the financial assets of the individual, $H \equiv kY + \frac{\bar{Y}}{r_0} (1 - e^{-r_0 \tau})$ is the human capital and $N \equiv \frac{\bar{c}}{r_0} (1 - e^{-r_0 \tau})$ are the reserves covering the subsistence level of consumption.

4.1.4 Optimal Policies

Plugging in the relevant partial derivatives from Appendix 4.A.1 into the FOCs (4.6) and (4.7) leads to

$$c_t^* = \frac{\hat{A}}{\int_0^\tau e^{C(X,s)} ds} + \bar{c} \quad (4.21)$$

and

$$\begin{aligned} A\pi_t^* = & \frac{1}{\gamma} \frac{\lambda_1}{\sigma_s^2} \hat{A} + \frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \frac{\int_0^\tau c_1(s) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} \hat{A} \\ & - \frac{\partial k}{\partial X} Y \frac{\rho_{sx}\sigma_x}{\sigma_s} - kY \frac{\rho_{sy}\sigma_y}{\sigma_s} \end{aligned} \quad (4.22)$$

where total wealth \hat{A} is defined above and $C(X, s) \equiv \frac{1}{\gamma} (c_0(s) + c_1(s)X)$.

Remarks

- Optimal consumption (4.21) consists of two parts. Only the first part varies over time; the subsistence part is constant. As a consequence, consumption varies less strongly than total wealth. Indeed, in the classical Merton (1969) model, consumption has the same variation

as financial wealth, which is implausible¹². In this framework with power utility ($\bar{c} = 0$) the relation is not one-to-one, but since the variation in $\int_0^\tau e^{C(X,s)} ds$ is low the relation is close. Adding a subsistence level of consumption eases this issue.

- For individuals close to the margin of subsistence ($\hat{A} \rightarrow 0$), optimal consumption converges to \bar{c} and its variation disappears.
- In contrast to models where the market price of risk of the risky asset is linear in X (Chapter 2), the optimal risky exposure (4.22) varies less strongly with X . For example, comparing the first component of the RHS of (4.22) with that of (2.24) shows that in this model, myopic demand varies only with total wealth, while in the other model, the component varies directly with X . As stated by Liu (2007), the additional risk premium at higher volatility levels and the higher risk offset each other¹³. A similar statement is true for state variable hedging demand.
- For individuals close to the margin of subsistence ($\hat{A} \rightarrow 0$), the first two parts of (4.22) vanish.
- The third term of optimal risky investment (2.24) is an additional state variable hedging demand that arises under labor income. It does not vanish for individuals close to the margin of subsistence. Furthermore, it even exists if labor income is locally riskfree ($\sigma_y = 0$) or the correlation between labor income and the risky asset is zero ($\rho_{sy} = 0$). Of course, it is necessary that the risky asset and the state variable are correlated ($\rho_{sx} \neq 0$). It is shown below that this part is negative for individuals with unfavorable income characteristics, which helps to explain the low equity exposure of low-educated and poor individuals.
- Furthermore, the third term of optimal risky investment (4.22) has a natural interpretation. In fact, partitioning the third term into

$$\underbrace{-\rho_{sx}}_{i)} \underbrace{\frac{\sigma_x}{\sigma_s}}_{ii)} \underbrace{\frac{\partial k}{\partial X}}_{iii)} Y$$

allows the following interpretation. Most importantly, *iii)* is the first derivative of the value per unit of labor income on X . In other words, this part gives the change in the value of one unit of labor income when the state variable moves. *ii)* is a multiplier that relates the strength of the shocks of the risky asset and the state variable. *i)* is simply plus or minus one and gives the direction the state variable moves in relation to the risky asset. Thus, it can be summarized that this third term is a hedge for the value of the future income stream to changes in the state of the economy.

¹²Cochrane (2007, p. 76).

¹³For more details with respect to the first two parts of the optimal policy, the reader is referred to Liu (2007). Although the focus of Liu is on utility over terminal wealth and incomplete markets, the statements remain valid for this framework.

- The last term is the hedging demand for labor income risk. This part does not vanish for individuals close to the margin of subsistence. For positive (negative) ρ_{sy} it will decrease (increase) the amount invested in the risky asset. Moreover, it will vanish if labor income returns are uncorrelated with the risky asset or labor income is locally riskfree ($\sigma_y = 0$).

4.1.5 Dynamics of Optimal Total Wealth

In Appendix 4.A.2 it is shown that optimal total wealth follows

$$\begin{aligned} \frac{d\hat{A}^*}{\hat{A}^*} = & \left(r_0 + \frac{1}{\gamma} \frac{\lambda_1^2}{\sigma_s^2} X + \frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 X \frac{\int_0^\tau c_1(s) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} - \frac{1}{\int_0^\tau e^{C(X,s)} ds} \right) dt \\ & + \left(\frac{1}{\gamma} \frac{\lambda_1}{\sigma_s} + \frac{1}{\gamma} \rho_{sx}\sigma_x \frac{\int_0^\tau c_1(s) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} \right) \sqrt{X} dW_s(t) \end{aligned} \quad (4.23)$$

Remarks

- Under the assumption of perfect correlation or locally riskfree labor income, the individual is able to hedge labor income risk entirely. Optimal total wealth follows a geometric Brownian motion with time-varying coefficients and will stay non-negative in all cases. Hence, given that initial total wealth $\hat{A} > 0$, the individual will be able to afford the subsistence level of consumption in all cases.
- As can be seen, total wealth follows the same dynamics as financial wealth in the case without labor income and a subsistence level of consumption. The individual takes into account the additional wealth due to human capital and the reduction in wealth due to the reserves covering the subsistence level and controls his total wealth in the same manner as financial wealth.
- In Appendix 3.A.1 of Chapter 3 the valuation of the future income stream is performed using the martingale approach. Since the assumption of complete markets implies that the market price of risk is unique and the risk-neutral valuation asks for the absence of arbitrage, it is not surprising that the value of the future labor income stream is a combination of the riskfree and the risky asset. As a consequence, the special relation of financial and non-financial assets allows them to be absorbed in one factor - total wealth.

4.1.6 Main Results

The most important results can be summarized in the following proposition.

PROPOSITION 4.1 *Given the assumptions $\hat{A}(0) > 0$, $\rho_{sx} \in \{-1, 1\}$ and either $\rho_{sy} \in \{-1, 1\}$ or $\sigma_y = 0$ one obtains*

$$J = \frac{e^{-\delta(T-\tau)} \left[\int_0^\tau e^{\frac{1}{\gamma}(c_0(s)+c_1(s)X)} ds \right]^\gamma (A + k(X, \tau)Y - R(\tau))^{1-\gamma}}{1 - \gamma}$$

with

$$\begin{aligned}\frac{\partial c_1(s)}{\partial s} &= k_0 + k_1 c_1(s) + k_2 c_1(s)^2 \\ \frac{\partial c_0(s)}{\partial s} &= k_3 + k_4 c_1(s)\end{aligned}$$

with initial conditions $c_0(0) = 0$ and $c_1(0) = 0$ and where

$$\begin{aligned}k_0 &\equiv \frac{1}{2} \frac{1-\gamma}{\gamma} \frac{\lambda_1^2}{\sigma_s^2}, \quad k_1 \equiv -\kappa_x + \frac{1-\gamma}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1, \quad k_2 \equiv \frac{1}{2} \frac{1}{\gamma} \sigma_x^2 \\ k_3 &\equiv (1-\gamma)r_0 - \delta, \quad k_4 \equiv \kappa_x \bar{X}\end{aligned}$$

The value of one unit of income is given by

$$k(\tau, X) = \int_0^\tau e^{d_0(s)+d_1(s)X} ds$$

where $d_0(s)$ and $d_1(s)$ are the solutions to the following system of ordinary differential equations

$$\begin{aligned}\frac{\partial d_1(s)}{\partial s} &= l_0 + l_1 d_1(s) + l_2 d_1(s)^2 \\ \frac{\partial d_0(s)}{\partial s} &= l_3 + l_4 d_1(s)\end{aligned}$$

with initial conditions $d_0(0) = 0$ and $d_1(0) = 0$ and where

$$\begin{aligned}l_0 &\equiv y_1 - \frac{\rho_{sy}\sigma_y}{\sigma_s} \lambda_1, \quad l_1 \equiv -\kappa_x - \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 + \rho_{xy}\sigma_x\sigma_y, \quad l_2 \equiv \frac{1}{2} \sigma_x^2 \\ l_3 &\equiv y_0 - r_0, \quad l_4 \equiv \kappa_x \bar{X}\end{aligned}$$

The net reserves follow

$$R(\tau) = \frac{\bar{c} - \bar{Y}}{r_0} (1 - e^{-r_0\tau})$$

Optimal consumption and risky investment are given by

$$c_t^* = \frac{\hat{A}}{\int_0^\tau e^{C(X,s)} ds} + \bar{c} \quad (4.24)$$

$$\begin{aligned}A\pi_t^* &= \frac{1}{\gamma} \frac{\lambda_1}{\sigma_s^2} \hat{A} + \frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \frac{\int_0^\tau c_1(s) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} \hat{A} \\ &\quad - \frac{\rho_{sx}\sigma_x}{\sigma_s} \left(\int_0^\tau d_1(s) e^{d_0(s)+d_1(s)X} ds \right) Y - \frac{\rho_{sy}\sigma_y}{\sigma_s} kY\end{aligned} \quad (4.25)$$

4.2 Model with Utility over Terminal Wealth

Following from this, a similar problem for an investor with utility over terminal wealth only is solved. Liu (2007) shows that without labor income for an individual with these preferences, the assumption $\rho_{sx} \in \{-1, 1\}$ is not necessary in order to get closed-form solutions. As stated in Appendix 2.A.2 of Chapter 2 for a model that can be solved by similar methods, this is not the case in the presence of labor income.

Expected utility is given by

$$E_t \left[\frac{(A_T - \bar{A})^{1-\gamma}}{1-\gamma} \right], \gamma > 1$$

Following the same steps as above, the following HJB-equation results

$$\begin{aligned} 0 = & J_t + J_A A(t) r_0 + J_A Y(t) + J_A \bar{Y} \\ & - J_X \kappa_x (X(t) - \bar{X}) + J_Y Y(t) [y_0 + y_1 X(t)] \\ & + \frac{1}{2} J_{AA}(t) \pi_t^* \lambda_1 X(t) + \frac{1}{2} J_{AX}(t) \pi_t^* \rho_{sx} \sigma_x \sigma_s X(t) \\ & + \frac{1}{2} J_{AY}(t) Y(t) \pi_t^* \rho_{sy} \sigma_s \sigma_y X(t) \\ & + J_{XY}(t) \rho_{xy} \sigma_x \sigma_y X(t) + \frac{1}{2} J_{YY}(t)^2 \sigma_y^2 X(t) + \frac{1}{2} J_{XX}(t) \sigma_x^2 X(t) \end{aligned}$$

where π_t^* is the FOC also given by (4.7). The solution of the optimization problem is similar to the consumption case and therefore omitted.

The results are summarized in the following proposition.

PROPOSITION 4.2 *Given the assumptions $\hat{A}(0) > 0$, $\rho_{sx} \in \{-1, 1\}$ and either $\rho_{sy} \in \{-1, 1\}$ or $\sigma_y = 0$ one obtains*

$$J = \frac{e^{-\delta(T-\tau)} e^{c_0(\tau) + c_1(\tau)X} (A + k(\tau, X)Y - R(\tau))^{1-\gamma}}{1-\gamma}$$

with $c_0(s)$ and $c_1(s)$ and $k(\tau, X)$ identical to Proposition 4.1. The net reserves follow

$$R(\tau) = -\frac{\bar{Y}}{r_0} (1 - e^{-r_0 \tau}) + e^{-r_0 \tau} \bar{A}$$

Optimal risky investment is given by

$$\begin{aligned} A\pi_t^* = & \frac{1}{\gamma} \frac{\lambda_1}{\sigma_s^2} \hat{A} + \frac{1}{\gamma} \frac{\rho_{sx} \sigma_x}{\sigma_s} c_1(\tau) \hat{A} \\ & - \int_0^\tau d_1(s) e^{d_0(s) + d_1(s)X} ds \frac{\rho_{sx} \sigma_x}{\sigma_s} Y - kY \frac{\rho_{sy} \sigma_y}{\sigma_s} \end{aligned} \quad (4.26)$$

where $\tau \equiv T - t$.

Remark

- The difference between the optimal investment rule in case of terminal wealth (4.26) and in case of intermediate consumption (4.22) stems exclusively from the second term¹⁴.

¹⁴The impact of the described part on total allocation is rather small. For this reason there are no considerable differences in the investment strategies of the cases with and without consumption. As a consequence, the discussion of the results refers to the consumption case but is valid in analogy for the case of terminal wealth.

4.3 Long-Horizon Stability of the Solution

The first equation of the system of ODE (4.15) - (4.16) is a Riccati differential equation with constant coefficients. In contrast to (4.10) there is no assumption that ensures that the solution is stable as $s \rightarrow \infty$.

Figures 4.1 and 4.2 show the phase plane analysis for the Riccati differential equation. Since l_2 is unambiguously greater than zero, the parabola opens upward and six cases arise. Figure 4.1 shows the cases where (4.15) has two real particular solutions. If $l_0 < 0$, two real solutions with different signs exist. As can be seen in Panel (a), $d_1(s)$ converges to the negative solution. Given two real solutions, if $l_0 > 0$ and $l_1 < 0$ two positive solutions exist. From Panel (b) it can be recognized that $d_1(s)$ converges to the smaller solution. Given two real solutions, if $l_0 > 0$ and $l_1 > 0$, two negative solutions exist. From Panel (c) it has to be noticed that this setting is unstable as $d_1(s)$ grows without bound.

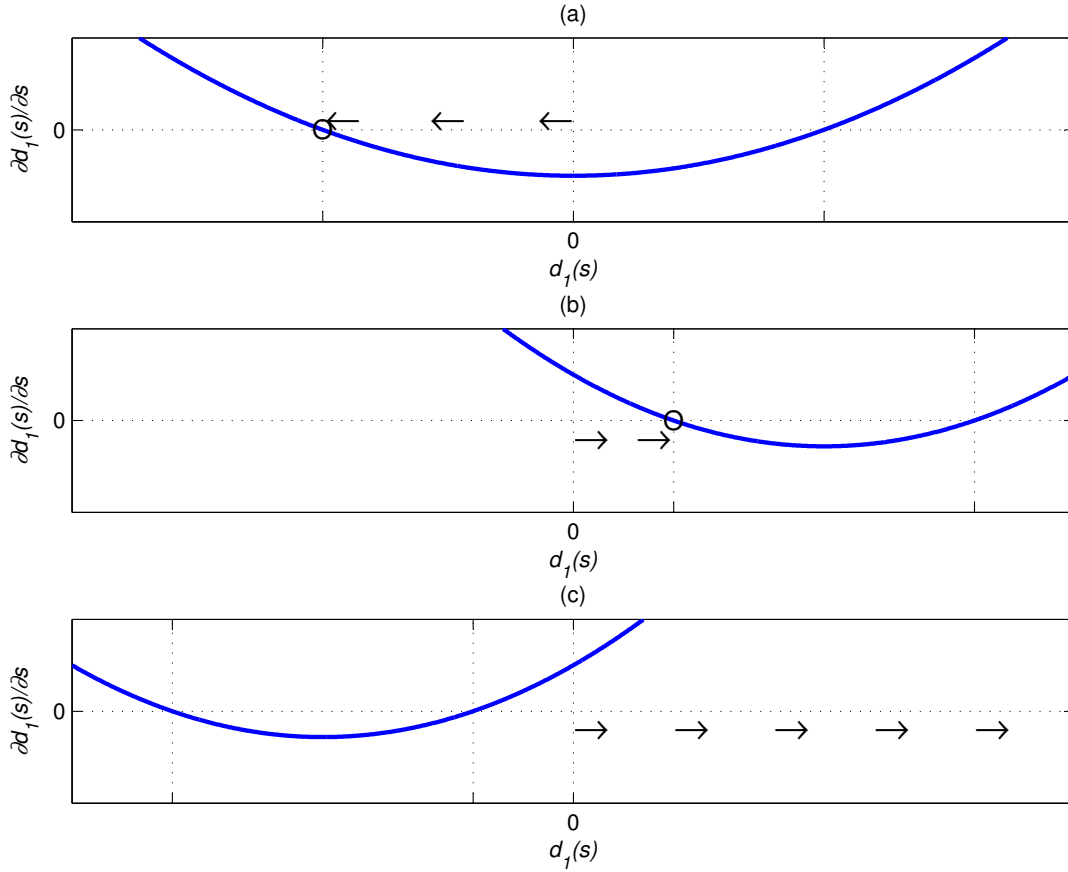


Figure 4.1: Phase Plane Analysis I

Panels (a), (b) and (c) show a phase plane analysis of the equation $\frac{\partial d_1(s)}{\partial s} = l_0 + l_1 d_1(s) + l_2 d_1(s)^2$ for $q_l > 0$. In all cases, two real particular solutions exist. In Panel (a), $l_0 < 0$, in Panel (b), $l_0 > 0$ and $l_1 < 0$ and in Panel (c) $l_0 > 0$ and $l_1 > 0$. In Panels (a) and (b), $d_1(s)$ converges to a stable solution marked by a circle.

Panels (a) and (b) from Figure 4.2 show the case where only one real particular solution exists. The discussion is analogous to Panels (b) and (c) from Figure 4.1. In Panel (c) there are no real

solutions and $d_1(s)$ grows without bound. Table 4.1 summarizes the stability analysis.

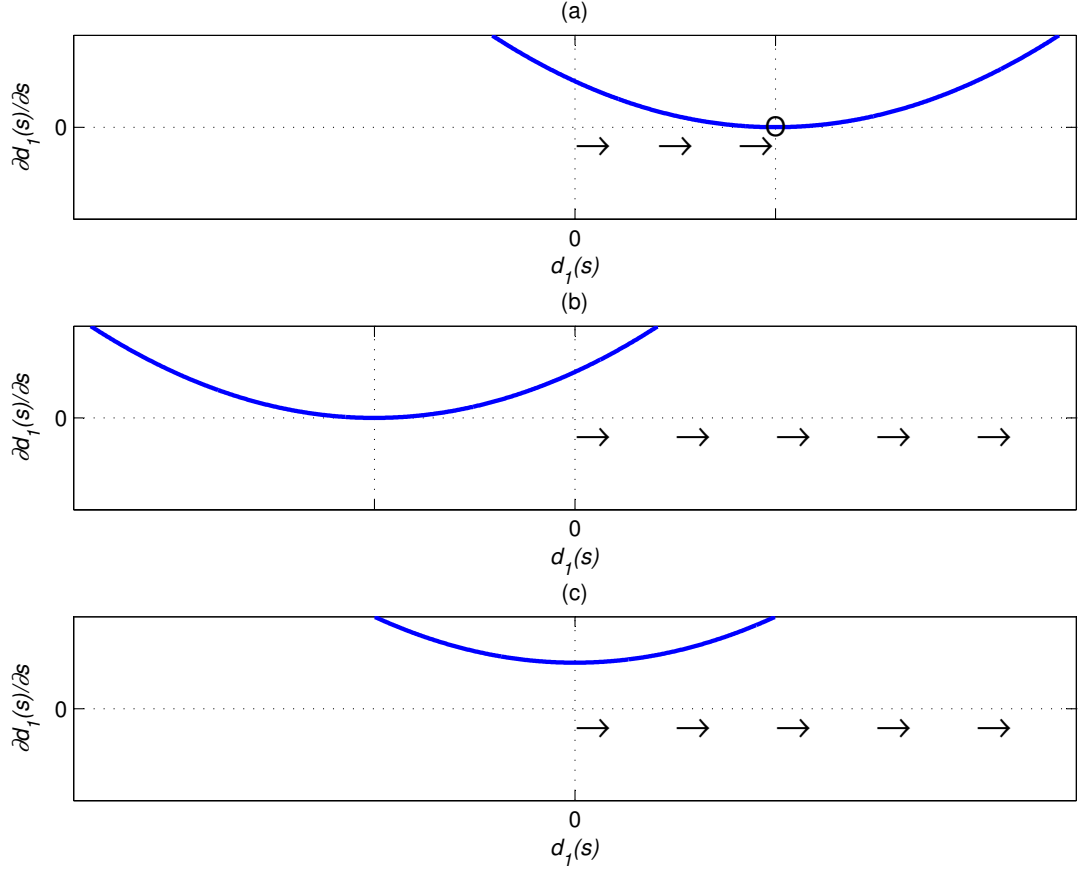


Figure 4.2: Phase Plane Analysis II

Panels (a), (b) and (c) show a phase plane analysis of the equation $\frac{\partial d_1(s)}{\partial s} = l_0 + l_1 d_1(s) + l_2 d_1(s)^2$. In Panels (a) and (b), one real particular solution exists ($q_l = 0$). Panel (c) shows the case without a particular solution ($q_l < 0$). In Panel (a), $l_1 < 0$ and in Panel (b), $l_1 > 0$. In Panel (a), $d_1(s)$ converges to a stable solution marked by a circle.

Number of Particular Solutions	Stable?
2	$q_l > 0, \quad l_0 < 0$ yes
	$q_l > 0, \quad l_0 > 0, \quad l_1 < 0$ yes
	$q_l > 0, \quad l_0 > 0, \quad l_1 > 0$ no
1	$q_l = 0, \quad l_1 < 0$ yes
	$q_l = 0, \quad l_1 > 0$ no
0	$q_l < 0$ no

Table 4.1: Stability Analysis

From the definition of $l_0 = y_1 - \frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1$ it can be seen that instable solutions do not occur

arbitrarily, but have an economic background. In fact, for negative l_0 the valuation of the future income stream is always stable. In other words, if the corresponding hedging portfolio has a higher drift than labor income, valuation is finite. On the other hand, if l_0 is positive the valuation of the income stream can become infinite. Loosely speaking, as long as l_0 is not too positive, equation (4.15) still has two particular solution. In this case

$$l_1 \equiv -\kappa_x - \frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1 + \rho_{xy}\sigma_x\sigma_y$$

is crucial for stability (see Figure 4.1).

It becomes obvious that a high mean-reversion coefficient is in favor of stability, as it pushes l_1 to negativity. Furthermore, from an empirical point of view $\rho_{sx} = -1$ is a more reasonable choice¹⁵ than $\rho_{sx} = 1$. Hence, the second term leads to a more unstable system. More importantly, $-\frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1$ has a direct connection to intertemporal hedging. Indeed, $\rho_{sx} < 0$ implies a higher Sharpe ratio after a decline in the value of the risky asset and this matches the goal of intertemporal hedging: to be in a good state after a decline of wealth and vice versa. If l_0 is also positive, the effects in the financial and non-financial market go in the same direction and the effect from l_0 is strengthened.

The last term is of minor importance as $\sigma_x\sigma_y$ is small in magnitude. The ρ_{xy} must be interpreted in analogy to Section 2.4.3 of Chapter 2. From the dynamics of optimal total wealth (4.32) of Appendix 4.A.2, it can be seen that

$$l_0 < 0 \Rightarrow \frac{\partial k}{\partial X} < 0, \quad \left(l_0 > 0 \Rightarrow \frac{\partial k}{\partial X} > 0 \right)$$

and a negative correlation ρ_{xy} imply a positive (negative) drift in total wealth that must be accounted for¹⁶. Since this part does not originate from a first order condition but simply from the cross product of labor income and state variable diffusion, it is comprehensible that there is no connection to intertemporal hedging.

In cases of one particular solution the interpretation is analogous. Finally, if l_0 is sufficiently high then this effect dominates anyway and no particular solution exists. In this case $d_1(s)$ is always unstable.

Thus, it can be concluded that instable solutions arise only if labor income and the risky asset share very distinct dynamics and the labor income process has an attractive risk-growth profile. From an equilibrium perspective there should be a connection between labor and capital markets and therefore the rather extreme parameter sets that which lead to instability are probably not in line with reality. This statement is similar to that in Kim and Omberg (1996, p. 151) for their instable solutions. The other explanation of Kim and Omberg is, of course, also applicable; namely that the real world includes constraints and costs that are not part of the model and prevent the value of the income stream to become infinite.

Since $d_1(s)$ grows faster than implied by a linear differential equation, in the case of instability

¹⁵ $\rho_{sx} = -1$ implies high volatility after a decline in the value of the risky asset.

¹⁶ See third element of the last line of (4.32).

$d_1(s)$ reaches infinity at a finite horizon¹⁷. The critical horizon s_c is given by

$$s_c = \begin{cases} \frac{1}{\eta_l} \ln \left(\frac{l_1 + \eta_l}{l_1 - \eta_l} \right) & , q_l > 0 \\ \frac{2}{l_1} & , q_l = 0 \\ \frac{\pi}{\eta_l} - \frac{2}{\eta_l} \varphi & , q_l < 0 \end{cases}$$

Nevertheless, it should be kept in mind that the employment phase is finite and has a horizon of approximately 40 years. As a consequence, parameters must have extreme values to end up with a critical horizon lower than the employment phase. A numerical example follows at the end of the next section.

4.4 Illustration of the Results of the Basic Model

As a starting point, results for an individual with locally riskfree labor income $\sigma_y = 0$ are discussed. This assumption simplifies the interpretation because some terms of the optimal policies and some terms in the system of ODEs drop out. In a second step, risky labor income is introduced and the impact of stochastic volatility in labor income on the optimal policies is discussed. In the case of risky labor income it is assumed that¹⁸ $\rho_{sy} = 1$. Table 4.2 shows all fixed parameters that are chosen for the numerical examples.

Financial Market			
<hr/>			
$r_0 = 0.0050$			
$\lambda_1 = 1.0000$	$\sigma_s = 1.0000$		
$\kappa_{x,1} = 0.1000$	$\bar{X} = 0.0400$	$\sigma_{x,1} = 0.0323$	
$\kappa_{x,2} = 0.4000$	$\bar{X} = 0.0400$	$\sigma_{x,2} = 0.0647$	
$\rho_{sx} = -1$			
<hr/>			
Individual			
<hr/>			
$\gamma = 4$	$\delta = 0.06$		
$\bar{y}_1 = 0.0500$	$\bar{y}_2 = 0.00250$	$\rho_{sy} = 1$	$\rho_{xy} = -1$
$A(0) = 50$	$Y(0) = 10$	$\bar{Y} = 40$	
$\bar{c} = 45$			

Table 4.2: Parameter Values

Parameters for the riskless rate, the risky asset and the volatility process are chosen at realistic values¹⁹. Specifically, the long-run equity premium (volatility) is given as 4% (20%) and the variation of the equity premium (volatility) will become clear from Figure 4.3.

¹⁷See Appendix 3.A.2 of Chapter 3 or Kim and Omberg (1996).

¹⁸The case $\rho_{sy} = -1$ can be derived in analogy.

¹⁹An empirical estimation of the model is beyond the scope of this thesis as it demands sophisticated methods as presented in Chacko and Viceira (2003), Jiang and Knight (2002) and Singleton (2001).

The speed of mean-reversion κ_x and the diffusion parameter σ_x can take two values²⁰. For $\kappa_x = 0.1$, mean reversion is low and shocks to volatility are more persistent. $\kappa_x = 0.4$ is a setting with faster mean reversion. For the sake of comparability σ_x is normalized so that the stationary gamma distributions of X are identical. This is ensured by choosing $\sigma_{x,i}$ according to

$$\sigma_{x,i} = \sqrt{\frac{2\kappa_{x,i}\bar{\sigma}_x^2}{\bar{X}}}, i \in \{1, 2\}$$

where $\bar{\sigma}_x^2$ is the unconditional variance of X and $\bar{\sigma}_x$ is set to 0.0145. It can be verified that for both values $(\kappa_{x,1}, \sigma_{x,1})$ and $(\kappa_{x,2}, \sigma_{x,2})$, the condition $2\kappa_x \bar{X} \geq \sigma_x^2$ is fulfilled. Hence, stochastic volatility X will not touch zero²¹.

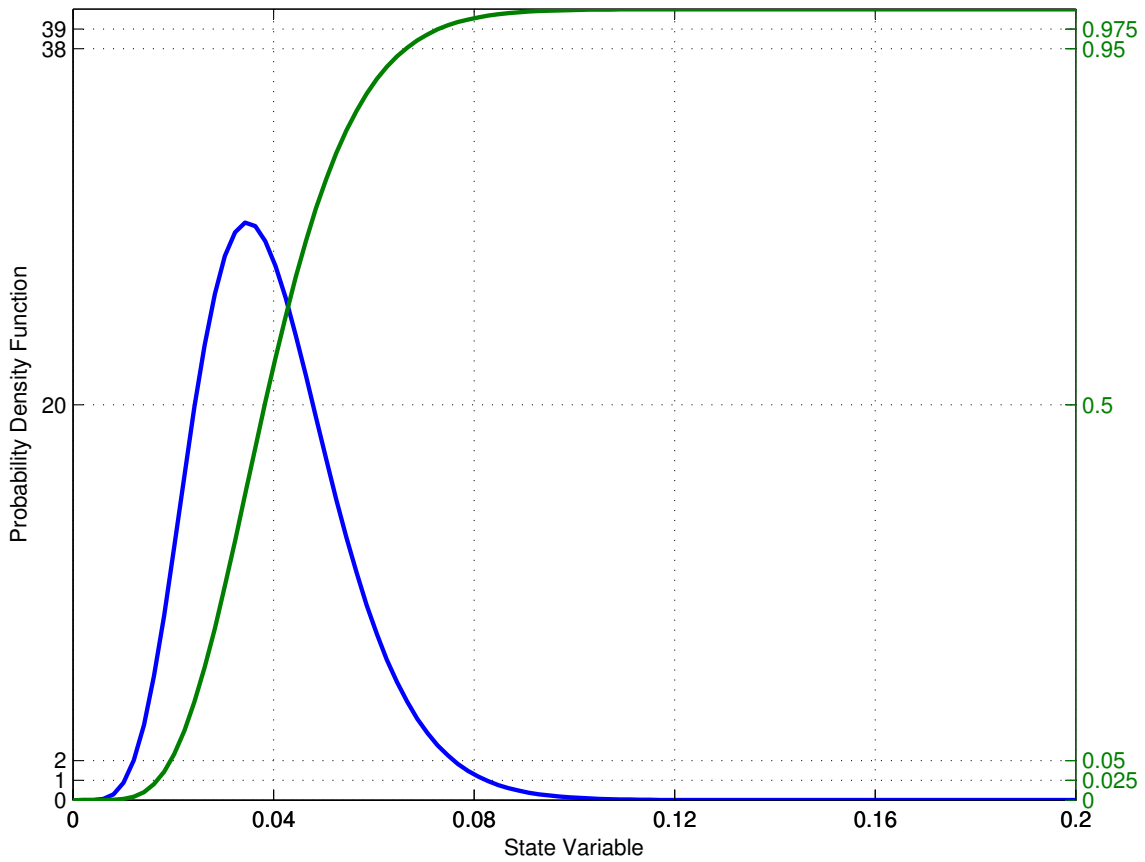


Figure 4.3: Distribution of the State Variable

This Figure shows the probability density function (left vertical axis) and the cumulative distribution function (right vertical axis) of the stationary distribution of the state variable. Parameters are given as in Table 4.2. The long-run mean of the state variable \bar{X} is equal to 0.04.

The stationary gamma distribution of the state variable is shown in Figure 4.3. The blue (green) line shows the probability density function (cumulative distribution function) and belongs to the

²⁰Empirical estimation of a model with stochastic volatility seems to be highly sensitive to the sample period and the frequency of the data. See for example Chacko and Viceira (2005) Table 1.

²¹All properties are well described in Cox et al. (1985). For a textbook treatment with the derivation of the first two moments of $X(t)$ see Shreve (2004, pp. 151-153).

vertical axis on the left (right). The grid points of the horizontal axis correspond to the points displayed in the subsequent figures, which contain the results of the optimal policies. It should be noticed that the states displayed in the right part of Figure 4.3 have only a small probability of occurring. Nevertheless, this area shows the results in case of an unusual volatility shock and thus, the sensitivity of the results in extreme states. In all figures the grid points of the horizontal axis show $(0, \bar{X}, 2\bar{X}, 3\bar{X}, 4\bar{X}, 5\bar{X})$.

For the sake of comparability, the labor income growth parameters are chosen so that the growth rates at the long-run mean are the same for all choices of y_1 . To be precise,

$$y_0 = \bar{y} - y_1 \bar{X}$$

where \bar{y} is the growth rate at the long-run mean and given in Table 4.2.

For the sake of clarity, we introduce the following definitions for the components of risky investment

$$\begin{aligned}
 A\pi_t^* = & \underbrace{\frac{1}{\gamma} \frac{\lambda_1}{\sigma_s^2} \hat{A}}_{\text{"myopic"}} + \underbrace{\frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \frac{\int_0^\tau c_1(s) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} \hat{A}}_{\text{"state variable hedging"}} \\
 & - \underbrace{\frac{\rho_{sx}\sigma_x}{\sigma_s} \left(\int_0^\tau d_1(s) e^{d_0(s)+d_1(s)X} ds \right) Y}_{\text{"indirect labor hedging"}} - \underbrace{\frac{\rho_{sy}\sigma_y}{\sigma_s} \left(\int_0^\tau e^{d_0(s)+d_1(s)X} ds \right) Y}_{\text{"direct labor hedging"}} \quad (4.27)
 \end{aligned}$$

As far as possible, the presented results are justified by an analytical argumentation and followed by an economic intuition.

4.4.1 Locally Riskfree Labor Income

In case of $\sigma_y = 0$, direct labor hedging demand is equal to zero. Moreover, the SODE (4.15) - (4.16) reduces to

$$\begin{aligned}
 \frac{\partial d_1(s)}{\partial s} &= l_0 + l_1 d_1(s) + l_2 d_1(s)^2 \\
 \frac{\partial d_0(s)}{\partial s} &= l_3 + l_4 d_1(s)
 \end{aligned}$$

where

$$\begin{aligned}
 l_0 &\equiv y_1, \quad l_1 \equiv -\kappa_x - \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1, \quad l_2 \equiv \frac{1}{2} \sigma_x^2 \\
 l_3 &\equiv y_0 - r_0, \quad l_4 \equiv \kappa_x \bar{X}
 \end{aligned}$$

Figure 4.4 shows the value of total wealth dependent on stochastic volatility. The blue (green) lines show $\kappa_x = 0.1$ ($\kappa_x = 0.4$). The lines with crosses (circles) belong to an individual with low (high) labor income growth. The panel to the left (right) contains the results for an individual with a negative (positive) sensitivity of the labor income drift on stochastic volatility of $y_1 = -0.2$ ($y_1 = 0.2$). It should be noticed that for a negative sensitivity of labor income growth to stochastic volatility, the value of total wealth decreases with volatility. Without labor income

volatility the sign of $d_1(s)$ depends exclusively on the sign of the sensitivity parameter y_1 . From the phase plane analysis it is known that for

$$l_0 < 0 \Rightarrow d_1(s) < 0, \quad (l_0 > 0 \Rightarrow d_1(s) > 0), \quad s > 0$$

In addition, from

$$\frac{\partial \hat{A}}{\partial X} = \frac{\partial k}{\partial X} Y = \int_0^\tau d_1(s) e^{d_0(s) + d_1(s)X} ds$$

it becomes clear that $k(\tau, X)$ is decreasing (increasing) in X if $y_1 < 0$ ($y_1 > 0$). The formal reasoning is confirmed by intuition. Indeed, in the case of a volatility shock, X rises and the individual with a negative sensitivity to X faces lower income growth as in the long-run mean. This clearly reduces the value of the income stream.

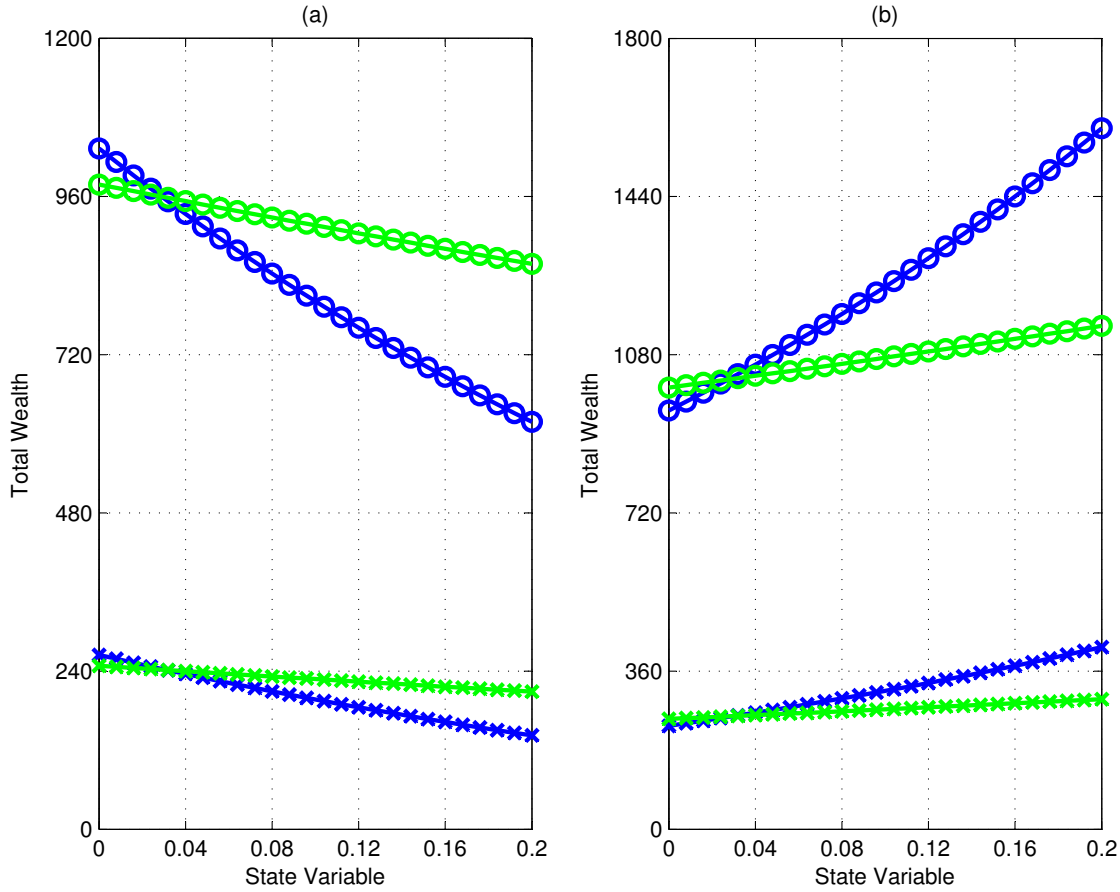


Figure 4.4: Total Wealth - Locally Riskfree Labor Income

This Figure shows total wealth \hat{A} dependent on stochastic volatility. Parameters are given as in Table 4.2. The blue (green) lines show a framework with $\kappa_x = 0.1$ ($\kappa_x = 0.4$) implying high (low) persistent shocks on volatility. The lines with crosses (circles) display results for an individual with low (high) income growth $\bar{y} = 0.25\%$ ($\bar{y} = 5\%$). In the panel to the left (right) the sensitivity of the labor income growth rate to stochastic volatility is positive (negative) and given by $y_1 = -0.2$ ($y_1 = 0.2$).

Furthermore, total wealth varies more strongly if volatility shocks are persistent. For low κ_x ,

$d_1(s)$ becomes higher in magnitude²². The intuition behind this observation is that in a framework with persistent shocks to the opportunity set, the individual is aware that it will take longer until the shocks reverse, which leads to longer phases away from the long-run mean and hence, to a more pronounced valuation of current income growth.

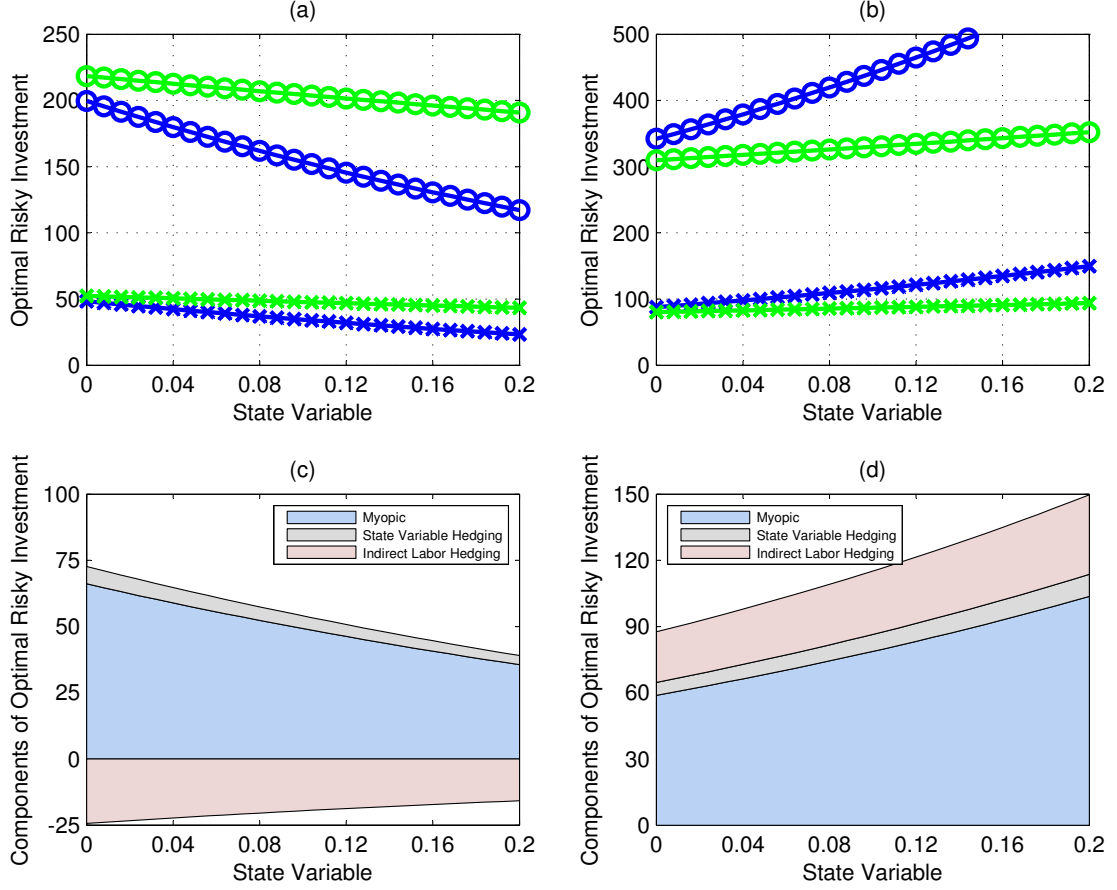


Figure 4.5: Optimal Risky Investment - Locally Riskfree Labor Income

This Figure shows the amount optimally invested in the risky asset $A\pi_t^*$ dependent on stochastic volatility. Parameters are given as in Table 4.2. The blue (green) lines show a framework with $\kappa_x = 0.1$ ($\kappa_x = 0.4$) implying high (low) persistent shocks on volatility. The lines with crosses (circles) display results for an individual with low (high) income growth $\bar{y} = 0.25\%$ ($\bar{y} = 5\%$). In the panels to the left (right) the sensitivity of the labor income growth rate to stochastic volatility is positive (negative) and given by $y_1 = -0.2$ ($y_1 = 0.2$). Panels (c) and (d) show the contribution of the parts described in equation (4.27) for the individual that is described by the blue line with crosses.

For identical growth rates of labor income, the *level* of total wealth at $X = \bar{X}$ is close. From an analytical point of view, unambiguous results with respect to the sign and the magnitude of the difference seem not available because κ_x and σ_x have an impact on $d_0(s)$ and $d_1(s)$ over several channels. However, the closeness of total wealth at $X = \bar{X}$ is intuitive as both cases have the

²²Since both coefficients l_1 and l_2 are affected by changes in κ_x and σ_x , two effects are at work. Unambiguous results are not available because of the dependence on ρ_{sx} , σ_s and λ_1 . For the chosen parameters, $d_1(s)$ is mainly affected by the changes in l_1 . It can be verified that in l_1 the change of κ_x dominates the change of σ_x .

same stationary distribution and labor income growth is identical at $X = \bar{X}$.

Figure 4.5 shows the results for optimal investment. As before, the left (right) panels show the results for an individual with a negative (positive) sensitivity of the labor income drift on stochastic volatility. The lower panels show the contribution of the components of (4.27) for an individual with income growth and persistent shocks²³. Myopic demand is well known and has an unambiguously positive contribution, which varies with total wealth. Because of the negative correlation between the risky asset and stochastic volatility ($\rho_{sx} = -1 < 0$) and because $c_1(s) < 0$, the contribution of state variable hedging demand is unambiguously positive and also varies with total wealth. The low importance of this part is in accordance with Table 5 in Chacko and Viceira (2005, p. 1392).

Indirect labor income hedging demand that is caused by the combination of the state variable and labor income shows an unambiguously positive *slope*. Indeed, the derivative of indirect labor hedging demand with respect to X is given by

$$-\frac{\rho_{sx}\sigma_x}{\sigma_s} \left(\int_0^\tau d_1(s)^2 e^{d_0(s)+d_1(s)X} ds \right) Y > 0$$

The results are intuitive as for $y_1 < 0$ ($y_1 > 0$) the lower (higher) growth rate of labor income leads to a lower (higher) valuation of labor income and the exposure must be adapted correspondingly.

If $y_1 < 0$ ($y_1 > 0$) the *level* of indirect labor hedging demand is unambiguously negative (positive). This follows directly from the positivity of the exponential function and Y and the sign of $d_1(s)$. The results are intuitive as in the case $y_1 < 0$, the individual faces lower income growth after a negative return of the risky asset. This is clearly a deterioration of her environment and by going short (at least for this part of the optimal investment rule) in the risky asset brings a positive return when times of low labor income growth arrive, which is the natural goal of intertemporal hedging. The interpretation for the case $y_1 > 0$ follows the same line of argumentation in the other direction.

The pattern for optimal consumption is more complicated as two dynamic effects are at work simultaneously. On the one hand, the numerator of (4.21) follows the direction of total wealth. On the other hand, the denominator $\int_0^\tau e^{c_0(s)+c_1(s)X} ds$ has an impact as X changes as well. The effects on total wealth are described above. Since $c_1(s) < 0$ for $s > 0$, the first derivative of the denominator with respect to X is

$$\int_0^\tau c_1(s) e^{c_0(s)+c_1(s)X} ds < 0$$

Hence, for $y_1 > 0$ higher volatility leads to an unambiguously positive effect on consumption because the numerator is increasing and the denominator is decreasing in X . If $y_1 < 0$ this is not the case, as both the numerator and the denominator of (4.21) are decreasing in X and the combination of these two effects becomes deciding. In fact, Figure 4.6 displays that for the chosen parameter values, consumption is falling in X . Loosely speaking, for the case $y_1 < 0$ consumption

²³This corresponds to the blue line with crosses in Panel (a) and (b) respectively. The results for the other shown cases are qualitatively similar and therefore omitted.

is falling in X if the percentage decline in total wealth is stronger than the percentage increase in the denominator.

The results are of importance. On the one hand, without labor income the Sharpe ratio and the consumption-wealth ratio have a clear relation. In particular, the consumption-wealth ratio is high when the Sharpe ratio is high. This must clearly not be the case in the presence of labor income. Moreover, if one goes a step further and interprets states of high volatility as recession states²⁴, then decreasing consumption in high states of X seems a property in line with reality.

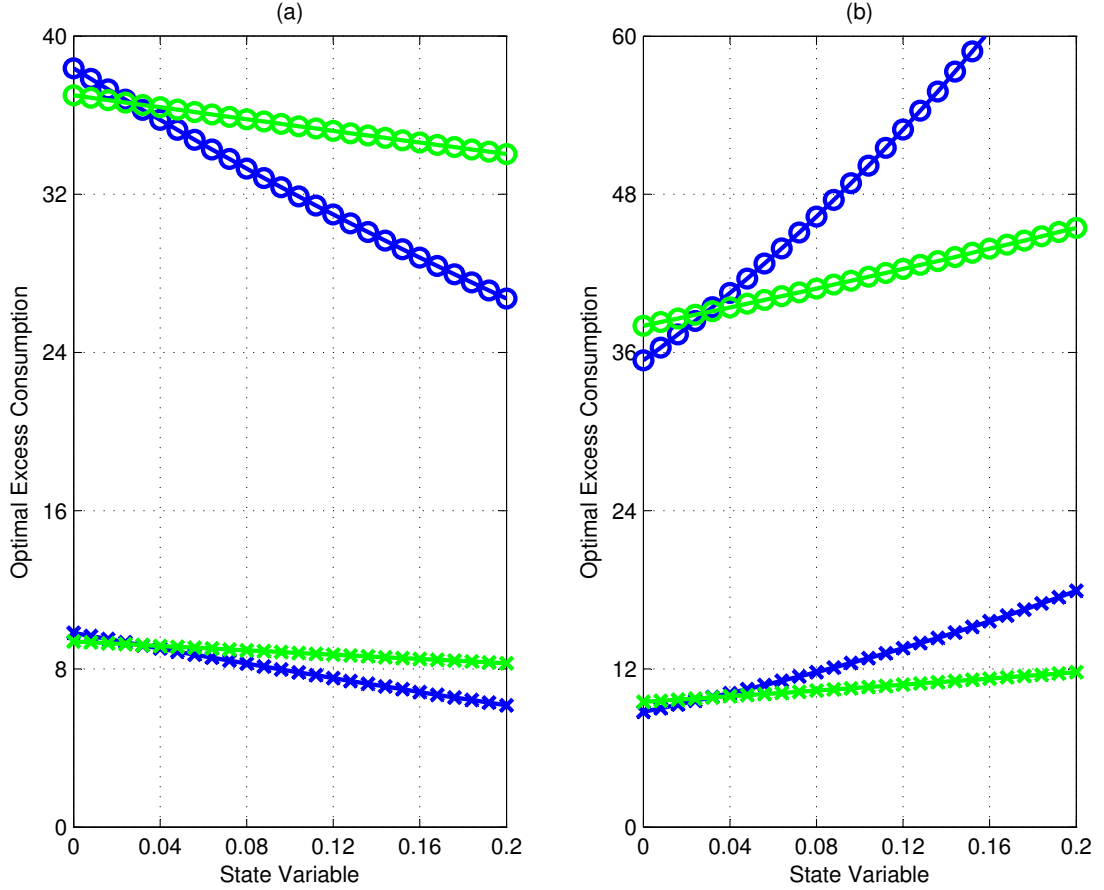


Figure 4.6: Optimal Excess Consumption - Locally Riskfree Labor Income

This Figure shows optimal consumption exceeding the subsistence level $c_t^* - \bar{c}$ dependent on stochastic volatility. Parameters are given as in Table 4.2. The blue (green) lines show a framework with $\kappa_x = 0.1$ ($\kappa_x = 0.4$) implying high (low) persistent shocks on volatility. The lines with crosses (circles) display results for an individual with low (high) income growth $\bar{y} = 0.25\%$ ($\bar{y} = 5\%$). In the panel to the left (right) the sensitivity of the labor income growth rate to stochastic volatility is positive (negative) and given by $y_1 = -0.2$ ($y_1 = 0.2$).

This section ends with a second look at the stability of the valuation of the labor income stream. It can be easily verified that the parameter values for the ‘blue’ individual imply $l_1 < 0$. Hence, in order to end up in a setting with a nirvana solution by altering only y_1 , one needs a $q_l < 0$.

²⁴Engle and Rangel (2008) separate stock market volatility in a persistent component with low-frequency and transitory shocks. It is shown that the low-frequency component of volatility has a clear connection to the business-cycle; specifically, low-frequency volatility is high when economic growth is low.

The critical value for the sensitivity of labor income growth which leads to $q_l = 0$ is given by $y_{1,c} = 3.0935$. Thus, it can be concluded that y_1 must be unrealistically high with the given set of parameters. Nevertheless, for even more persistent volatility, $l_1 > 0$ cannot be excluded in general and the non-normal cases of the solution may be of interest for future research.

4.4.2 Risky Labor Income

The main differences to the locally riskfree labor income case are the additional direct labor income hedging part in the optimal investment policy (4.27) and the change in l_0 . The additional term in l_1 is small in magnitude and hence, rather irrelevant. In order to show the impact of σ_y , various values are chosen. Furthermore, it is assumed that²⁵ $\rho_{sy} = 1$.

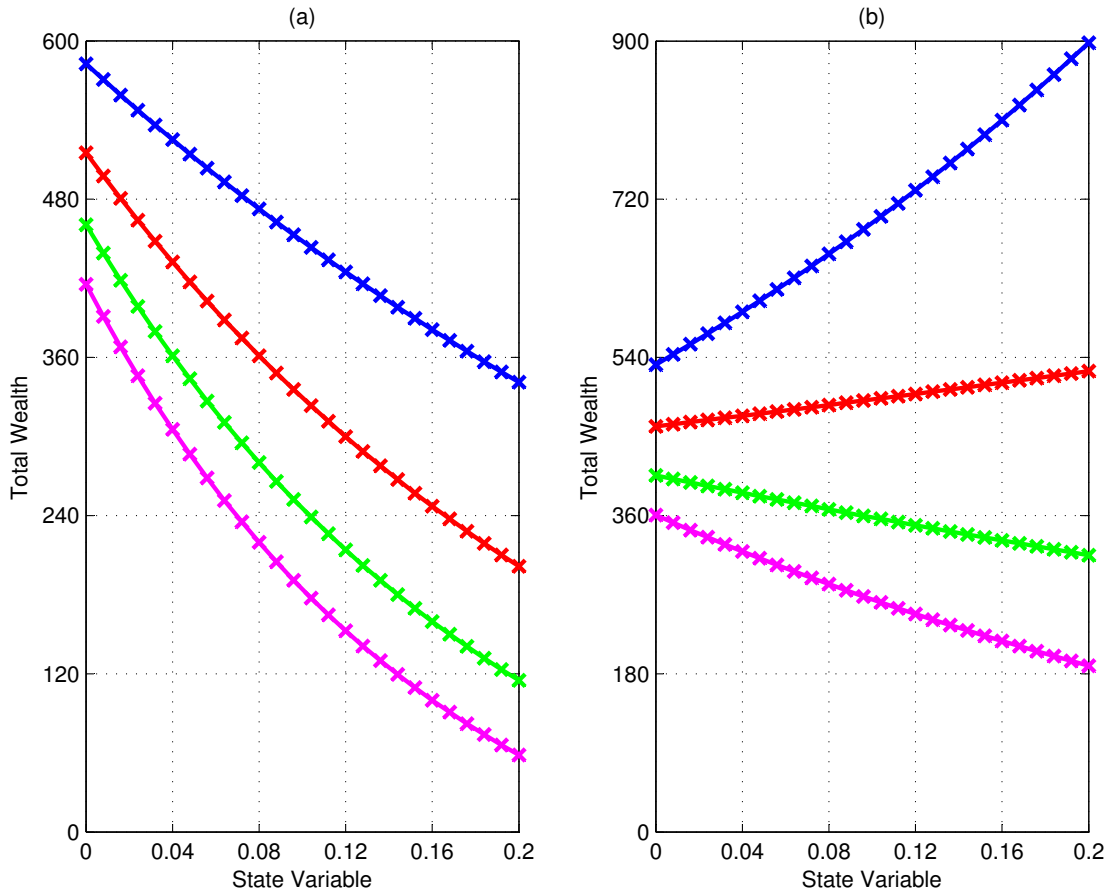


Figure 4.7: Total Wealth - Risky Labor Income

This Figure shows total wealth \hat{A} dependent on stochastic volatility. Parameters are given as in Table 4.2, $\bar{y} = 3\%$ and $\kappa_x = 0.1$. The blue (red, green, purple) lines show a framework with labor income volatility $\sigma_y = 0$ (0.15, 0.30, 0.45). In the panel to the left (right) the sensitivity of the labor income growth rate to stochastic volatility is positive (negative) and given by $y_1 = -0.2$ ($y_1 = 0.2$).

Figure 4.7 exhibits total wealth dependent on stochastic volatility. The blue (red, green, purple) line shows the results for $\sigma_y = 0$ (0.15, 0.30, 0.45). Through the smaller l_0 and thus smaller $d_1(s)$,

²⁵Results for $\rho_{sy} = -1$ can be derived in analogy.

total wealth reacts more negatively to higher stochastic volatility. Even in the case of a positive sensitivity parameter y_1 , the *slope* of total wealth is negative if $l_0 < 0 \Leftrightarrow \sigma_y > \frac{y_1}{\lambda_1 \rho_{sy}} \sigma_s$.

As expected, the *level* of total wealth becomes smaller for higher income volatility. In fact, from the discussion of the dynamics of total wealth it is known that the individual controls total wealth in the same manner as an investor without labor income and subsistence consumption. As non-financial income becomes risky, the individual will need additional (short) positions in the risky asset, which leads to a correspondingly lower valuation of the labor income stream²⁶.

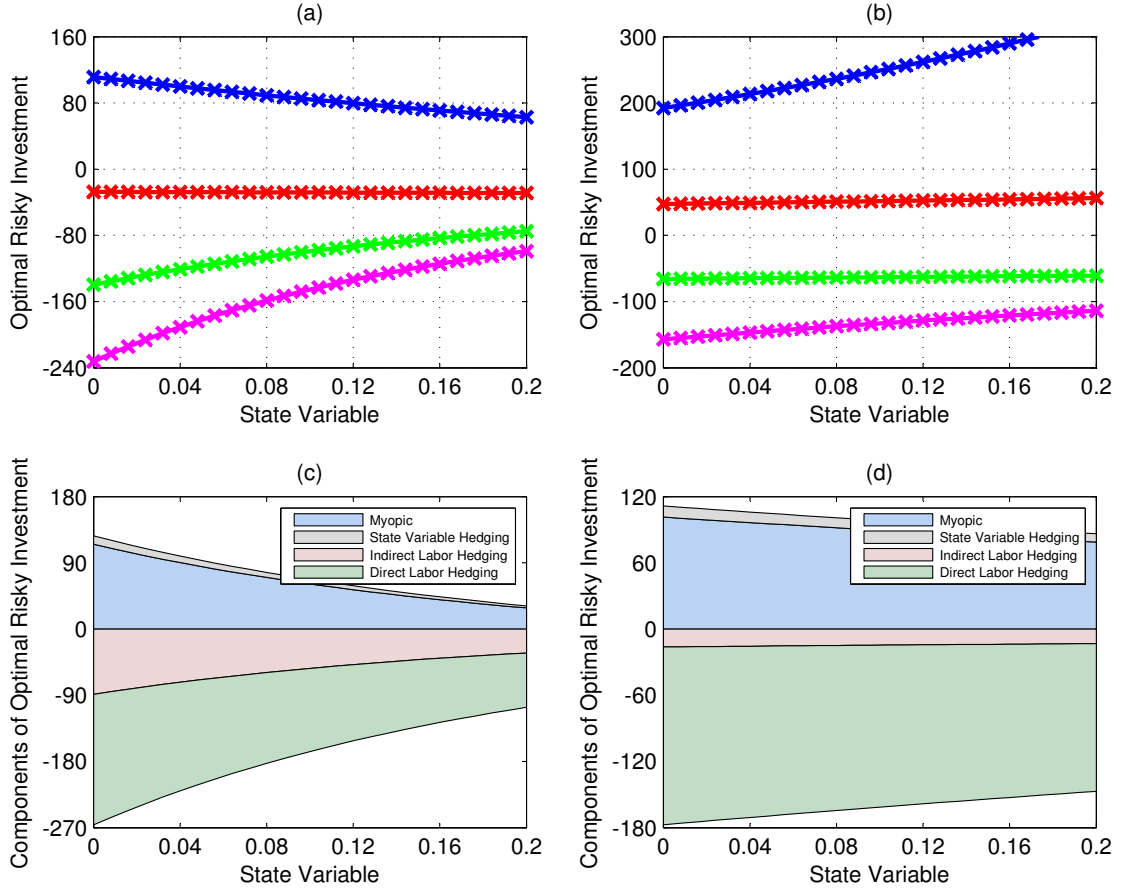


Figure 4.8: Optimal Risky Investment - Risky Labor Income

This Figure shows the amount optimally invested in the risky asset $A\pi_t^*$ dependent on stochastic volatility. Parameters are given as in Table 4.2, $\bar{y} = 3\%$ and $\kappa_x = 0.1$. The blue (red, green, purple) lines show a framework with labor income volatility $\sigma_y = 0$ (0.15, 0.30, 0.45). In the panels to the left (right) the sensitivity of the labor income growth rate to stochastic volatility is positive (negative) and given by $y_1 = -0.2$ ($y_1 = 0.2$). Panels (c) and (d) show the contribution of the parts described in equation (4.27) for the individual that is described by the green line.

This perspective is confirmed by the risk-neutral derivation of k in the Appendix of Chapter 3. In fact, it is shown that the excess expected return (over the riskfree rate) of stochastic human

²⁶The result is valid for reasonable parameter values. However, for very high σ_x , the level of total wealth can rise with higher income volatility due to the term $\rho_{xy}\sigma_x\sigma_y$ in l_1 .

capital $G \equiv kY$ must be in accordance with the market price of risk. Adapted to the market price as given by (4.1), the second part of the RHS of 3.12 becomes

$$\frac{\partial G}{\partial Y} Y \frac{\rho_{sy} \sigma_y}{\sigma_s} \lambda_1 X = kY \frac{\rho_{sy} \sigma_y}{\sigma_s} \lambda_1 X > 0$$

From the positivity of k , σ_y , σ_s and λ_1 , a positive ρ_{sy} in combination with high σ_y asks for a high premium. As a consequence, the value of G must be low similar to a financial asset that is discounted at a high rate.

From the positivity of k and ρ_{sy} it follows immediately that the direct labor hedging part reduces the exposure to the risky asset. Moreover, for realistic parameter values, total wealth and indirect labor hedging demand also become smaller. Thus, if $\rho_{sy} = 1$ the exposure in the risky asset is reduced over several channels.

The optimal investment policies are shown in Figure 4.8. In Panel (a), it can be seen that for $\sigma_y = 0$ the optimal policy is, as described above, decreasing in stochastic volatility. For $\sigma_y > 0$ this does not have to be the case. On the one hand, myopic and state variable hedging demand become smaller because of the decline in total wealth. On the other hand, hedging demands, where labor income is involved, become less negative. The latter effects dominates the first if σ_y is large enough.

Panel (b) shows an interesting pattern; specifically, that the amount invested in the risky asset increases in stochastic volatility for low and high σ_y and is rather insensitive to changes in X for intermediate values of σ_y .

The interpretation for the case $\sigma_y = 0$ is given above and the increasing pattern is comprehensible. For high σ_y the sensitivity of total wealth to stochastic volatility becomes negative and this effect reduces the exposure in the risky asset over myopic and state variable hedging demand as X rises. On the other hand, for high σ_y the (negative) indirect and direct labor hedging demand become important. At a certain level of σ_y , the importance of these parts is high enough that they induce a stronger rise in the risky asset exposure and overcompensate the reduction from myopic and state variable demand.

The discussion of the *slope* and the *level* of indirect labor hedging demand is analogous to the locally riskfree case and thus omitted.

As can be seen from (4.27), direct labor hedging demand is negative, as ρ_{sy} is assumed to be positive. The *slope* of direct labor income hedging demand is given by

$$-\frac{\rho_{sy} \sigma_y}{\sigma_s} \left(\int_0^\tau d_1(s) e^{d_0(s) + d_1(s)X} ds \right) Y$$

It can be clearly recognized that the sign of the slope is determined by the sign of $-d_1(s)$, which is equal to the opposite of the sign of l_0 . As shown above, l_0 is negative for high σ_y .

Finally, Figure 4.8 shows that total exposure in the risky asset becomes negative at a certain level of labor income volatility²⁷. The reasons are the decline in total wealth combined with the growing importance of the negative indirect and direct labor hedging demand.

²⁷In the presence of short selling restrictions it can be expected that such individuals would hold no risky securities at all.

Figure 4.9 displays optimal consumption. The two effects - changes in total wealth and $\int_0^\tau e^{C(X,s)} ds$ - that are at work at the same time are identical to the locally riskfree labor income framework. In fact, the denominator (4.6) is the same for all cases and the numerator is simply total wealth. This explains the displayed patterns.

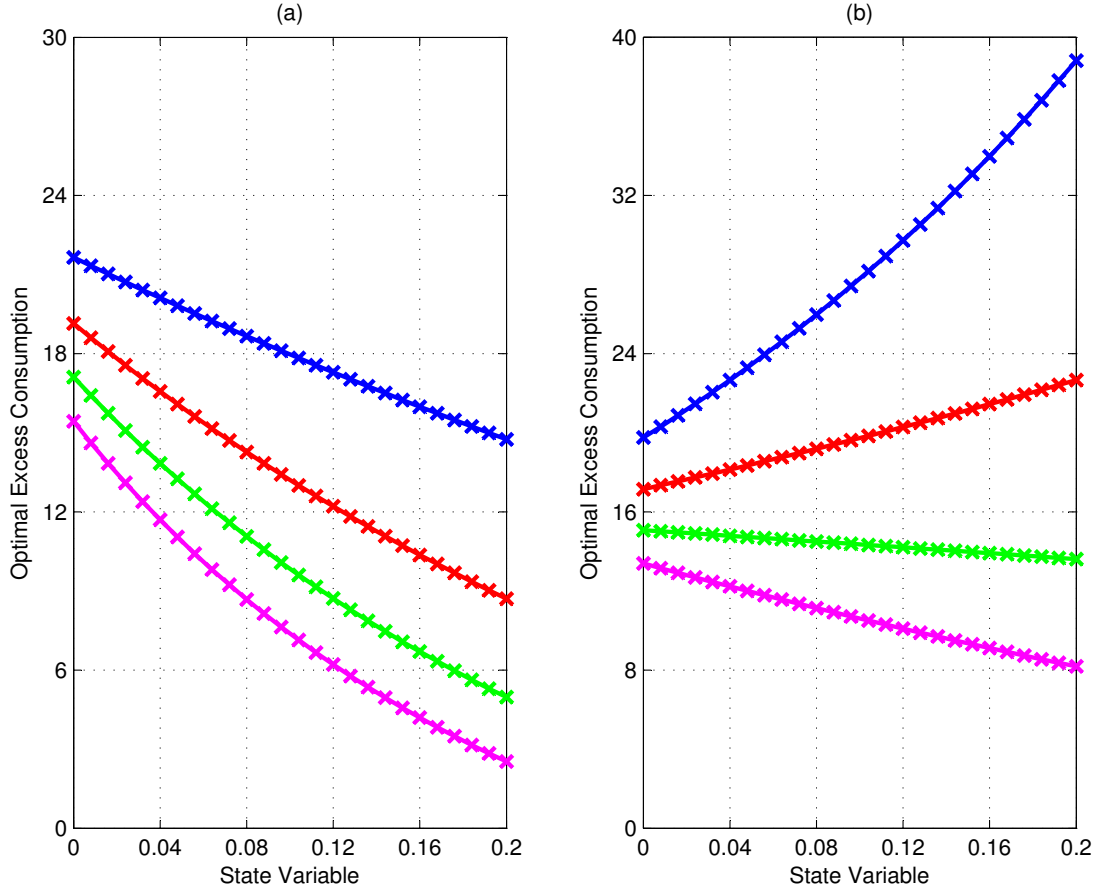


Figure 4.9: Optimal Excess Consumption - Risky Labor Income

This Figure shows optimal consumption exceeding the subsistence level $c_t^* - \bar{c}$ dependent on stochastic volatility. Parameters are given as in Table 4.2, $\bar{y} = 3\%$ and $\kappa_x = 0.1$. The blue (red, green, purple) lines show a framework with labor income volatility $\sigma_y = 0$ (0.15, 0.30, 0.45). In the panel to the left (right) the sensitivity of the labor income growth rate to stochastic volatility is positive (negative) and given by $y_1 = -0.2$ ($y_1 = 0.2$).

It can be summarized that the inclusion of a labor income stream does have an impact on the optimal policies over several channels. With appropriate parameter values as presented above, many interesting consumption and investment patterns can be established. In particular, it is shown that individuals with unfavorable labor income characteristics (such as negative sensitivity of income growth on volatility and/or risky labor income and positive correlation with the risky asset) do not want to invest a positive amount in the risky asset and reduce consumption in times of high volatility. These insights are a resolution as to why persons with low labor income prospects and/or low financial wealth do not hold any equity at all²⁸. Moreover, the results show

²⁸For example, in Figure 2 of Campbell (2006) it can be clearly recognized that a considerable fraction of the

that for an individual with $l_0 \equiv y_1 - \frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1 > 0$, optimal consumption and optimal investment grow with higher volatility and that in states of high volatility extreme values will occur. This seems to be rather unrealistic behavior. Hence, in analogy to Chapter 3 it can be stated that a positive sensitivity of labor income growth ($y_1 > 0$) without labor income volatility leads to unreasonably extreme results. However, as soon as the income volatility also rises with a certain strength, realistic optimal policies result.

4.5 Life-Cycle Model

In this extension it is assumed that the life-cycle of the individual consists of two phases²⁹. In the first phase, the phase of employment, the individual is working and receives labor income as described above. In the second phase, the phase of retirement, the individual is retired and receives no non-financial income³⁰. The most important aspect of this extension is that the individual has a limited working period. At the end of this period human capital is exhausted and the individual has to ensure the second phase without non-financial income. It is assumed that the retirement date is fixed at T_1 . The full planning horizon is given by T_2 . Finally, we assume that the subsistence level of consumption may differ in the two periods and denote the subsistence level during the working period with \bar{c}_e and during retirement with \bar{c}_r .

The solution approach asks to work backwards from the end of the planning horizon, i.e. it is necessary to solve the problem of the phase of retirement first. In a second step, the problem of the phase of employment is solved and linked to the retirement phase³¹.

Phase of Retirement

Without labor income the HJB is a simpler version of (4.5) and is given by

$$\begin{aligned}
0 = & J_t + \sup_c \left[e^{-\delta t} \frac{(c_t - \bar{c}_r)^{1-\gamma}}{1-\gamma} - J_A c_t \right] \\
& + \sup_\pi \left[J_A \pi(t) A(t) \lambda_1 X(t) + \frac{1}{2} J_{AA} \pi(t)^2 A(t)^2 \sigma_s^2 X(t) \right. \\
& \left. + J_{AX} \pi(t) A(t) \rho_{sx} \sigma_s \sigma_x X(t) \right. \\
& \left. + J_{AA} A(t) r_0 - J_X \kappa_x (X(t) - \bar{X}) + \frac{1}{2} J_{XX} \sigma_x^2 X(t) \right]
\end{aligned} \tag{4.28}$$

The FOC with respect to consumption are unchanged and given by (4.6). The FOC with respect to investment is given by

$$\pi_t^* = -\frac{J_A}{J_{AA} A(t)} \frac{\lambda_1}{\sigma_s^2} - \frac{J_{AX}}{J_{AA} A(t)} \frac{\rho_{sx} \sigma_x}{\sigma_s} \tag{4.29}$$

population does not participate in the stock market.

²⁹Life-cycle models have been of interest since Jagannathan and Kocherlakota (1996). More recent work includes Cocco et al. (2005), Lynch and Tan (2009) and Koijen et al. (2010).

³⁰Of course, one could assume that the individual receives pension benefits. For example, Lynch and Tan (2009) simply assume that the individual receives 93% percent of his final income. Moos and Müller (2010) analyze a life-cycle model with a pension system in a constant opportunity set framework. This system could be integrated as well, but this is omitted for the sake of simplicity.

³¹For more details the reader is referred to Huang and Milevsky (2008).

Plugging in the FOCs (4.6) and (4.29) into the HJB equation (4.28) yields

$$\begin{aligned} 0 = & J_t + \frac{\gamma}{1-\gamma} e^{-\frac{\delta}{\gamma}t} J_A^{1-\frac{1}{\gamma}} - J_A \bar{c}_r + J_A A(t) r_0 - J_X \kappa_x (X(t) - \bar{X}) \\ & + \frac{1}{2} J_{AA}(t) \pi_t^* \lambda_1 X(t) + \frac{1}{2} J_{AX} A(t) \pi_t^* \rho_{sx} \sigma_x \sigma_s X(t) + \frac{1}{2} J_{XX} \sigma_x^2 X(t) \end{aligned}$$

The value function that solves this problem has the following form

$$J = \frac{e^{-\delta(T_2-\tau_2)} \left[\int_0^{\tau_2} e^{\frac{1}{\gamma}(c_0(s)+c_1(s)X)} ds \right]^\gamma (A - R(\tau_2))^{1-\gamma}}{1-\gamma}, \quad t > T_1$$

where $\tau_2 \equiv T_2 - t$. Following the steps described in Appendix 2.A.2 of Chapter 2, one will recover (4.9) one-to-one. Hence, the system of ODE that determine $c_0(s)$ and $c_1(s)$ is unchanged. This is intuitive since this system does not involve any parameters of the labor income process or the subsistence level of consumption.

For the reserves covering the subsistence level of consumption, (4.19) changes to

$$0 = \int_0^{\tau_2} e^{C(X,s)} ds \left\{ \frac{\partial R_r}{\partial \tau_2} - \bar{c}_r + r_0 R_r \right\}$$

The initial condition $R_r(\tau_2 = 0) = 0$ implies

$$R_r(\tau_2) = \frac{\bar{c}_r}{r_0} (1 - e^{-r_0 \tau_2})$$

Phase of Employment

The HJB-equation of the phase of employment is given by (4.5). As a consequence, the solution of this part of the model is analogous to the basic model. The solution for $c_0(s)$ and $c_1(s)$ are similar to the basic model. The only difference emerges from the extended horizon.

The reserves are still governed by (4.19)

$$0 = \int_0^{\tau_1} e^{C(X,s)} ds \left\{ \frac{\partial R_e}{\partial \tau_1} + \bar{Y} - \bar{c} + r_0 R_e \right\}$$

but one has to take into account the initial condition

$$R_e(\tau_1 = 0) = R_r(\tau_2 = T_2 - T_1) = \frac{\bar{c}_r}{r_0} (1 - e^{-r_0 \tau_2})$$

where $\tau_1 \equiv T_1 - t$. The solution is given by

$$R_e(\tau_1) = \frac{\bar{c}_r}{r_0} (e^{-r_0 \tau_1} - e^{-r_0 \tau_2}) + \frac{\bar{c}_e - \bar{Y}}{r_0} (1 - e^{-r_0 \tau_1})$$

Since the value function of the phase of employment is still given by (4.8), $k(X, \tau_1 = 0) = 0$ and the reserves are properly linked; the value function is a smooth function at the jump date.

Results

The results are summarized in the following proposition.

PROPOSITION 4.3 *Given the assumptions $\hat{A}(0) > 0$, $\rho_{sx} \in \{-1, 1\}$ and either $\rho_{sy} \in \{-1, 1\}$ or $\sigma_y = 0$, one obtains*

$$R(t) = \begin{cases} \frac{\bar{c}_r}{r_0} (e^{-r_0 \tau_1} - e^{-r_0 \tau_2}) + \frac{\bar{c}_e - \bar{Y}}{r_0} (1 - e^{-r_0 \tau_1}) & , \quad 0 \leq t < T_1 \\ \frac{\bar{c}_r}{r_0} (1 - e^{-r_0 \tau_2}) & , \quad T_1 \leq t \leq T_2 \end{cases}$$

Optimal consumption and risky investment are given by

$$c_t^* = \begin{cases} \frac{\hat{A}_e}{\int_0^{\tau_2} e^{C(X,s)} ds} + \bar{c}_e & , 0 \leq t < T_1 \\ \frac{\hat{A}_r}{\int_0^{\tau_2} e^{C(X,s)} ds} + \bar{c}_r & , T_1 \leq t \leq T_2 \end{cases}$$

$$A\pi_t^* = \begin{cases} \frac{1}{\gamma} \frac{\lambda_1}{\sigma_s^2} \hat{A}_e + \frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \frac{\int_0^{\tau_2} c_1(s) e^{C(X,s)} ds}{\int_0^{\tau_2} e^{C(X,s)} ds} \hat{A}_e \\ - \frac{\partial k(X, \tau_1)}{\partial X} Y \frac{\rho_{sx}\sigma_x}{\sigma_s} - k(X, \tau_1) Y \frac{\rho_{sy}\sigma_y}{\sigma_s} & , 0 \leq t < T_1 \\ \frac{1}{\gamma} \frac{\lambda_1}{\sigma_s^2} \hat{A}_r + \frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \frac{\int_0^{\tau_2} c_1(s) e^{C(X,s)} ds}{\int_0^{\tau_2} e^{C(X,s)} ds} \hat{A}_r & , T_1 \leq t \leq T_2 \end{cases}$$

where $\tau_1 \equiv T_1 - t$, $\tau_2 \equiv T_2 - t$, $\hat{A}_e \equiv A + kY - R$ and $\hat{A}_r \equiv A - R$.

Remarks

- Since $\bar{c}_r \geq 0$ by assumption, the individual is in need of further reserves and thus $R(t)$ of the life-cycle model is higher as in the basic model.
- Compared to the basic model, optimal consumption is unambiguously smaller because the higher reserves for future subsistence consumption lead to a lower numerator and the denominator

$$\int_0^{\tau_2} e^{C(X,s)} ds = \int_0^{\tau_1} e^{C(X,s)} ds + \underbrace{\int_{\tau_1}^{\tau_2} e^{C(X,s)} ds}_{\geq 0}$$

is higher because of the longer horizon.

- If $\bar{c}_e \neq \bar{c}_r$ then there is a jump in the consumption strategy at retirement. Nevertheless, there is no jump in consumption exceeding the subsistence level.
- During the phase of employment, the impact on the optimal risky investment is more complicated as different effects work at the same time. The lower total wealth due to the higher reserves reduces myopic and state variable demand. Nevertheless, the longer horizon also has an impact on state variable hedging demand.
- Since $k(\tau_1 = 0) = \frac{\partial k(\tau_1=0)}{\partial X} = 0$, there is no jump in the investment strategy at retirement.

4.5.1 Illustration of the Results of the Life-Cycle Model

For the sake of simplicity we focus on the case with locally riskfree labor income. Parameters are as in Table 4.2, $\kappa_x = 0.1$ and $\bar{y} = 0.05$. The phase of retirement is assumed to be 20 years and \bar{c}_e and \bar{c}_r are assumed to be given by 45 and 40 respectively. The results of a life-cycle optimizing individual (crosses) are compared with an individual who does not account for the retirement period (circles).

Figures 4.10 and 4.11 show the results for an individual at the beginning of the phase of employment. The effect of the phase of retirement on total wealth is rather simple and displayed in Figure 4.10. The impact stems exclusively from the additional amount necessary to cover the

subsistence level of consumption in the retirement phase. Hence, total wealth is unambiguously lower for the life-cycle optimizer.

The effects on the optimal policies are more interesting and shown in the upper panels of Figure 4.11. On the one hand, the analytical results show that the longer horizon has an impact on state variable hedging demand. On the other hand, panels (a) and (b) reveal only differences that are similar to the differences in total wealth.

The optimal investment policy is also given by (4.27) with direct labor hedging demand equal to zero because of the assumption of locally riskfree labor income. The effect on myopic demand is only driven by the lower total wealth and is smaller for the life-cycle optimizer. State variable hedging demand is also affected by the lower total wealth. Nevertheless, the extended horizon has an effect on $c_1(s)$ and $c_0(s)$, but this effect is small in magnitude³². In fact, in the numerical example

$$\frac{\int_0^{60} c_1(s) e^{C(X,s)} ds}{\int_0^{60} e^{C(X,s)} ds} = -3.4106$$

compared to

$$\frac{\int_0^{40} c_1(s) e^{C(X,s)} ds}{\int_0^{40} e^{C(X,s)} ds} = -3.0742$$

In addition, because the ratio σ_x/σ_s in state variable hedging demand is small, the impact on optimal risky investment is of minor importance.

Indirect labor income hedging demand is not affected and is the same for both types of investors. Thus, it can be summarized that the changes in the optimal investment policy are driven almost entirely by changes in total wealth and hence a life-cycle optimizer invests clearly less in the risky asset.

Panels (c) and (d) of Figure 4.11 show the impact on optimal consumption. The numerator of optimal consumption is equal to total wealth, which is reduced for the life-cycle investor. Furthermore, as already stated, the denominator is higher because of the longer horizon. As a result, optimal consumption is unambiguously reduced for the life-cycle investor.

³²Strict analytical results seem not to be available. Nevertheless, an upper boundary for state variable hedging demand can be calculated. Defining c_1^- as the stable long-run solution of the Riccati differential equation and keeping in mind that $c_1(s) < c_1^-$, it follows that

$$\frac{\int_0^\infty c_1(s) e^{C(X,s)} ds}{\int_0^\infty e^{C(X,s)} ds} < \frac{\int_0^\infty c_1^- e^{C(X,s)} ds}{\int_0^\infty e^{C(X,s)} ds} = \frac{c_1^- \int_0^\infty e^{C(X,s)} ds}{\int_0^\infty e^{C(X,s)} ds} = c_1^-$$

Hence, $\frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} c_1^-$ gives an upper boundary for the effect of state variable hedging demand. In the numerical example $c_1^- = -4.9086$.

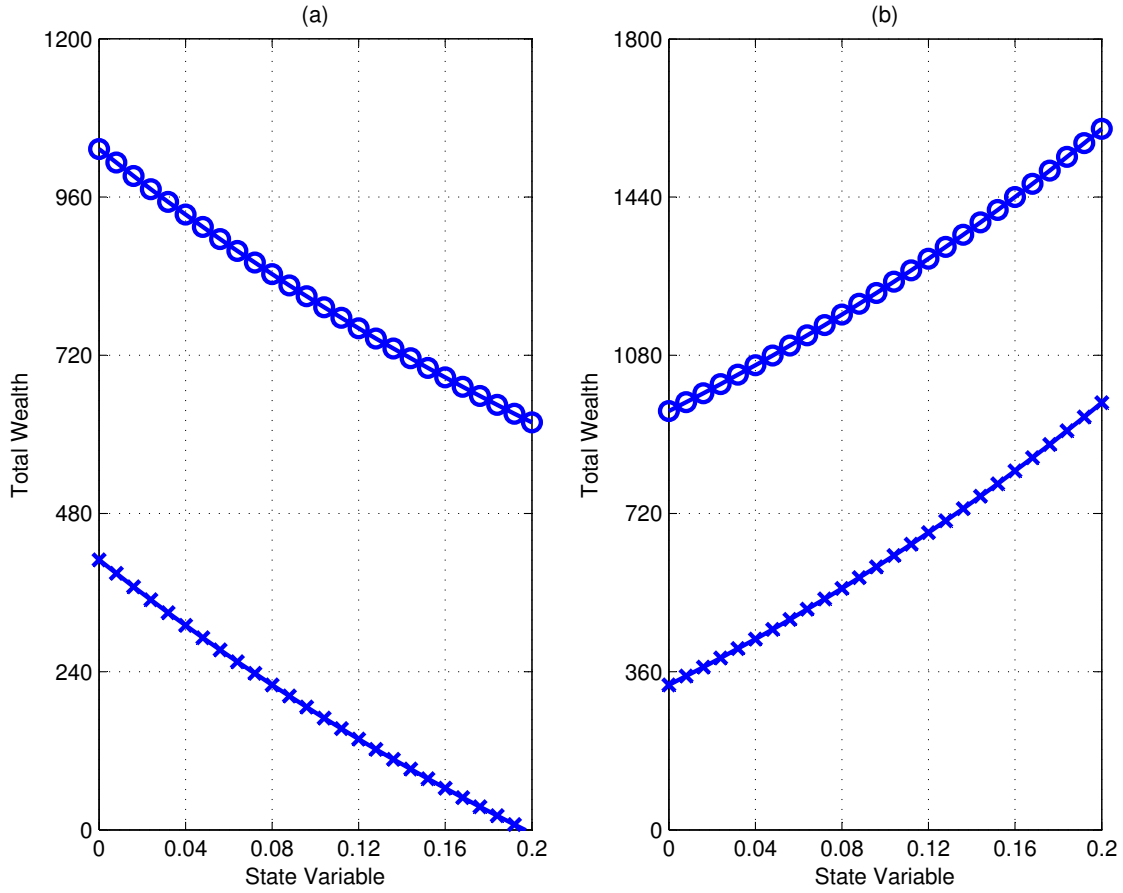


Figure 4.10: Total Wealth - Life-Cycle Model

This Figure shows total wealth \hat{A} dependent on stochastic volatility. Parameters are given as in Table 4.2 and in the text at the beginning of Section 4.5.1. The lines with crosses (circles) show the results for an individual with (without) a phase of retirement. In the panel to the left (right) the sensitivity of the labor income growth rate to stochastic volatility is negative (positive) and given by $y_1 = -0.2$ ($y_1 = 0.2$).

It can be stated that the additional retirement period without non-financial income has an effect on the optimal policies that is intuitive. The consideration of a retirement period offer a further explanation as to why risky investment is lower, as predicted by non life-cycle models. Moreover, in combination with a negative sensitivity of labor income growth on stochastic volatility, the results indicate equity exposure that is on a reasonable level and risky investment falls in states of high volatility. Furthermore, the assumption of HARA utility implies that only a fraction of optimal consumption varies with total wealth. In the final example for the individual with a negative sensitivity of labor income growth on X , consumption exceeding the subsistence level makes about a quarter of total consumption. Hence, consumption volatility is considerably lower than the volatility of total wealth³³.

³³The problem that consumption is as volatile as wealth is analogous to the problem to the equity premium puzzle in the asset pricing literature. For more details see the last paragraphs of Section 2.4.4 in Chapter 2.

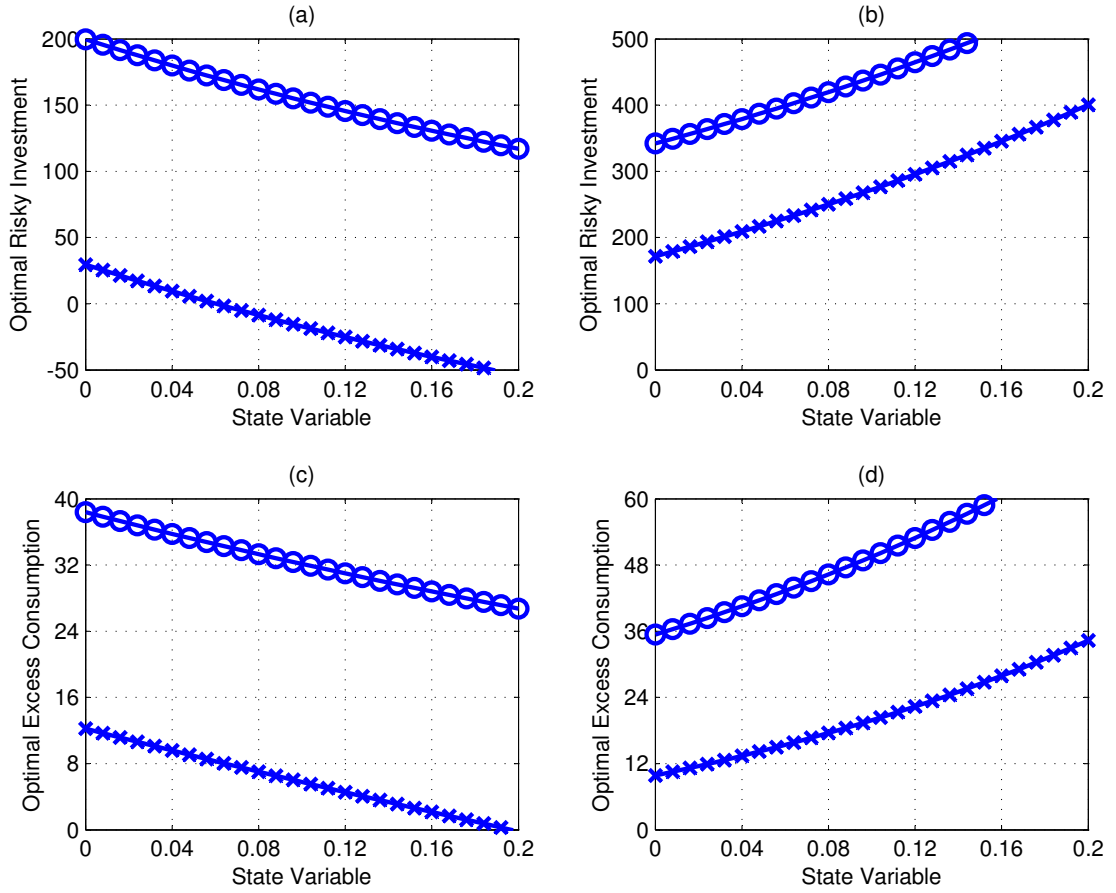


Figure 4.11: Optimal Policies - Life-Cycle Model

This Figure shows the optimal policies dependent on stochastic volatility. Parameters are given as in Table 4.2 and in the text at the beginning of Section 4.5.1. The lines with crosses (circles) show the results for an individual with (without) a phase of retirement. The upper panels exhibit the amount optimally invested in the risky asset $A\pi_t^*$, the lower panels display optimal consumption exceeding the subsistence level $c_t^* - \bar{c}$. In the panels to the left (right) the sensitivity of the labor income growth rate to stochastic volatility is negative (positive) and given by $y_1 = -0.2$ ($y_1 = 0.2$).

4.6 Conclusion

The most important results are the following:

1. The impact of time variation in non-financial income on optimal investment and consumption is important. Assuming time variation in the financial market and ignoring it for non-financial income leads to considerably distinct results.
2. The inclusion of time variation in labor income leads to an adaption of state variable hedging demand. In fact, state variable hedging demand can be separated into the usual part that arises in the absence of labor income and a new part. This part grows monotonically with planning horizon and can have either sign.

3. A negative sensitivity of labor income growth on the state variable can induce falling risky investment and consumption even if the Sharpe ratio of the risky asset is increasing in the state variable.
4. The SODE for the valuation of the labor income stream includes a Riccati differential equation. If the labor income process has very advantageous properties, the value of the future labor income stream can be infinite. However, these cases ask for unrealistically extreme parameter values.
5. The extension of the basic model to a life-cycle model with a phase of retirement is a simple and comprehensible instrument to reduce the value of total wealth. As a consequence, risky investment and excess consumption are on a reasonable level.
6. The reduction of total wealth in the life-cycle model implies that the importance of myopic and state variable hedging demand is reduced compared to the two labor income hedging demands.

Finally, it can be stated that the comments at the end of Chapter 2 remain valid. A higher risk aversion or a higher fraction of stochastic income on total income would lead to results that are even more affected by the labor income dynamics.

4.A Appendix

4.A.1 Solution of the HJB-Equation

The relevant partial derivatives are given by³⁴

$$\begin{aligned}
J_\tau &= e^{-\delta(T-\tau)} \left(\begin{aligned} &\frac{\delta}{1-\gamma} [\dots]^\gamma (\dots)^{1-\gamma} \\ &+ \frac{1}{1-\gamma} [\dots]^{\gamma-1} (\dots)^{1-\gamma} \int_0^\tau \left(\frac{\partial c_0(s)}{\partial s} + \frac{\partial c_1(s)}{\partial s} X \right) e^{C(X,s)} ds \\ &+ \frac{\gamma}{1-\gamma} [\dots]^{\gamma-1} (\dots)^{1-\gamma} + [\dots]^\gamma (\dots)^{-\gamma} \left(\frac{\partial k}{\partial \tau} Y - \frac{\partial R}{\partial \tau} \right) \end{aligned} \right) \\
J_A &= e^{-\delta(T-\tau)} [\dots]^\gamma (\dots)^{-\gamma}, J_{AA} = -\gamma e^{-\delta(T-\tau)} [\dots]^\gamma (\dots)^{-\gamma-1} \\
J_Y &= e^{-\delta(T-\tau)} [\dots]^\gamma (\dots)^{-\gamma} k, J_{YY} = -\gamma e^{-\delta(T-\tau)} [\dots]^\gamma (\dots)^{-\gamma-1} k^2 \\
J_X &= e^{-\delta(T-\tau)} \left(\begin{aligned} &\frac{1}{1-\gamma} [\dots]^{\gamma-1} (\dots)^{1-\gamma} \int_0^\tau c_1(s) e^{C(X,s)} ds \\ &+ [\dots]^\gamma (\dots)^{-\gamma} \frac{\partial k}{\partial X} Y \end{aligned} \right) \\
J_{XX} &= e^{-\delta(T-\tau)} \left(\begin{aligned} &-\frac{1}{\gamma} [\dots]^{\gamma-2} (\dots)^{1-\gamma} \left[\int_0^\tau c_1(s) e^{C(X,s)} ds \right]^2 \\ &+ \frac{1}{1-\gamma} [\dots]^{\gamma-1} (\dots)^{1-\gamma} \int_0^\tau \frac{1}{\gamma} c_1^2(s) e^{C(X,s)} ds \\ &+ 2 [\dots]^{\gamma-1} (\dots)^{-\gamma} \frac{\partial k}{\partial X} Y \int_0^\tau c_1(s) e^{C(X,s)} ds \\ &- \gamma [\dots]^\gamma (\dots)^{-\gamma-1} \left(\frac{\partial k}{\partial X} Y \right)^2 + [\dots]^\gamma (\dots)^{-\gamma} \frac{\partial^2 k}{\partial X^2} Y \end{aligned} \right) \\
J_{AX} &= e^{-\delta(T-\tau)} \left(\begin{aligned} &[\dots]^{\gamma-1} (\dots)^{-\gamma} \int_0^\tau c_1(s) e^{C(X,s)} ds \\ &- \gamma [\dots]^\gamma (\dots)^{-\gamma-1} \frac{\partial k}{\partial X} Y \end{aligned} \right) \\
J_{AY} &= -\gamma e^{-\delta(T-\tau)} [\dots]^\gamma (\dots)^{-\gamma-1} k \\
J_{XY} &= e^{-\delta(T-\tau)} \left(\begin{aligned} &[\dots]^{\gamma-1} (\dots)^{-\gamma} k \int_0^\tau c_1(s) e^{C(X,s)} ds \\ &- \gamma [\dots]^\gamma (\dots)^{-\gamma-1} \frac{\partial k}{\partial X} k Y + [\dots]^\gamma (\dots)^{-\gamma} \frac{\partial k}{\partial X} \end{aligned} \right)
\end{aligned}$$

where for the sake of brevity we define

$$\begin{aligned}
[\dots] &\equiv \left[\int_0^\tau e^{\frac{1}{\gamma}(c_0(s)+c_1(s)X)} ds \right] \\
(\dots) &\equiv (A + k(\tau, X)Y - R(\tau))
\end{aligned}$$

and where

$$C(X, \tau) \equiv \frac{1}{\gamma} (c_0(\tau) + c_1(\tau)X) \quad (4.30)$$

It should be noted that for J_τ , the following rule was applied:

$$f(a, b) = \int_b^a g(x) dx = G(a) - G(b)$$

\Rightarrow

$$\frac{\partial f(a, b)}{\partial a} = \frac{\partial G(a)}{\partial a} = g(a) - g(b) + g(b) = \int_b^a \frac{\partial g(x)}{\partial x} dx + g(b)$$

Only the initial conditions $c_0(0) = c_1(0) = 0$ ensure that J_τ contains $\frac{\gamma}{1-\gamma} [\dots]^{\gamma-1} (\dots)^{1-\gamma}$ and this term is inevitable to find a solution for the HJB.

³⁴More details with respect to the derivation of J_τ can be found at the bottom of the page.

To our knowledge, closed-form solutions for the general problem do not exist. For the sake of analytical solvability one must assume $\rho_{sx} \in \{-1, 1\}$ and $\rho_{sy} \in \{-1, 1\}$ or $\rho_{sx} \in \{-1, 1\}$ and $\sigma_y = 0$.

The solution follows the same steps as in Chapter 2 one-to-one and the reader is referred to Appendix 2.A.2 for more details. The final ODE that can be separated is given by

$$\begin{aligned}
0 = & -\frac{\delta}{1-\gamma}(\dots)[\dots] - \frac{1}{1-\gamma}(\dots) \int_0^\tau \left(\frac{\partial c_0(s)}{\partial s} + \frac{\partial c_1(s)}{\partial s} X \right) e^{C(X,s)} ds - \left(\frac{\partial k}{\partial \tau} Y - \frac{\partial R}{\partial \tau} \right) [\dots] \\
& - \bar{c}[\dots] + r_0(\dots)[\dots] - r_0(kY - R)[\dots] + \bar{Y}[\dots] + Y[\dots] + (y_0 + y_1 X)kY[\dots] \\
& - \frac{1}{1-\gamma} \kappa_x(X - \bar{X})(\dots) \int_0^\tau c_1(s) e^{C(X,s)} ds - \kappa_x(X - \bar{X}) \frac{\partial k}{\partial X} Y[\dots] \\
& + \frac{1}{2} \frac{1}{\gamma} \frac{\lambda_1^2}{\sigma_s^2} X(\dots)[\dots] + \frac{1}{2} \frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 X \int_0^\tau c_1(s) e^{C(X,s)} ds (\dots) \\
& - \frac{1}{2} \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 X \frac{\partial k}{\partial X} Y[\dots] - \frac{1}{2} \frac{\rho_{sy}\sigma_y}{\sigma_s} \lambda_1 X kY[\dots] \\
& + \frac{1}{2} \frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 X(\dots) \int_0^\tau c_1(s) e^{C(X,s)} ds - \frac{1}{2} \rho_{sx}^2 \sigma_x^2 X \frac{\partial k}{\partial X} Y \int_0^\tau c_1(s) e^{C(X,s)} ds \\
& - \frac{1}{2} \rho_{sy}\sigma_y \rho_{sx}\sigma_x X kY \int_0^\tau c_1(s) e^{C(X,s)} ds \\
& - \frac{1}{2} \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 X \frac{\partial k}{\partial X} Y[\dots] - \frac{1}{2} \rho_{sx}^2 \sigma_x^2 X \frac{\partial k}{\partial X} Y \int_0^\tau c_1(s) e^{C(X,s)} ds \\
& - \frac{1}{2} \frac{\rho_{sy}\sigma_y}{\sigma_s} \lambda_1 X kY[\dots] - \frac{1}{2} \rho_{sx}\sigma_x \rho_{sy}\sigma_y X kY \int_0^\tau c_1(s) e^{C(X,s)} ds \\
& + \rho_{xy}\sigma_x \sigma_y X kY \int_0^\tau c_1(s) e^{C(X,s)} ds + \rho_{xy}\sigma_x \sigma_y X \frac{\partial k}{\partial X} Y[\dots] \\
& + \frac{1}{2} \frac{1}{\gamma} \frac{1}{1-\gamma} \sigma_x^2 X(\dots) \int_0^\tau c_1^2(s) e^{C(X,s)} ds \\
& + \sigma_x^2 X \frac{\partial k}{\partial X} Y \int_0^\tau c_1(s) e^{C(X,s)} ds + \frac{1}{2} \sigma_x^2 X \frac{\partial^2 k}{\partial X^2} Y[\dots]
\end{aligned} \tag{4.31}$$

4.A.2 The Dynamics of Total Wealth

From the definition $\hat{A} \equiv A + k(X, t)Y - R(t)$, application of Ito's lemma yields the dynamics of total wealth

$$d\hat{A} = dA + \frac{\partial k}{\partial X} Y dX + \frac{1}{2} \frac{\partial^2 k}{\partial X^2} Y dX^2 + \frac{\partial k}{\partial t} Y dt + k dY + \frac{\partial k}{\partial X} dX dY - \frac{\partial R}{\partial t} dt$$

Plugging in (4.2) - (4.4) and the optimal policies (4.24) and (4.25) leads to

$$\begin{aligned}
d\hat{A}^* = & \left[\frac{1}{\gamma} \frac{\lambda_1^2}{\sigma_s^2} X \hat{A}^* + \frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 X \frac{\int_0^\tau c_1(s) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} \hat{A}^* - \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 X \frac{\partial k}{\partial X} Y - \frac{\rho_{sy}\sigma_y}{\sigma_s} \lambda_1 X kY \right] dt \\
& + r_0 \hat{A}^* - r_0(kY - R) + \bar{Y} + Y - \frac{1}{\int_0^\tau e^{C(X,s)} ds} \hat{A}^* - \bar{c} \\
& + \left(\frac{1}{\gamma} \frac{\lambda_1}{\sigma_s} \sqrt{X} \hat{A}^* + \frac{1}{\gamma} \rho_{sx}\sigma_x \sqrt{X} \frac{\int_0^\tau c_1(s) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} \hat{A}^* \right) dW_s(t) \\
& - \rho_{sx}\sigma_x \sqrt{X} \frac{\partial k}{\partial X} Y - \rho_{sy}\sigma_y \sqrt{X} kY \\
& - \frac{\partial k}{\partial X} Y \kappa_x(X - \bar{X}) dt + \frac{\partial k}{\partial X} Y \sigma_x \sqrt{X} dW_x(t) + \frac{1}{2} \sigma_x^2 X \frac{\partial^2 k}{\partial X^2} Y dt - \frac{\partial k}{\partial \tau} Y dt \\
& + kY(y_0 + y_1 X) dt + kY \sigma_y \sqrt{X} dW_y(t) + \frac{\partial k}{\partial X} \rho_{xy}\sigma_x \sigma_y XY dt + \frac{\partial R}{\partial \tau} dt
\end{aligned} \tag{4.32}$$

Arranging in proper order

$$\begin{aligned}
d\hat{A}^* &= \left(r_0 + \frac{1}{\gamma} \frac{\lambda_1^2}{\sigma_s^2} X + \frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 X \frac{\int_0^\tau c_1(s) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} - \frac{1}{\int_0^\tau e^{C(X,s)} ds} \right) \hat{A}^* dt \\
&+ \left(\frac{1}{\gamma} \frac{\lambda_1}{\sigma_s} + \frac{1}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \frac{\int_0^\tau c_1(s) e^{C(X,s)} ds}{\int_0^\tau e^{C(X,s)} ds} \right) \hat{A}^* \sqrt{X} dW_s(t) \\
&+ \left[\begin{aligned} &-\frac{\partial k}{\partial \tau} + 1 - r_0 k - \frac{\partial k}{\partial X} \kappa_x (X - \bar{X}) + k(y_0 + y_1 X) \\ &-\frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 \frac{\partial k}{\partial X} X - \frac{\rho_{sy}\sigma_y}{\sigma_s} \lambda_1 k X + \frac{\partial k}{\partial X} \rho_{xy} \sigma_x \sigma_y X + \frac{1}{2} \sigma_x^2 X \frac{\partial^2 k}{\partial X^2} \end{aligned} \right] Y dt \\
&+ \left[\frac{\partial R}{\partial \tau} + \bar{Y} + r_0 R - \bar{c} \right] dt \\
&+ [dW_y(t) - \rho_{sy} dW_s(t)] \sigma_y \sqrt{X} k Y + [dW_x(t) - \rho_{sx} dW_s(t)] \sigma_x \sqrt{X} \frac{\partial k}{\partial X} Y
\end{aligned}$$

The last line is equal to zero due to the assumptions about perfect dependence (4.c.1) - (4.c.2) and locally riskfree labor income (4.c.3), i.e. $dW_x(t) = \rho_{sx} dW_s(t)$ and $dW_y(t) = \rho_{sy} dW_s(t)$ or $\sigma_y = 0$. Inspecting the parts in the square brackets, one can identify (4.13) and (4.19), which are also equal to zero. Now, the dynamics of (4.23) follow directly.

4.A.3 A System of Two Ordinary Differential Equations

The system of two ordinary differential equations given by

$$\begin{aligned}
\frac{\partial f_1(s)}{\partial s} &= m_0 + m_1 f_1(s) + m_2 f_1(s)^2 \\
\frac{\partial f_0(s)}{\partial s} &= m_3 + m_4 f_1(s)
\end{aligned}$$

with initial condition $f_1(0) = f_0(0) = 0$ has the following solution³⁵. Defining $q_l \equiv m_1^2 - 4m_0m_2$ and $\eta_m \equiv \sqrt{|q_m|}$.

Case I $q_m > 0$

$$\begin{aligned}
f_1(s) &= \frac{2m_0(1 - e^{-\eta_m s})}{2\eta_m - (m_1 + \eta_m)(1 - e^{-\eta_m s})} \\
\int f_1(s) ds &= \frac{2m_0}{\eta_m - k_1} s + \frac{4k_0}{\eta_m^2 - k_1^2} \ln \left(\frac{2\eta_m - (m_1 + \eta_m)(1 - e^{-\eta_m s})}{2\eta_m} \right) \\
f_0(s) &= m_3 s + m_4 \left(\frac{2m_0}{\eta_m - k_1} s + \frac{4m_0}{\eta_m^2 - k_1^2} \ln \left(\frac{2\eta_m - (m_1 + \eta_m)(1 - e^{-\eta_m s})}{2\eta_m} \right) \right. \\
&\quad \left. - \frac{4m_0}{\eta_m^2 - k_1^2} \ln(1) \right) \\
&= m_3 s + \frac{2m_0 m_4}{\eta_m - m_1} \tau + \frac{4m_0 m_4}{\eta_m^2 - m_1^2} \ln \left(\frac{2\eta_m - (m_1 + \eta_m)(1 - e^{-\eta_m s})}{2\eta_m} \right)
\end{aligned}$$

Absolute value within the natural logarithm is not necessary as $2\eta_m - (m_1 + \eta_m)(1 - e^{-\eta_m s}) > 0$ for $s < s_c$ where s_c is the critical horizon³⁶.

³⁵All solutions are verified by the use of Mathematica (Version 7.0.1.0).

³⁶More information with respect to the critical horizon can be found in Appendix 3.A.2 of Chapter 3.

Case II $q_m = 0$

$$\begin{aligned}
f_1(s) &= -\frac{1}{m_2 \left(s - \frac{2}{m_1}\right)} - \frac{m_1}{2m_2} \\
\int f_1(s) ds &= -\frac{m_1}{2m_2}s - \frac{1}{m_2} \ln \left(\frac{\left| m_1 \left(s - \frac{2}{m_1}\right) \right|}{2} \right) \\
f_0(s) &= m_3s + m_4 \left(-\frac{m_1}{2m_2}s - \frac{1}{m_2} \ln \left(\frac{\left| m_1 \left(s - \frac{2}{m_1}\right) \right|}{2} \right) + 0 + \frac{1}{m_2} \ln(1) \right) \\
&= m_3s + m_4 \left(-\frac{m_1}{2m_2}s - \frac{1}{m_2} \ln \left(\frac{\left| m_1 \left(s - \frac{2}{m_1}\right) \right|}{2} \right) \right)
\end{aligned}$$

Case III $q_m < 0$

$$\begin{aligned}
f_1(s) &= \frac{\eta}{2m_2} \tan(\omega s + \varphi) - \frac{m_1}{2m_2} \\
\int f_1(s) d\tau &= -\frac{m_1}{2m_2}s - \frac{1}{m_2} \ln [\cos(\omega s + \varphi)] \\
f_0(s) &= m_3s + m_4 \left(-\frac{m_1}{2m_2}s - \frac{1}{m_2} \ln [\cos(\omega s + \varphi)] + 0 + \frac{1}{m_2} \ln [\cos(\varphi)] \right) \\
&= m_3s + m_4 \left(-\frac{m_1}{2m_2}s - \frac{1}{m_2} \ln [\cos(\omega s + \varphi)] + \frac{1}{m_2} \ln [\cos(\varphi)] \right)
\end{aligned}$$

Chapter 5

Portfolio and Consumption Decisions with Labor Income and a Volatility Premium and Non-Constant Labor Income Parameter Values

The models of the preceding chapters implied that all parameter values are constant. While this can be seen as a reasonable assumption for the financial market, labor income growth is unlikely to be constant over the employment phase. In fact, Cocco et al. (2005) point out variations in labor income growth over the life cycle¹. Munk and Sørensen (2010) include a similar growth profile in their model.

In this chapter the assumption of constant labor income parameters is relaxed. In the preceding chapters with constant parameter values it was shown that the solutions of the ordinary differential equations, which determine the sign of indirect labor hedging demand, are monotone and do not change sign over the horizon. In the case of non-constant parameter values this does not have to be the case.

It is assumed that the setting is identical to Chapter 4. Nevertheless, the methods described in this chapter could also be applied on the models of Chapters 2 and 3. The primary objective is to show the impact of non-constant labor income parameters on the valuation of the future income stream and the implications for total wealth and the optimal policies.

The rest of this chapter is organized as follows. In Section 5.1 the model with non-constant labor income parameters is introduced. Section 5.2 and 5.3 contain the result of two special cases that can be solved in closed-form. In Section 5.2 only the part of labor income growth that does not vary with the state variable is subject to time-dependence. In Section 5.3 the parameters that determine the effect of the state variable are non-constant. The final section concludes. Appendix 5.A.1 contains the solution of a Riccati differential equation with the initial condition unequal to zero.

¹See Figure 1 and Table 2 in Cocco et al. (2005).

5.1 Model

In the basic model it is assumed that all parameters are constant. However, labor income dynamics are unlikely to be constant over the employment phase. For example, labor income growth might be higher at an early age or older employees could be more exposed to a deterioration of the economic environment.

Under non-constant labor income parameters, it is necessary to pay more attention to the change in variable from t to τ . With non-constant labor income parameters the dynamics of the stochastic part of labor income are given by

$$\frac{dY(t)}{Y(t)} = [y_0(t) + y_1(t)X(t)]dt + \sigma_y(t)\sqrt{X(t)}dW_y(t)$$

and the term in the brackets of equation (4.13) from Chapter 4 is given by

$$0 = \left\{ \begin{aligned} &\frac{\partial k}{\partial t} + 1 - r_0k - \kappa_x X \frac{\partial k}{\partial X} + \kappa_x \bar{X} \frac{\partial k}{\partial \bar{X}} + k[y_0(t) + y_1(t)X] \\ &- \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 \frac{\partial k}{\partial X} X - \frac{\rho_{sy}\sigma_y(t)}{\sigma_s} \lambda_1 kX + \rho_{xy}\sigma_x\sigma_y(t) \frac{\partial k}{\partial X} X + \frac{1}{2}\sigma_x^2 \frac{\partial^2 k}{\partial X^2} X \end{aligned} \right\} \quad (5.1)$$

A function of the form

$$k(X, t, T) = \int_t^T e^{d_0(u) + d_1(u)X} du \quad (5.2)$$

with initial conditions $d_1(u=t) = d_0(u=t) = 0$ will solve equation (5.1).

(5.2) has a natural interpretation. In fact, k is the function that values the future income stream relative to its current value $Y(t)$. The initial conditions imply that the immediate point in time is weighted by $e^0 = 1$ which is intuitive. Future income is considered with respect to its growth rate and risk and the derivation of k with the martingale method in Appendix 3.A.1 of Chapter 3 confirms this notion. A similar statement is made by Wachter (2002). By solving the consumption-investment problem with the martingale method she noticed that wealth can be viewed as a bond that pays consumption as its coupon and hence the total value of wealth is simply the sum over all future consumption². In analogy, the future labor income stream is an asset that pays Y as its coupon.

As in the previous chapters, the transformation $s \equiv u - t$ is made which leads to

$$k(X, \tau) = \int_0^\tau e^{d_0(s|t) + d_1(s|t)X} ds \quad (5.3)$$

Equation (5.3) follows from $ds = du$, the upper boundary $T - t = \tau$ and the lower boundary $t - t = 0$. The operator $(s|t)$ shows that the time variable is s but the parameters depend on the initial point in time t .

It should be noticed that in (5.2) $u \in [t, T]$ while in (5.3) $s \in [0, \tau]$. In words, t (T) of the original time domain matches with 0 (τ) of the new domain.

²See Wachter (2002, p. 69).

Furthermore, because the relevant partial derivatives of (5.3) are given by³

$$\begin{aligned} k_\tau &= \int_0^\tau \left(\frac{\partial d_0(s|t)}{\partial s} + \frac{\partial d_1(s|t)}{\partial s} X \right) e^{d_0(s|t)+d_1(s|t)X} ds + 1 \\ k_X &= \int_0^\tau d_1(s|t) e^{d_0(s|t)+d_1(s|t)X} ds \\ k_{XX} &= \int_0^\tau d_1(s|t)^2 e^{d_0(s|t)+d_1(s|t)X} ds \end{aligned}$$

and $\partial k / \partial t = -\partial k / \partial \tau$, (5.1) can be written as

$$0 = \int_0^\tau e^{d_0(s|t)+d_1(s|t)X} \left\{ \begin{aligned} & - \left(\frac{\partial d_0(s|t)}{\partial s} + \frac{\partial d_1(s|t)}{\partial s} X \right) - r_0 - \kappa_x d_1(s|t) X \\ & + \kappa_x \bar{X} d_1(s|t) + [y_0(s|t) + y_1(s|t) X] \\ & - \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 d_1(s|t) X - \frac{\rho_{sy}\sigma_y(s|t)}{\sigma_s} \lambda_1 X \\ & + \rho_{xy}\sigma_x\sigma_y(s|t) d_1(s|t) X + \frac{1}{2}\sigma_x^2 d_1(s|t)^2 X \end{aligned} \right\} ds \quad (5.4)$$

Matching coefficients on X and the constant term leads to a system of two ordinary differential equations

$$\frac{\partial d_1(s|t)}{\partial s} = l_0(s|t) + l_1(s|t) d_1(s|t) + l_2 d_1(s|t)^2 \quad (5.5)$$

$$\frac{\partial d_0(s|t)}{\partial s} = l_3(s|t) + l_4 d_1(s|t) \quad (5.6)$$

with initial conditions $d_1(s=0|t) = d_0(s=0|t) = 0$ and where

$$l_0(s|t) \equiv y_1(s|t) - \frac{\rho_{sy}\sigma_y(s|t)}{\sigma_s} \lambda_1, \quad l_1(s|t) \equiv -\kappa_x - \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 + \rho_{xy}\sigma_x\sigma_y(s|t)$$

$$l_2 \equiv \frac{1}{2}\sigma_x^2$$

$$l_3(s|t) \equiv y_0(s|t) - r_0, \quad l_4 \equiv \kappa_x \bar{X}$$

It can be recognized that SODE (5.5) - (5.6) is similar to SODE (4.15) - (4.16) with the exception of the time variation in the coefficients l_0 , l_1 and l_3 .

5.2 Time Dependence in y_0 only

Including time dependence in y_0 only does not bring any severe difficulties. In particular, (5.5) has only constant coefficients and thus the solutions of $d_1(s)$ from Chapter 4 are still valid. Since $d_1(s)$ does not depend on the initial point in time t , (5.6) is given by

$$\frac{\partial d_0(s|t)}{\partial s} = y_0(s|t) - r_0 + l_4 d_1(s)$$

Since this equation can be solved by integration, $d_0(s|t)$ is available in closed-form as long as $y_0(s|t)$ is integrable in closed-form. Based on the work of Cocco et al. (2005), Munk and

³More details with respect to the derivation of J_τ can be found in Appendix 2.A.2 of Chapter 2

Sørensen (2010) assume $y_0(s|t)$ has the following structure

$$\begin{aligned} y_0(s|t) &= a_0 + 2a_1(t+s) + 3a_2(t+s)^2 \\ &= a_0 + 2a_1t + 2a_1s + 3a_2t^2 + 6a_2ts + 3a_2s^2 \end{aligned}$$

Defining $\hat{a}_0(t) \equiv a_0 + 2a_1t + 3a_2t^2$ and $\hat{a}_1(t) \equiv a_1 + 3a_2t$ leads to

$$y_0(s|t) = \hat{a}_0(t) + 2\hat{a}_1(t)s + 3a_2s^2$$

Hence, the solutions are given by

$$\begin{aligned} d_1(s) &= \begin{cases} \frac{2l_0(1-e^{-\eta_l s})}{2\eta_l - (l_1 + \eta_l)(1-e^{-\eta_l s})}, & , q_l > 0 \\ -\frac{1}{l_2(s-\frac{2}{l_1})} - \frac{l_1}{2l_2} & , q_l = 0 \\ \frac{\eta_l}{2l_2} \tan(\omega s + \varphi) - \frac{l_1}{2l_2} & , q_l < 0 \end{cases} \\ d_0(s|t) &= (\hat{a}_0(t) - r_0)s + \hat{a}_1(t)s^2 + a_2s^3 \\ &+ \begin{cases} +\frac{2l_0l_4}{\eta_l - l_1}s + \frac{4l_0l_4}{\eta_l^2 - l_1^2} \ln\left(\frac{2\eta_l - (l_1 + \eta_l)(1-e^{-\eta_l s})}{2\eta_l}\right) & , q_l > 0 \\ -\frac{l_1l_4}{2l_2}s - \frac{l_4}{l_2} \ln\left(\frac{|l_1(s-\frac{2}{l_1})|}{2}\right) & , q_l = 0 \\ -\frac{l_1l_4}{2l_2}s + \frac{l_4}{l_2} [\cos(\omega s + \varphi) - \cos(\varphi)] & , q_l < 0 \end{cases} \end{aligned}$$

Remarks

- Since the solution of $d_1(s)$ is unchanged, the properties from Chapter 4 remain true, i.e. $d_1(s)$ is monotone and does not change sign over the horizon. As a consequence, the qualitative properties of total wealth and the optimal policies of the aforementioned chapter are unchanged.
- However, more realistic growth pattern can considerably affect the magnitude of total wealth and the optimal policies. This is intuitive since

$$k = \int_0^\tau e^{d_0(s|t) + d_1(s)X} ds$$

and

$$k_X = \int_0^\tau d_1(s) e^{d_0(s|t) + d_1(s)X} ds$$

show the impact of changes in $d_0(s|t)$. Imagine two individuals with equal average income growth but the first individual has a higher growth rate during the first phase of employment. For this individual, high growth rates in the beginning lead to a rapid increase in $d_0(s|t)$ and the higher $e^{d_0(s|t)}$ generates a larger area under the integral. For this reason and because $d_1(s)$ is the same, both k and k_X become greater for the first individual compared to the second and the results amplify. This statement is illustrated in the next section.

5.2.1 Illustration of the Results

Parameters of the financial market and the individual are displayed in Table 5.1. For the sake of simplicity, the individual is assumed to have locally riskfree labor income. All other values are similar to Table 4.2 of Chapter 4.

Financial Market		
$r_0 = 0.0050$		
$\lambda_1 = 1.0000$	$\sigma_s = 1.0000$	
$\kappa_x = 0.1000$	$\bar{X} = 0.0400$	$\sigma_x = 0.0323$
$\rho_{sx} = -1$		
Individual		
$\gamma = 4$	$\delta = 0.06$	
$\bar{y} = 0.03$	$\sigma_y = 0$	
$A(0) = 50$	$Y(0) = 10$	$\bar{Y} = 40$
$\bar{c} = 45$		

Table 5.1: Parameter Values

As shown in Table 5.2, three scenarios are looked at⁴. In the first case *i*), the individual has constant labor income growth and this case therefore coincides with the model of Chapter 4. In the second case *ii*), the individual has a linear trend in labor income growth. In particular, labor income growth is high at the beginning and decreases over time. In the third case *iii*), the quadratic term generates an even more pronounced growth in the beginning and lower growth towards the end of the horizon.

<i>i</i>)	$a_0 = 0.0380 (0.0220)$	$a_1 = a_2 = 0$	
<i>ii</i>)	$a_0 = 0.0980 (0.0820)$	$a_1 = -0.0015$	$a_2 = 0$
<i>iii</i>)	$a_0 = 0.1113 (0.0953)$	$a_1 = -0.0025$	$a_2 = 0.8333 \cdot 10^{-5}$

Table 5.2: $y_0(t)$ Growth Parameters

For the sake of comparability, we restrict

$$\int_0^T a_0 + 2a_1t + 3a_2t^2 dt \quad (5.7)$$

to be constant. Moreover, it should be noticed that for the case *i*)

$$a_0 = \bar{y} - y_1 \bar{X}$$

For the second (third) case, a_1 (a_1 and a_2) were chosen exogenously and a_0 is chosen so that (5.7) is fulfilled. The term in the brackets of Table 5.2 belongs to the individual with $y_1 = 0.2$

⁴The signs chosen are identical to Munk and Sørensen (2010, p. 455).

and the regular term belongs to the individual with $y_1 = -0.2$. The growth profiles of $y_0(t)$ are displayed in Figure 5.1.

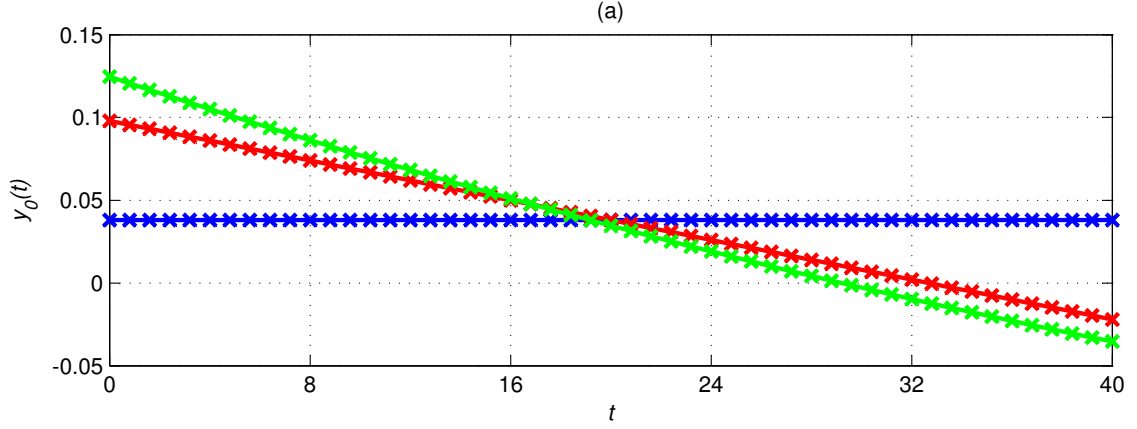


Figure 5.1: Growth Profiles

This Figure shows the growth profile as displayed in Table 5.2 for the individual with a negative sensitivity of the labor income growth rate to stochastic volatility of $y_1 = -0.2$. Parameters are given as in Table 5.1. The blue (red, green) line shows the case *i*) (*ii*), *iii*) as described in Table 5.2.

Figure 5.1 contains the growth profile for the individual with a negative sensitivity of labor income growth onto X . For the individual with a positive sensitivity the growth profile looks identical with the exception of a change in level. It should be recognized that for the cases *ii*) and *iii*), income growth at the beginning of the working period is considerably higher and turns negative towards the end of the working period.

Figure 5.2 shows the value of total wealth. It can be clearly recognized that the statement from above is confirmed, i.e. high labor income growth at the beginning of the working period matters indeed. The numerical example reveals that compared to the base case *i*), cases *ii*) and *iii*) show that total wealth approximately doubles. The result is intuitive since labor income is a flow, and a fast growth at the beginning leads to more income over the phase of employment even if final income is the same. Besides, because $d_1(s)$ is unchanged the statements from Section 4.4.1 of Chapter 4 with respect to the properties of total wealth remain valid.

Figure 5.3 shows the impact on optimal investment. Since myopic demand and state variable hedging demand are only affected by the changes in total wealth they are omitted. In particular, the increased value of total wealth increases these two demands.

Panels (c) and (d) show the impact on indirect labor hedging demand which becomes larger in magnitude. For the individual to the right, since indirect labor hedging demand in case of $y_1 > 0$ is positive, all components of risky investment are unambiguously greater and hence, risky investment in cases *ii*) and *iii*) is higher compared to case *i*).

This does not have to be the case for the individual to the left. On the one hand, $y_1 < 0$ implies negative indirect labor hedging demand, which lowers risky investment for the cases *ii*) and *iii*) compared to case *i*). On the other hand, because of the higher total wealth, myopic demand

and state variable hedging demand increase. As a consequence there are ambiguous effects for the case $y_1 < 0$ and the change in risky investment depends on the specific parameter values. In the numerical example the impact of higher myopic and state variable demand is stronger and hence, risky investment also rises.

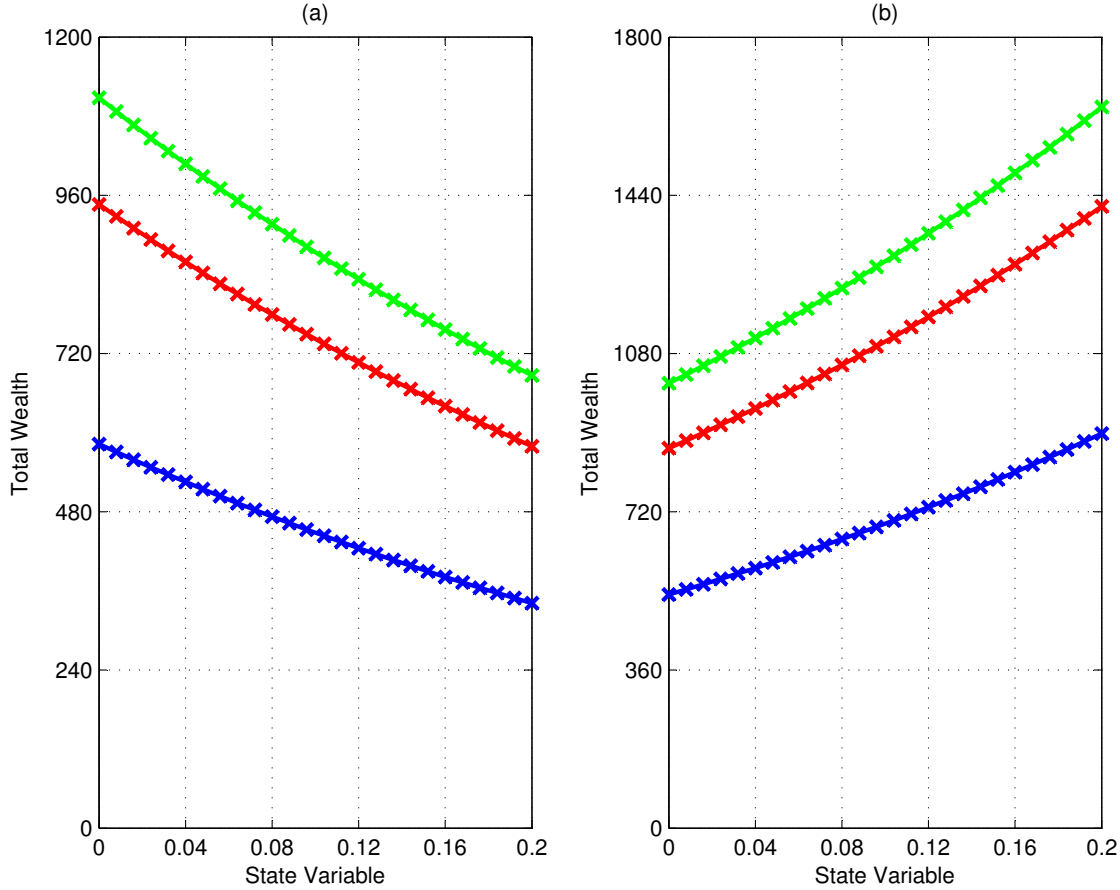


Figure 5.2: Total Wealth - Locally Riskfree Labor Income

This Figure shows total wealth \hat{A} dependent on stochastic volatility. Parameters are given as in Table 5.1. The blue (red, green) lines show the cases *i*), *ii*) and *iii*) as described in Table 5.2. In the panel to the left (right) the sensitivity of the labor income growth rate to stochastic volatility is positive (negative) and given by $y_1 = -0.2$ ($y_1 = 0.2$).

The differences in excess consumption are simple and a figure with the results is thus omitted. In fact, from optimal consumption (4.24) it can be recognized that the denominator is the same for all cases. Hence, changes in excess consumption stem exclusively from changes in total wealth, which are described above.

It can be summarized that the inclusion of more realistic growth pattern for labor income leads to a considerably higher value of the future labor income stream. With the exception of scenarios with strongly negative y_1 and a high fraction of stochastic labor income, higher total wealth will induce higher risky investment and higher consumption.

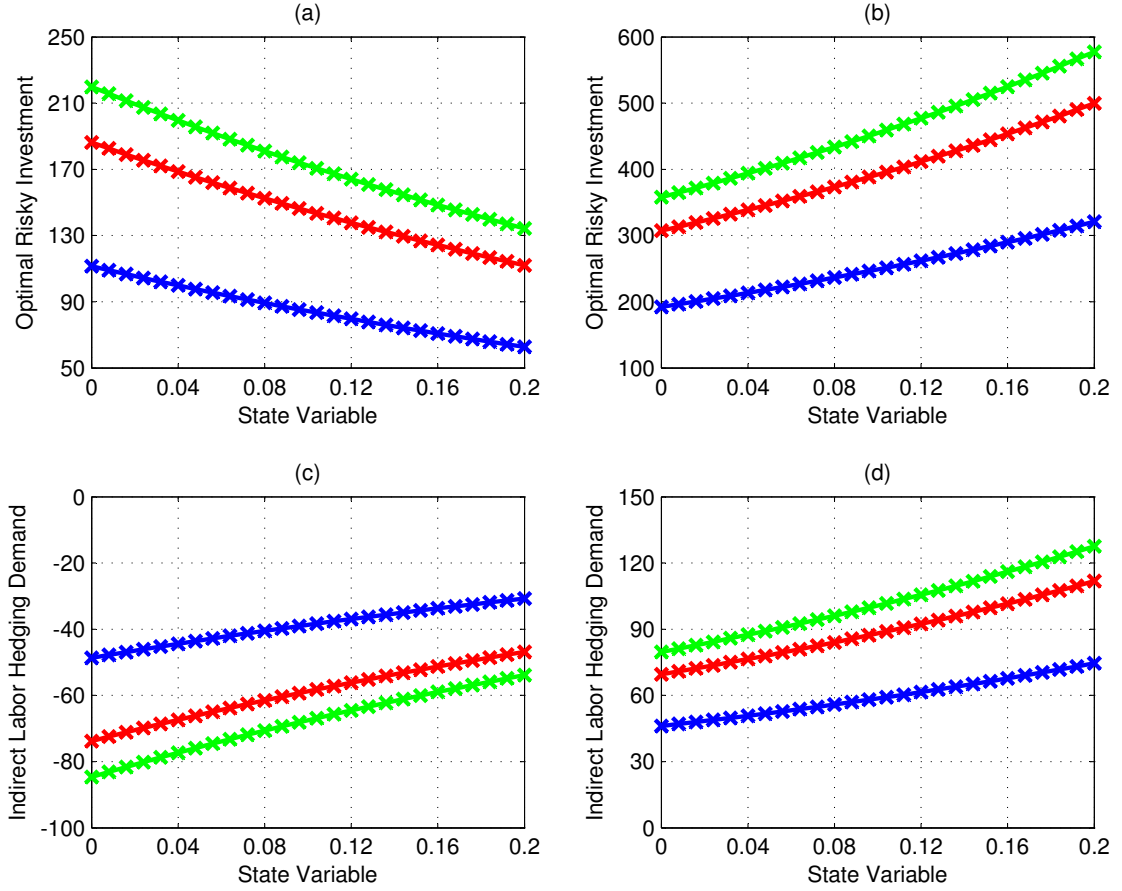


Figure 5.3: Optimal Risky Investment - Locally Riskfree Labor Income

Panels (a) and (b) show the amount optimally invested in the risky asset $A\pi_t^*$ dependent on stochastic volatility. Panels (c) and (d) display indirect labor hedging demand as described in equation (4.27) of Chapter 4. Parameters are given as in Table 5.1. The blue (red, green) lines show the cases *i*), *ii*) and *iii*) as described in Table 5.2. In the panel to the left (right) the sensitivity of the labor income growth rate to stochastic volatility is positive (negative) and given by $y_1 = -0.2$ ($y_1 = 0.2$).

5.3 Time Dependence in y_1 and σ_y

Time variation in y_1 and σ_y is more complicated. To our knowledge, closed-form solutions to Riccati differential equations with non-constant coefficients are only available in very few special cases⁵. Nevertheless, one is able to find closed form solutions for $d_1(t)$ if y_1 and σ_y are piecewise constant. The assumption of piecewise constant y_1 and the assumption of constant labor income growth at the long-run mean $X = \bar{X}$ imply that y_0 must also be piecewise constant. Furthermore, also assuming piecewise constant y_0 enables different income growth for different time periods, which is a desirable feature on its own.

For the sake of simplicity, we will assume that the employment phase is divided into two parts

⁵See Boyle et al. (2002) for an example.

and

$$\begin{aligned} y_0(t) &= y_{j,0}, & y_1(t) &= y_{j,1}, & \sigma_y(t) &= \sigma_{j,y}, & t &\leq T_1 \\ y_0(t) &= y_{s,0}, & y_1(t) &= y_{s,1}, & \sigma_y(t) &= \sigma_{s,y}, & T_1 < t &\leq T_2 \end{aligned}$$

where T_2 is the end of the planning horizon and T_1 is the jump date.

For the second period, $T_1 < t \leq T_2$, the solution of the SODE is easy because for this period the solution is given by the constant parameters solution of the previous chapter with

$$\begin{aligned} l_{s,0} &\equiv y_{s,1} - \frac{\rho_{sy}\sigma_{s,y}}{\sigma_s}\lambda_1, & l_{s,1} &\equiv -\kappa_x - \frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1 + \rho_{xy}\sigma_x\sigma_{s,y} \\ l_{s,3} &\equiv y_{s,0} - r_0 \end{aligned}$$

Including the first period, $0 \leq t \leq T_1$, makes the solution more extensive. As shown in (5.5), for $0 \leq s \leq T_1 - t$

$$\begin{aligned} \frac{\partial d_{j,1}(s)}{\partial s} &= l_{j,0} + l_{j,1}d_{j,1}(s) + l_{j,2}d_{j,1}(s)^2 \\ \frac{\partial d_{j,0}(s)}{\partial s} &= l_3 + l_4d_{j,1}(s) \end{aligned}$$

with initial conditions $d_1(s=0) = d_0(s=0) = 0$ and where

$$\begin{aligned} l_{j,0} &\equiv y_{j,1} - \frac{\rho_{sy}\sigma_{j,y}}{\sigma_s}\lambda_1, & l_{j,1} &\equiv -\kappa_x - \frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1 + \rho_{xy}\sigma_x\sigma_{j,y} \\ l_{j,3} &\equiv y_{j,0} - r_0 \end{aligned}$$

For $T_1 - t < s \leq T_2 - t$, the system is given by

$$\begin{aligned} \frac{\partial d_{s,1}(s|t)}{\partial s} &= l_{s,0} + l_{s,1}d_{s,1}(s|t) + l_{s,2}d_{s,1}(s|t)^2 \\ \frac{\partial d_{s,0}(s|t)}{\partial s} &= l_3 + l_4d_{s,1}(s|t) \end{aligned}$$

Initial conditions for the second period are given by $d_{s,1}(s = T_1 - t|t) = d_{j,1}(s = T_1 - t)$ and $d_{s,0}(s = T_1 - t|t) = d_{j,0}(s = T_1 - t)$. In other words, the solution of a two period problem involves solving a Riccati differential equation with an initial value not equal to zero. The detailed derivation of the solution can be found in Appendix 5.A.1.

For $T_1 < t \leq T_2$, the problem consists only of the second period and thus the solution is completely analogous to the constant coefficient case where l_0 (l_1) has to be replaced by $l_{s,0}$ ($l_{s,1}$). Hence,

$$d_1(s) = d_{s,1}(s) = \begin{cases} \frac{2l_{s,0}(1-e^{-\eta_{s,l}s})}{2\eta_{s,l} - (l_{s,1} + \eta_{s,l})(1-e^{-\eta_{s,l}s})} & , \quad q_{s,l} > 0 \\ -\frac{1}{l_2\left(s - \frac{2}{l_{s,1}}\right)} - \frac{l_{s,1}}{2l_2} & , \quad q_{s,l} = 0 \\ \frac{\eta_{s,l}}{2l_2} \tan(\omega_{s,l}s + \varphi_{s,l}) - \frac{l_{s,1}}{2l_2} & , \quad q_{s,l} < 0 \end{cases}$$

where $q_{s,l} \equiv l_{s,1}^2 - 4l_{s,0}l_{s,1}$ and $\eta_{s,l} \equiv \sqrt{|q_{s,l}|}$ and where $\omega_{s,l} \equiv \frac{\eta_{s,l}}{2}$ and $\varphi_{s,l} \equiv \arctan\left(\frac{l_{s,1}}{\eta_{s,l}}\right)$. For

the case $0 \leq t \leq T_1$, the solution for $0 \leq s \leq T_1 - t$ is given by

$$d_1(s|t) = d_{j,1}(s) = \begin{cases} \frac{2l_{j,0}(1-e^{-\eta_{j,l}s})}{2\eta_{j,l}-(l_{j,1}+\eta_{j,l})(1-e^{-\eta_{j,l}s})} & , q_{j,l} > 0 \\ -\frac{1}{l_2\left(s-\frac{2}{l_{j,1}}\right)} - \frac{l_{j,1}}{2l_2} & , q_{j,l} = 0 \\ \frac{\eta_{j,l}}{2l_2} \tan(\omega_{j,l}s + \varphi_{j,l}) - \frac{l_{j,1}}{2l_2} & , q_{j,l} < 0 \end{cases}$$

and for $T_1 - t < s \leq T_2 - t$

$$d_1(s|t) = d_{j,1}(\tau_1) + \begin{cases} \frac{2\hat{l}_{s,0}(1-e^{-\eta_{s,l}(s-\tau_1)})}{2\eta_{s,l}-(\hat{l}_{s,1}+\eta_{s,l})(1-e^{-\eta_{s,l}(s-\tau_1)})} & , q_{s,l} > 0 \\ -\frac{1}{l_2\left((s-\tau_1)-\frac{2}{\hat{l}_{s,1}}\right)} - \frac{\hat{l}_{s,1}}{2l_2} & , q_{s,l} = 0 \\ \frac{\eta_{s,l}}{2l_2} \tan(\hat{\omega}_{s,l}(s-\tau_1) + \hat{\varphi}_{s,l}) - \frac{\hat{l}_{s,1}}{2l_2} & , q_{s,l} < 0 \end{cases}$$

where $\tau_1 \equiv T_1 - t$, $\hat{l}_{s,0} \equiv l_{s,0} + l_{s,1}d_{j,1}(\tau_1) + l_2d_{j,1}(\tau_1)^2$ and $\hat{l}_{s,1} \equiv l_{s,1} + 2l_2d_{s,1}(\tau_1)$ and where $\hat{\omega}_{s,l}, \hat{\varphi}_{s,l}, q_{j,l}$ and $\eta_{j,l}$ are defined in analogy to the above⁶.

The solution for $d_0(s)$ follows the same steps. For $T_1 < t \leq T_2$

$$\begin{aligned} d_0(s) &= d_{s,0}(s) \\ &= \begin{cases} l_{s,3}s + \frac{2l_{s,0}l_4}{\eta_l - l_{s,1}}s + \frac{4l_{s,0}l_4}{\eta_l^2 - l_{s,1}^2} \ln\left(\frac{2\eta_l - (l_{s,1} + \eta_l)(1-e^{-\eta_l s})}{2\eta_l}\right) & , q_{s,l} > 0 \\ l_{s,3}s - \frac{l_{s,1}l_4}{2l_2}s - \frac{l_4}{l_2} \ln\left(\frac{\left|l_{s,1}\left(s-\frac{2}{l_{s,1}}\right)\right|}{2}\right) & , q_{s,l} = 0 \\ l_{s,3}s - \frac{l_{s,1}l_4}{2l_2}s + \frac{l_4}{l_2} [\cos(\omega_{s,l}s + \varphi_{s,l}) - \cos(\varphi_{s,l})] & , q_{s,l} < 0 \end{cases} \end{aligned}$$

For the case $0 \leq t \leq T_1$, the solution for $0 \leq s \leq T_1 - t$

$$\begin{aligned} d_0(s|t) &= d_{j,0}(s) \\ &= \begin{cases} l_{j,3}s + \frac{2l_{j,0}l_4}{\eta_l - l_{j,1}}s + \frac{4l_{j,0}l_4}{\eta_l^2 - l_{j,1}^2} \ln\left(\frac{2\eta_l - (l_{j,1} + \eta_l)(1-e^{-\eta_l s})}{2\eta_l}\right) & , q_{j,l} > 0 \\ l_{j,3}s - \frac{l_{j,1}l_4}{2l_2}s - \frac{l_4}{l_2} \ln\left(\frac{\left|l_{j,1}\left(s-\frac{2}{l_{j,1}}\right)\right|}{2}\right) & , q_{j,l} = 0 \\ l_{j,3}s - \frac{l_{j,1}l_4}{2l_2}s + \frac{l_4}{l_2} [\cos(\omega_{j,l}s + \varphi_{j,l}) - \cos(\varphi_{j,l})] & , q_{j,l} < 0 \end{cases} \end{aligned}$$

⁶It should be noted that $q_{s,l} = l_{s,1}^2 - 4l_{s,0}l_{s,2}$ is unchanged. In fact, $\hat{l}_{s,1}^2 - 4\hat{l}_{s,0}l_{s,2} = l_{s,1}^2 + 4l_2l_{s,1}y + 4l_2^2y^2 - 4l_2[l_{s,0} + l_{s,1}y + l_2y^2] = l_{s,1}^2 - 4l_{j,0}l_{j,2}$. This result is intuitive because Appendix 5.A.1 shows that the shape and the vertical position of the parabola are not changed by the transformation.

and for $T_1 - t < s \leq T_2 - t$

$$d_0(s|t) = d_{j,0}(\tau_1) + l_{s,3}[s - \tau_1] + l_4 d_{j,1}(\tau_1)[s - \tau_1] + \begin{cases} + \frac{2\hat{l}_{s,0}l_4}{\eta_s - \hat{l}_{s,1}}[s - \tau_1] + \frac{4\hat{l}_{s,0}l_4}{\eta_s^2 - \hat{l}_{s,1}^2} \ln \left(\frac{2\eta_s - (\hat{l}_{s,1} + \eta_s)(1 - e^{-\eta_s[s - \tau_1]})}{2\eta_s} \right) & , q_{s,l} > 0 \\ - \frac{\hat{l}_{s,1}l_4}{2l_2}[s - \tau_1] - \frac{l_4}{l_2} \ln \left(\frac{|\hat{l}_{s,1}([s - \tau_1] - \frac{2}{\hat{l}_{s,1}})|}{2} \right) & , q_{s,l} = 0 \\ - \frac{\hat{l}_{s,1}l_4}{2l_2}[s - \tau_1] + \frac{l_4}{l_2} [\cos(\hat{\omega}_{s,l}[s - \tau_1] + \hat{\varphi}_{s,l}) - \cos(\hat{\varphi}_{s,l})] & , q_{s,l} < 0 \end{cases}$$

Remarks

- Piecewise constant y_1 and σ_y allow $d_1(s|t)$ to evolve differently over the two periods. Most notably, $d_1(s|t)$ can change sign at a certain point of the horizon.
- Combined with the monotonically decreasing hedging demand, this can generate a variety of patterns of the risky asset allocation over the horizon.
- Introducing more subperiods could generate even more sophisticated pattern.

5.3.1 Illustration of the Results

The primary differences that arise in the presence of piecewise constant labor income parameters are changes in the valuation of the income stream $k(X, \tau)$. For this reason, the illustration of the results focus on this factor. A detailed discussion of total wealth and the optimal policies is omitted. Nevertheless, important implications for total wealth and the optimal policies are pointed out explicitly. Parameters for the financial market and the individual are given as in Table 5.1. It should be kept in mind that for the sake of simplicity it is assumed that labor income is locally riskfree.

Furthermore, it is assumed that the entire planning horizon is $T_2 = 40$ years and the jump date is at $T_1 = 20$ years.

i)	$y_{j,1} = -0.2$	$y_{s,1} = -0.2$
ii)	$y_{j,1} = -0.4$	$y_{s,1} = 0$
iii)	$y_{j,1} = 0$	$y_{s,1} = -0.4$

Table 5.3: Parameter Values - Piecewise Constant y_1

As can be seen from Table 5.3, we restrict the discussion on a setting where the individual has a negative sensitivity on the state variable⁷. The first scenario *i*) is similar to the constant parameter case since for both periods y_1 is the same. In the second case *ii*) the individual faces a strong negative sensitivity in the first period and no sensitivity during the second period. The third case *iii*) is similar to the second one, but the periods are interchanged. Over the entire

⁷Further results can be derived in analogy.

horizon, the sensitivity to changes in X is on average the same. For the sake of comparability, labor income growth at the long-run mean $X = \bar{X}$ is the same

$$y_{i,0} = \bar{y} - y_{i,1}\bar{X}, \quad i \in \{j, s\}$$

Figure 5.4 shows the development of $k(X, \tau)$ over the time horizon. The blue (red, green) lines display case $i)$ ($ii)$, $iii)$). In order to see the impact of different states, the lines with crosses (circles, squares) show the state $X = \bar{X}$ ($X = 0$, $X = 0.08$).

The state variable dimension does not show any surprising results. Since labor income growth is high (low) for low (high) states of X , the lines with circles (squares) lie above (below) the line with crosses. Furthermore, it should be noticed that for the second case (red lines), there is no sensitivity of k over states for a horizon below 20 years. This is intuitive as this individual faces no state variable sensitivity for the second period.

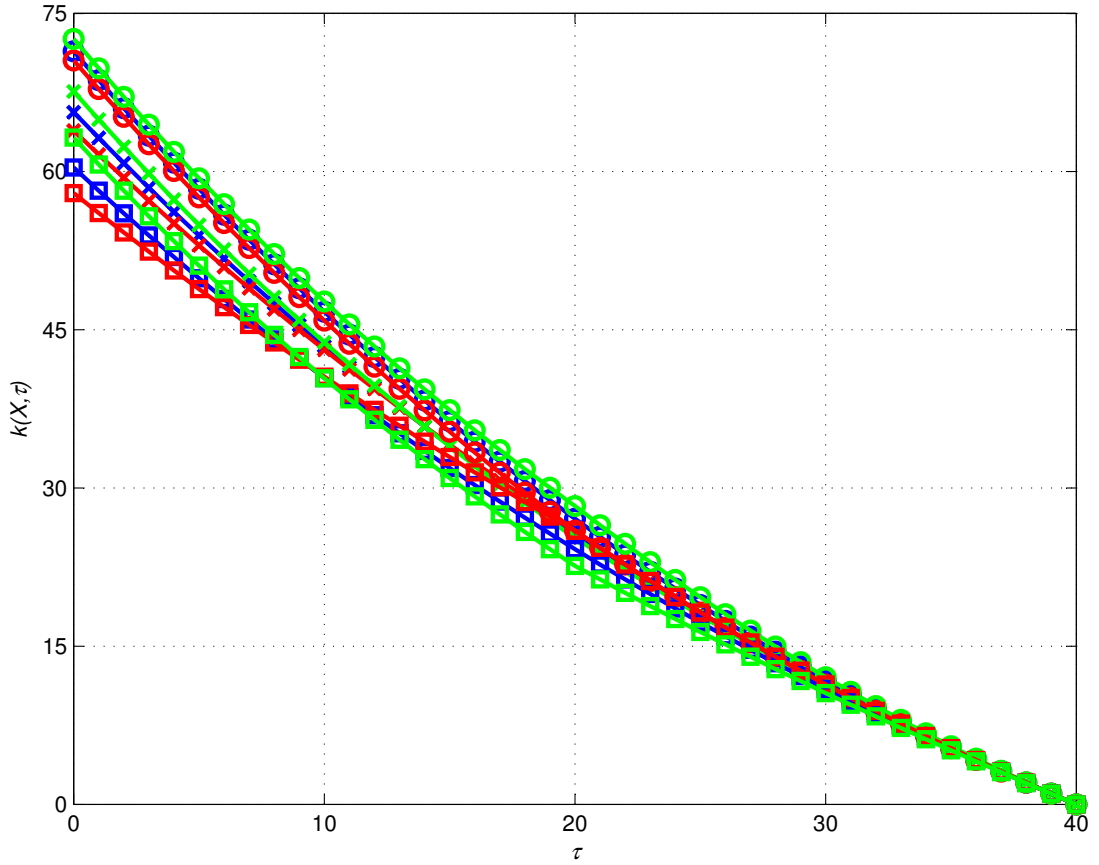


Figure 5.4: $k(X, \tau)$ Piecewise Constant y_1

This Figure shows $k(X, \tau)$ for the model with piecewise constant labor income parameters. Parameter values are given as in Table 5.1 and Table 5.3. The blue (red, green) lines show the cases $i)$, $ii)$ and $iii)$ as described in Table 5.3. The line with crosses (circles, squares) shows the state $X = \bar{X}$ ($X = 0$, $X = 0.08$).

The differences between the three cases are rather small given the state. This result is intuitive because the growth rate at the long-run mean is the same for all cases by assumption. Since

total wealth and optimal consumption depend only on $k(X, \tau)$ (but not on $\partial k(X, \tau) / \partial X$), they are only slightly affected by the inclusion of piecewise constant labor income parameters.

The changes in $\partial k(X, \tau) / \partial X$ are far more interesting and displayed in Figure 5.5. Since this term is determined for indirect labor hedging demand, differences in the optimal investment policies will arise.

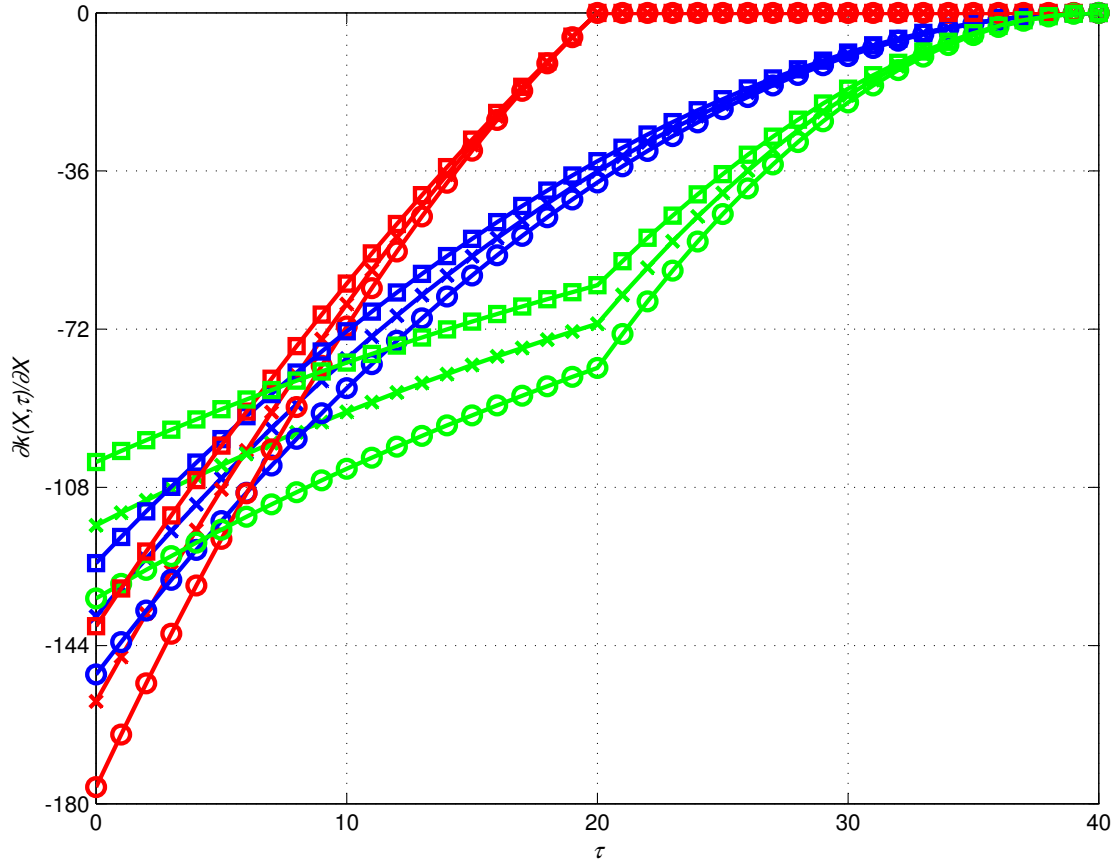


Figure 5.5: $\partial k(X, \tau) / \partial X$ Piecewise Constant y_1

This Figure shows $\partial k(X, \tau) / \partial X$ for the model with piecewise constant labor income parameters. Parameter values are given as in Table 5.1 and Table 5.3. The blue (red, green) lines show the cases *i*), *ii*) and *iii*) as described in Table 5.3. The line with crosses (circles, squares) shows the state $X = \bar{X}$ ($X = 0$, $X = 0.08$).

It can be recognized that the development of

$$k_X = \int_0^\tau d_1(s|t) e^{d_0(s|t) + d_1(s|t)X} ds \quad (5.8)$$

is directly influenced by $d_1(s|t)$. The blue lines show that for the constant parameter case, the development of $\partial k(X, \tau) / \partial X$ is a smooth function over the horizon. Thus, the statement from the phase plane analysis of Chapter 4 that k and k_X are monotone over time is confirmed. For the red case, $\partial k(X, \tau) / \partial X$ is strong in magnitude for long horizon and decreases fast with the horizon. At the jump date, the value is equal to zero, which is intuitive as there is no sensitivity for the second period. As a consequence, indirect labor hedging demand vanishes after the first

period and the optimal investment policy is solely determined by myopic and state variable hedging demand.

The third case is the most interesting. Even if the sensitivity of labor income growth on state variable is zero during the first phase, $\partial k(X, \tau) / \partial X$ is not zero. Furthermore, the first phase $\partial k(X, \tau) / \partial X$ is more stable over time and remains important even for intermediate horizons. This pattern can be explained as follows:

- During the first period, $0 \leq \tau < T_1 = 20$, $\partial k(X, \tau) / \partial X$ changes only because of the longer horizon that leads to changes in $d_0(s|t)$. In fact, defining $\tau_1 \equiv T_1 - t$, $\tau_2 \equiv T_2 - t$ and rewriting (5.8) as

$$k_X = \underbrace{\int_0^{\tau_1} d_1(s|t) e^{d_0(s|t) + d_1(s|t)X} ds}_{=0} + \int_{\tau_1}^{\tau_2} d_1(s|t) \underbrace{e^{d_0(s|t) + d_1(s|t)X}}_{\omega} ds$$

reveals that the first part on the RHS is zero because from the analytical derivation it can be seen that $d_1(s|t) = 0$, $s \leq \tau_1$. Nevertheless, in the second term on the RHS $d_0(s|t)$ changes with horizon. In the numerical example, l_3 is assumed to be positive and thus $d_0(s|t)$ becomes higher for longer horizons. The weighting factor ω therefore gives more weight to $d_1(s|t)$ and k_X has a higher magnitude for longer horizon.

- As shown in the analytical part, the problem for the second period is analogous to the constant parameter framework. Compared to the constant case *i*), the higher magnitude of $y_{s,1} = -0.4$ leads to a faster decrease of k_X during the second period as τ goes towards zero.

The third case reveals that even if the labor income of a young individual is not exposed to changes in the state variable, the individual has to take them into consideration if labor income at a later phase of life is exposed to changes in the states.

5.4 Conclusion

The inclusion of non-constant labor income parameters give new insights. In addition to the results of the basic model presented in Chapter 4 the most important results are the following:

1. Time dependence in y_0 is rather simple to implement. In fact, $d_0(s)$ is available in closed-form as long as $y_0(s|t)$ is integrable in closed-form.
2. The inclusion of high labor income growth at the beginning of the working period leads to a higher valuation of the future income stream. As a consequence, the importance of labor income on the optimal policies increases.
3. Time dependence in y_1 or σ_y is difficult to implement because closed-form solutions of Riccati differential equations with time-varying coefficients only exist in a few special cases. Nevertheless, closed-form solutions can be found for piecewise constant parameters.

4. The analytical results show that non-constant labor income parameters allow for more sophisticated patterns of k and especially k_X over the time horizon. Under constant parameters, k_X is either positive or negative over the entire horizon. This does not have to be the case under non-constant parameters.
5. Even if labor income from young individuals is not exposed to changes in the state of the economy, the valuation of the future income stream depends on X if income is exposed to changes in the state at a later time period. In this case, the sensitivity of the value of the future income stream of a young individual is rather stable over time.

5.A Appendix

5.A.1 Riccati Differential Equation with $y(0) \neq 0$

The solution of the model of Section 5.3 includes the task of solving a Riccati differential equation with the initial condition not equal to zero. Nevertheless, the solution to this problem is rather simple because every Riccati differential equation with initial value \bar{y} can be transformed into an equation with an initial value equal to zero. In fact, starting from

$$\frac{dy(t)}{dt} = m_0 + m_1 y(t) + m_2 y(t)^2$$

with initial condition $y(T) = \bar{y}$, the simple transformation⁸

$$\hat{y}(t) \equiv y(t) - \bar{y}$$

leads to $\hat{y}(T) = 0$ and

$$\frac{d\hat{y}(t)}{dt} = \frac{dy(t)}{dt}$$

Hence,

$$\begin{aligned} \frac{d\hat{y}(t)}{dt} &= m_0 + m_1 (\hat{y}(t) + \bar{y}) + m_2 (\hat{y}(t) + \bar{y})^2 \\ &= \hat{m}_0 + \hat{m}_1 \hat{y}(t) + \hat{m}_2 \hat{y}(t)^2 \end{aligned} \tag{5.9}$$

where $\hat{m}_0 \equiv m_0 + m_1 \bar{y} + m_2 \bar{y}^2$, $\hat{m}_1 \equiv m_1 + 2m_2 \bar{y}$ and $\hat{m}_2 \equiv m_2$. Now, the standard formulas can be applied on (5.9).

⁸Graphically, the transformation is simply a horizontal shift of the parabola from \bar{y} in to the origin.

Chapter 6

Conclusion

Section 6.1 of this final chapter aims to give a short overview of the most important results. A detailed discussion is not intended and more information can be found at the end of the introduction and at the end of the corresponding chapters. In Section 6.2, we take a second look at the critical assumptions that must be taken for the sake of closed-form solutions and point out possible topics for future research.

6.1 Summary

In this thesis, the consumption and portfolio optimization problem of an investor facing time-varying investment opportunities and dynamic non-financial income is solved with analytical methods. The models focus on this dimension and additional features are either completely neglected (as, for example, life-time uncertainty) or presented as extensions (life-cycle models with a phase of retirement). This approach allows the impact of dynamic labor income to be studied in a pure form and allows us to evaluate its importance.

In Chapter 2, a model with time variation in the expected return of the risky asset and time variation in labor income growth is introduced. Time variation is driven by a state variable that follows an Ornstein-Uhlenbeck process. The financial market setting is identical to Wachter (2002) and Campbell et al. (2004). The framework was extended to stochastic labor income volatility in Chapter 3. The subsequent chapter presents a model with stochastic volatility for the risky asset and an affine volatility premium. Similarly to the risky asset, labor income is also assumed to have time-varying growth and volatility. In this model, stochastic volatility follows a CIR-process and a similar model without labor income is presented by Liu (2007) as a special case of his general model. In Chapter 5, the assumption of constant labor income parameters over the time horizon is relaxed.

From a technical point of view, the assumption of perfect correlation of the state variable and the risky asset and the assumption of perfect correlation of labor income and the risky asset or locally riskfree labor income (complete markets) allow for a separation of the complicated HJB-equation into ordinary differential equations. This is common to all models presented in this thesis and the same statement is valid for models with time-varying investment opportunities

without labor income. However, the resulting systems of ordinary differential equations are different from case to case and have to be solved by appropriate methods. The separability of the HJB-equation is intuitive. The reason is the assumption of complete markets, which enables the individual to control total wealth from becoming negative and this implies that she is able to afford future subsistence consumption in all cases.

The most striking result is that counter-cyclical non-financial income growth (income growth is low when expected returns are high) or pro-cyclical income volatility (income volatility is high when expected returns are high) lead to a strong reduction of investment in the risky asset. In fact, calibrated on realistic data, even for frameworks that yield a strong investment in the risky asset, as for example Wachter (2002) or Campbell et al. (2004), risky investment is considerably reduced and can even turn negative. In fact, if the dynamics of the labor income stream have a particular relation to the financial assets, an investment strategy with low/no risky investment can be optimal. Hence, it can be stated that dynamic labor income is a simple and comprehensible instrument to explain why some people do not participate in the stock market. In other words, as opposed to common financial advice, it makes completely sense for some individuals to disclaim from risky investment. Moreover, it could be shown that consumption can fall even in states of rising expected returns.

The extension to non-constant labor income parameters showed that time variation in labor income is important even if current labor income does not vary with the economic states. In fact, if labor income is exposed to variation in the economic states towards the end of the life-cycle this has an impact on the behavior of a young individual anyhow.

The valuation of the future income stream is the central issue and determines the impact of dynamic labor income on the optimal policies. This task involves solving ordinary differential equations. It could be shown that certain combinations of parameter values lead to solutions of the differential equations that do not converge in the long-run. These settings are in favor of extreme results and this should be considered a warning for numerical studies of the consumption-investment problem with labor income that are calibrated on empirical results.

6.2 Open Issues and Future Research

The assumption of complete markets allowed to derive analytical solutions in closed-form. Admittedly, these assumptions are not completely in line with reality and it seems a natural next step to approach similar models with weaker assumptions. However, it was shown that in these cases the HJB-equations cannot be solved by analytical methods and one has to rely on numerical methods. Lynch and Tan (2009) is an example but their model includes multiple feature and the sensitivity with respect to the states is neglected.

In the models of this thesis it is assumed that all the variation in the states is driven by one factor. Several studies have shown that one factor is not able to reproduce all characteristics of financial assets or multifactor models lead to better results¹. The inclusion of multiple factors

¹For example, Campbell and Vuolteenaho (2004) use a four factor model to describe the economy.

does not affect the separation property of the HJB-equation. In other words, as long as the assumption of complete markets is fulfilled the HJB can be separated. However, the solution asks to solve systems of linear differential equations and/or systems of Riccati differential equations. Although these systems are generally solvable², the results are rather difficult to interpret. Nevertheless, an extension to a two factor model which allows for more autonomy of the financial and the non-financial market seems desirable. As shown in, for example, Buraschi et al. (2010) two-factor models can be interpreted in reasonable depth.

Munk and Sørensen (2010) analyze a joint bond-stock model with dynamic labor income. In their model the long-term bond is described by a Vasicek model and the growth rate of labor income varies with the (stochastic) short rate. One important advantage of this model is that the long-term bond and the state variable (the short rate) are perfectly negatively correlated by construction. Hence, market incompleteness arises only from the non-perfect correlation of labor income and the financial market. A simplified version of the model of Munk and Sørensen (2010) could be analyzed in more depth and the model could be extended to more sophisticated quadratic term structure models. Labor growth would not have to be affine in the short rate but only in the factor that drives the quadratic term structure model. Thus, more flexible relations of the bond and the labor market could be analyzed.

Short sale and borrowing constraints were not included in the models. In fact, the models of the thesis imply that the individual is able to borrow against future labor income. As shown by Koo (1998) and Munk (2000) borrowing constraints lead to a lower valuation of the future income stream especially for individuals with low financial wealth. Hence, by the inclusion of additional constraints risky investment and consumption could be reduced further. Moreover, in our models optimal investment in the risky asset can be negative in the steady state or can turn negative in some states. This is not specific to our models and the models of, for example, Kim and Omberg (1996), Wachter (2002) and many others share this property. The primary intention of the thesis was to evaluate whether dynamic labor income matters and it was shown that compared to the effects of classical state variable hedging demand and myopic demand, the impact of dynamic labor income is important. The inclusion of these kinds of constraints would make the model more realistic and comparable with the results of our models. However, extended models cannot be solved with analytical methods and one must rely on numerical procedures.

²See Grasselli and Tebaldi (2008).

Bibliography

- A. Ang and G. Bekaert. Stock Return Predictability: Is it there? *Review of Financial Studies*, 20(3):651, 2007. ISSN 0893-9454.
- D. Avramov and S. Hore. Rationalizing Momentum Interactions. *Working Paper*, 2007.
- D. Avramov and G. Zhou. Bayesian Portfolio Analysis. *Annual Review of Financial Economics*, 2(1), 2010. ISSN 1941-1367.
- P. Balduzzi and A.W. Lynch. Transaction Costs and Predictability: Some Utility Cost Calculations. *Journal of Financial Economics*, 52(1):47–78, 1999. ISSN 0304-405X.
- N. Barberis. Investing for the Long Run when Returns Are Predictable. *The Journal of Finance*, 55(1):225–264, 2000. ISSN 1540-6261.
- B. Bick, H. Kraft, and C. Munk. Investment, Income, and Incompleteness. *Working Paper*, 2009.
- Z. Bodie, R.C. Merton, and W.F. Samuelson. Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model. *Journal of Economic Dynamics and Control*, 16(3-4):427–449, 1992. ISSN 0165-1889.
- T. Bollerslev. Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics*, 31(3):307–327, 1986. ISSN 0304-4076.
- A. Bommier. Risk Aversion, Intertemporal Elasticity of Substitution and Correlation Aversion. *Economics Bulletin*, 4(29):1–8, 2007. ISSN 1545-2921.
- W. Boyle, W. Tian, and F. Guan. The Riccati Equation in Mathematical Finance. *Journal of Symbolic Computation*, 33(3):343–355, 2002. ISSN 0747-7171.
- D.T. Breeden. An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities. *Journal of Financial Economics*, 7(3):265–296, 1979. ISSN 0304-405X.
- M.J. Brennan. The Role of Learning in Dynamic Portfolio Decisions. *European Finance Review*, 1(3):295, 1998. ISSN 1572-3097.
- M.J. Brennan and Y. Xia. Tay’s as good as cay. *Finance Research Letters*, 2(1):1–14, 2005. ISSN 1544-6123.

- A. Buraschi, P. Porchia, and F. Trojani. Correlation Risk and Optimal Portfolio Choice. *The Journal of Finance*, 65(1):393–420, 2010. ISSN 1540-6261.
- J.Y. Campbell. Household Finance. *The Journal of Finance*, 61(4):1553–1604, 2006. ISSN 1540-6261.
- J.Y. Campbell and S.B. Thompson. Predicting Excess Stock Returns out of Sample: Can Anything Beat the Historical Average? *Review of Financial Studies*, 2007. ISSN 0893-9454.
- J.Y. Campbell and L. Viceira. Erratum: Consumption and Portfolio Decisions when Expected Returns are Time-Varying. Technical report, mimeo, Harvard University, 2000.
- J.Y. Campbell and L.M. Viceira. Consumption and Portfolio Decisions When Expected Returns are Time Varying. *Quarterly Journal of Economics*, 114(2):433–495, 1999. ISSN 0033-5533.
- J.Y. Campbell and T. Vuolteenaho. Bad Beta, Good Beta. *American Economic Review*, 94(5):1249–1275, 2004. ISSN 0002-8282.
- J.Y. Campbell, G. Chacko, J. Rodriguez, and L.M. Viceira. Strategic Asset Allocation in a Continuous-Time VAR Model. *Journal of Economic Dynamics and Control*, 28(11):2195–2214, 2004. ISSN 0165-1889.
- G. Chacko and L.M. Viceira. Spectral GMM Estimation of Continuous-Time Processes. *Journal of Econometrics*, 116(1-2):259–292, 2003. ISSN 0304-4076.
- G. Chacko and L.M. Viceira. Dynamic Consumption and Portfolio Choice with Stochastic Volatility in Incomplete Markets. *Review of Financial Studies*, 18(4):1369, 2005. ISSN 0893-9454.
- J.F. Cocco, F.J. Gomes, and P.J. Maenhout. Consumption and Portfolio Choice over the Life Cycle. *Review of Financial Studies*, 18(2):491, 2005. ISSN 0893-9454.
- J.H. Cochrane. *Asset Pricing*. Princeton Univ Pr, 2005. ISBN 0691121370.
- J.H. Cochrane. *Portfolio Theory - A New Chapter in the New Revision of Asset Pricing*. 2007.
- J.H. Cochrane. The Dog that Did not Bark: A Defense of Return Predictability. *Review of Financial Studies*, 21(4):1533, 2008. ISSN 0893-9454.
- I. Cooper and R. Priestley. Time-Varying Risk Premiums and the Output Gap. *Review of Financial Studies*, 22(7):2801, 2009. ISSN 0893-9454.
- J.C. Cox and C. Huang. Optimal Consumption and Portfolio Policies when Asset Prices Follow a Diffusion Process. *Journal of Economic Theory*, 49(1):33–83, 1989. ISSN 0022-0531.
- J.C. Cox, J.E. Ingersoll Jr, and S.A. Ross. A Theory of the Term Structure of Interest Rates. *Econometrica: Journal of the Econometric Society*, 53(2):385–407, 1985. ISSN 0012-9682.

- Q. Dai and K.J. Singleton. Specification Analysis of Affine Term Structure Models. *The Journal of Finance*, 55(5):1943–1978, 2000. ISSN 1540-6261.
- C.H. Dow. *Scientific Stock Speculation*. Magazine of Wall Street, 1920.
- D. Duffie. *Dynamic Asset Pricing Theory*. Princeton Univ Pr, 2001. ISBN 069109022X.
- D. Duffie and L.G. Epstein. Asset Pricing with Stochastic Differential Utility. *Review of Financial Studies*, 5(3):411, 1992a. ISSN 0893-9454.
- D. Duffie and L.G. Epstein. Stochastic Differential Utility. *Econometrica: Journal of the Econometric Society*, 60(2):353–394, 1992b. ISSN 0012-9682.
- D. Duffie, W. Fleming, H.M. Soner, and T. Zariphopoulou. Hedging in Incomplete Markets with HARA Utility. *Journal of Economic Dynamics and Control*, 21(4-5):753–782, 1997. ISSN 0165-1889.
- P.H. Dybvig and H. Liu. Lifetime Consumption and Investment: Retirement and Constrained Borrowing. *Journal of Economic Theory*, 145(3):885–907, 2010. ISSN 0022-0531.
- K.E. Dynan, J. Skinner, and S.P. Zeldes. Do the Rich Save More? *Journal of Political Economy*, 112(2):397–444, 2004. ISSN 0022-3808.
- R.F. Engle. Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation. *Econometrica: Journal of the Econometric Society*, 50(4):987–1007, 1982. ISSN 0012-9682.
- R.F. Engle and J.G. Rangel. The Spline-GARCH Model for Low-Frequency Volatility and its Global Macroeconomic Causes. *Review of Financial Studies*, 21(3):1187, 2008. ISSN 0893-9454.
- E.F. Fama and K.R. French. Dividend Yields and Expected Stock Returns. *Journal of Financial Economics*, 22(1):3–25, 1988. ISSN 0304-405X.
- L. Garlappi, R. Uppal, and T. Wang. Portfolio Selection with Parameter and Model Uncertainty: A Multi-Prior Approach. *Review of Financial Studies*, 20(1):41, 2007. ISSN 0893-9454.
- A. Goyal and I. Welch. A Comprehensive Look at the Empirical Performance of Equity Premium Prediction. *Review of Financial Studies*, 21(4):1455, 2008. ISSN 0893-9454.
- IS Gradshteyn and IM Ryzhik. *Table of Integrals, Series, and Products*. Academic Press, San Diego, 2000.
- M. Grasselli and C. Tebaldi. Solvable Affine Term Structure Models. *Mathematical Finance*, 18(1):135–153, 2008. ISSN 1467-9965.
- H. He and H.F. Pagès. Labor Income, Borrowing Constraints, and Equilibrium Asset Prices. *Economic Theory*, 3(4):663–696, 1993. ISSN 0938-2259.

- H. He and N.D. Pearson. Consumption and Portfolio Policies with Incomplete Markets and Short-sale Constraints: The Infinite Dimensional Case. *Journal of Economic Theory*, 54(2): 259–304, 1991. ISSN 0022-0531.
- J. Heaton and D. Lucas. Portfolio Choice in the Presence of Background Risk. *The Economic Journal*, 110(460):1–26, 2000a. ISSN 1468-0297.
- J. Heaton and D. Lucas. Portfolio Choice and Asset Prices: The Importance of Entrepreneurial Risk. *The Journal of Finance*, 55(3):1163–1198, 2000b. ISSN 1540-6261.
- F. Herzog, G. Dondi, H. Geering, and L. Schumann. Continuous-Time Multivariate Strategic Asset Allocation. *Working Paper*, pages 1–34, 2004.
- S.L. Heston. A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *Review of Financial Studies*, 6(2):327–343, 1993. ISSN 0893-9454.
- S. Hore. Equilibrium Predictability and Other Return Characteristics. *Working Paper*, 2008.
- H. Huang and M.A. Milevsky. Portfolio Choice and Mortality-Contingent Claims: The General HARA Case. *Journal of Banking & Finance*, 32(11):2444–2452, 2008. ISSN 0378-4266.
- H. Huang, M.A. Milevsky, and J. Wang. Portfolio Choice and Life Insurance: The CRRA Case. *Journal of Risk and Insurance*, 75(4):847–872, 2008. ISSN 1539-6975.
- J.E. Ingersoll. *Theory of Financial Decision Making*. Rowman & Littlefield Pub Inc, 1987. ISBN 0847673596.
- A. Inoue and L. Kilian. In-Sample or Out-of-Sample Tests of Predictability: Which One Should We Use? *Econometric Reviews*, 23(4):371–402, 2004. ISSN 0747-4938.
- R. Jagannathan and N.R. Kocherlakota. Why Should Older People Invest Less in Stocks than Younger People. *Federal Reserve Bank of Minneapolis Quarterly Review*, 20:11–20, 1996. ISSN 0271-5287.
- G.J. Jiang and J.L. Knight. Estimation of Continuous-Time Processes via the Empirical Characteristic Function. *Journal of Business and Economic Statistics*, 20(2):198–212, 2002. ISSN 0735-0015.
- R. Kan and G. Zhou. Optimal Portfolio Choice with Parameter Uncertainty. *Journal of Financial and Quantitative Analysis*, 42(03):621–656, 2007. ISSN 0022-1090.
- I. Karatzas, J.P. Lehoczky, and S.E. Shreve. Optimal Portfolio and Consumption Decisions for a Small Investor on a Finite Horizon. *SIAM Journal on Control and Optimization*, 25:1557, 1987.
- T.S. Kim and E. Omberg. Dynamic Nonmyopic Portfolio Behavior. *Review of Financial Studies*, 9(1):141–161, 1996. ISSN 0893-9454.

- R.S.J. Koijen, T.E. Nijman, and B.J.M. Werker. When Can Life Cycle Investors Benefit from Time-Varying Bond Risk Premia? *Review of Financial Studies*, 23(2):741, 2010. ISSN 0893-9454.
- H.K. Koo. Consumption and Portfolio Selection with Labor Income: A Continuous Time Approach. *Mathematical Finance*, 8(1):49–65, 1998. ISSN 1467-9965.
- R. Korn. *Optimal Portfolios: Stochastic Models for Optimal Investment and Risk Management in Continuous Time*. World Scientific Pub Co Inc, 1997. ISBN 9810232152.
- M. Leippold and L. Wu. Asset Pricing Under the Quadratic Class. *Journal of Financial and Quantitative Analysis*, 37(02):271–295, 2002. ISSN 0022-1090.
- M. Leippold and L. Wu. Design and Estimation of Quadratic Term Structure Models. *European Finance Review*, 7(1):47, 2003. ISSN 1572-3097.
- M. Lettau and S. Ludvigson. Resurrecting the (C) CAPM: A Cross-Sectional Test when Risk Premia Are Time-Varying. *Journal of Political Economy*, 109(6):1238–1287, 2001. ISSN 0022-3808.
- M. Lettau and S. Van Nieuwerburgh. Reconciling the Return Predictability Evidence. *Review of Financial Studies*, 2007. ISSN 0893-9454.
- J. Lintner. The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets. *The Review of Economics and Statistics*, pages 13–37, 1965. ISSN 0034-6535.
- H. Liu and M. Loewenstein. Optimal Portfolio Selection with Transaction Costs and Finite Horizons. *Review of Financial Studies*, 15(3):805, 2002. ISSN 0893-9454.
- J. Liu. Portfolio Selection in Stochastic Environments. *Review of Financial Studies*, 20(1):1, 2007. ISSN 0893-9454.
- N. Liu and L. Zhang. Is the Value Spread a Useful Predictor of Returns? *Journal of Financial Markets*, 11(3):199–227, 2008. ISSN 1386-4181.
- A.W. Lo and J. Wang. Implementing Option Pricing Models when Asset Returns Are Predictable. *Journal of Finance*, 50(1):87–129, 1995. ISSN 0022-1082.
- A.W. Lynch and S. Tan. Labor Income Dynamics at Business-Cycle Frequencies: Implications for Portfolio Choice. *NBER Working Paper*, 2009.
- H. M. Markowitz. Portfolio Selection. *The Journal of Finance*, 7:77–91, 1952.
- R. Mehra and E.C. Prescott. The Equity Premium: A Puzzle. *Journal of Monetary Economics*, 15(2):145–161, 1985. ISSN 0304-3932.

- R.C. Merton. Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case. *The Review of Economics and Statistics*, 51(3):247–257, 1969. ISSN 0034-6535.
- R.C. Merton. Optimum Consumption and Portfolio Rules in a Continuous-Time Model. *Journal of Economic Theory*, 3(4):373–413, 1971. ISSN 0022-0531.
- R.C. Merton. An Intertemporal Capital Asset Pricing Model. *Econometrica: Journal of the Econometric Society*, 41(5):867–887, 1973. ISSN 0012-9682.
- R.C. Merton. On Estimating the Expected Return on the Market: An Exploratory Investigation. *Journal of Financial Economics*, 8(4):323–361, 1980. ISSN 0304-405X.
- D. Moos and H. Müller. A Life Cycle Model with Pension Benefits and Taxes. *Working Paper University of St. Gallen*, 2010.
- C. Munk. Optimal Consumption/Investment Policies with Undiversifiable Income Risk and Liquidity Constraints. *Journal of Economic Dynamics and Control*, 24(9):1315–1343, 2000. ISSN 0165-1889.
- C. Munk and C. Sørensen. Dynamic Asset Allocation with Stochastic Income and Interest Rates. *Journal of Financial Economics*, 96(3):433–462, 2010. ISSN 0304-405X.
- C. Munk, C. Sørensen, and T. Nygaard Vinther. Dynamic Asset Allocation under Mean-Reverting Returns, Stochastic Interest Rates, and Inflation Uncertainty: Are Popular Recommendations Consistent with Rational Behavior? *International Review of Economics & Finance*, 13(2):141–166, 2004. ISSN 1059-0560.
- B.K. Øksendal. *Stochastic Differential Equations: An Introduction with Applications*. Springer Verlag, 6th edition, 2003. ISBN 3540047581.
- L. Pástor and R.F. Stambaugh. The Equity Premium and Structural Breaks. *The Journal of Finance*, 56(4):1207–1239, 2001. ISSN 1540-6261.
- S.R. Pliska. A Stochastic Calculus Model of Continuous Trading: Optimal Portfolios. *Mathematics of Operations Research*, 11(2):371–382, 1986. ISSN 0364-765X.
- S.R. Pliska and J. Ye. Optimal Life Insurance Purchase and Consumption/Investment under Uncertain Lifetime. *Journal of Banking & Finance*, 31(5):1307–1319, 2007. ISSN 0378-4266.
- AD Polyanin and VF Zaitsev. *Handbook of Exact Solutions for Differential Equations*. CRC Press, Boca Raton, FL, 1995.
- P.A. Samuelson. Lifetime Portfolio Selection by Dynamic Stochastic Programming. *The Review of Economics and Statistics*, 51(3):239–246, 1969. ISSN 0034-6535.
- R. Schöbel and J. Zhu. Stochastic Volatility with an Ornstein–Uhlenbeck Process: An Extension. *European Finance Review*, 3(1):23, 1999. ISSN 1572-3097.

- M. Schroder and C. Skiadas. Optimal Consumption and Portfolio Selection with Stochastic Differential Utility. *Journal of Economic Theory*, 89(1):68–126, 1999. ISSN 0022-0531.
- W. F. Sharpe. Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. *Journal of Finance*, 19(3):425–442, 1964. ISSN 0022-1082.
- S.E. Shreve. *Stochastic Calculus for Finance: Continuous-Time Models*. Springer Verlag, 2004. ISBN 0387401016.
- K.J. Singleton. Estimation of Affine Asset Pricing Models Using the Empirical Characteristic Function. *Journal of Econometrics*, 102(1):111–141, 2001. ISSN 0304-4076.
- J.H. Stock and M.W. Watson. Business Cycle Fluctuations in US Macroeconomic Time Series. *Handbook of Macroeconomics*, 1:3–64, 1999. ISSN 1574-0048.
- J. Tobin. The Theory of Portfolio Selection. In F.P.R Brechling and Frank Hahn, editors, *The Theory of Interest Rates*, pages 3–51. Macmillan, London, 1965.
- O. Vasicek. An Equilibrium Characterization of the Term Structure. *Journal of Financial Economics*, 5(2):177–188, 1977. ISSN 0304-405X.
- L.M. Viceira. Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income. *The Journal of Finance*, 56(2):433–470, 2001. ISSN 1540-6261.
- J. Wachter. Asset Allocation. *Annual Review of Financial Economics*, 2(1):175–206, 2010. ISSN 1941-1367.
- J.A. Wachter. Portfolio and Consumption Decisions under Mean-Reverting Returns: An Exact Solution for Complete Markets. *Journal of Financial and Quantitative Analysis*, 37(01):63–91, 2002. ISSN 0022-1090.
- J.A. Wachter and M. Warusawitharana. Predictable Returns and Asset Allocation: Should a Skeptical Investor Time the Market? *Journal of Econometrics*, 148(2):162–178, 2009. ISSN 0304-4076.
- Y. Xia. Learning about Predictability: The Effects of Parameter Uncertainty on Dynamic Asset Allocation. *The Journal of Finance*, 56(1):205–246, 2001. ISSN 1540-6261.
- Daniel Zwillinger. *Handbook of Differential Equations*. Academic Pr, 3rd edition, 1998.

Curriculum vitae

Daniel Moos, born on the 5th of August 1980 in Princeton/NJ as son of Ulrich and Heidi Moos; two brothers Rudolf and Markus.

Education

2006 - 2011	Doctoral Studies in Economics, University of St. Gallen (HSG)
2004 - 2006	Studies in Quantitative Economics and Finance (MiQEF) (MA HSG), University of St. Gallen (HSG)
2001 - 2004	Studies in Economics (BA HSG), University of St. Gallen (HSG)
1996 - 2000	Matura Typus C, Kantonsschule Baden (AG)

Work Experience

2006 - 2010	Analyst and Portfolio Manager, Kraus Partner Investment AG, Zürich
2005 -	Research and Teaching Assistant, Department of Mathematics and Statistics, University of St. Gallen (HSG)