Freedom and Contests

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Hendrik Rommeswinkel

from

Germany

Approved on the application of

Prof. Dr. Martin Kolmar

and

Prof. Dr. Clemens Puppe

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Hendrik Rommeswinkel

Summary

The following dissertation has two parts. The topic of the first part is the role of freedom in normative and positive economics. This part deals with questions such as “How can one measure freedom in economic models?”, “How can one design institutions that give individuals the most freedom?”, and “Do individuals intrinsically value freedom?”. To answer these questions, a game-theoretic approach is endorsed. One chapter provides a measure of freedom which captures many intuitions we have about this elusive concept. By various examples, it is shown how this measure can be used for policy evaluation in normative economics. Another chapter demonstrates the importance of freedom in positive economics by use of an experiment. Experimental subjects played a game where their choices affected the freedom they had at later stages of the game. Using the freedom measure from the previous chapter, it was then estimated to which degree individuals value freedom intrinsically.

The topic of the second part is the interaction between groups when competing in a contest. This part analyzes how the technology with which group members aggregate their efforts influences the success of the group. Again, a game-theoretic model is used. One chapter examines the role of complementarity among group members’ efforts in more detail. Another chapter focuses on the interaction of various technological properties with group size to determine which technological properties benefit larger or smaller groups.

Zusammenfassung

Die folgende Dissertation hat zwei Teile. Das Thema des ersten Teils ist die Rolle von Freiheit in normativer und positiver Ökonomik. Dieser Teil beschäftigt sich mit Fragen, wie: „Wie kann man Freiheit in ökonomischen Modellen messen?“, „Wie kann man Institutionen schaffen, die Individuen maximale Freiheit geben?“ und „Bevorzugen Individuen intrinsisch mehr Freiheit?“ Um diese Fragen zu beantworten, wird ein spieltheoretischer Ansatz verwendet. Ein Kapitel entwickelt ein Maß für Freiheit, welches viele Intuitionen dieses schwer fassbaren Konzepts widerspiegelt. Mittels verschiedener Beispiele wird gezeigt, wie das Maß in normativer Ökonomik verwendet werden kann. In einem weiteren Kapitel wird mittels eines Experimentes die Wichtigkeit von Freiheit in positiver Ökonomik demonstriert. Teilnehmer des Experiments spielten ein Spiel in welchem ihr Verhalten ein Einfluss auf ihre Freiheit im restlichen Verlauf des Spiels hat. Mittels des Freiheitsmaßes wurde dann die intrinsische Wertschätzung für Freiheit geschätzt.

Das Thema des zweiten Teils ist die Interaktion zwischen Gruppen in einem Wettkampf. Dieser Teil analysiert wie die Technologie mit welcher Gruppenmitglieder ihre Anstrengungen aggregieren den Gruppenerfolg beeinflussen. Erneut wird ein spieltheoretischer Ansatz verwendet. Ein Kapitel untersucht die Rolle von Komplementarität zwischen den Anstrengungen im Detail. Ein weiteres Kapitel beschäftigt sich damit, welche technologischen Eigenschaften große oder kleine Gruppen bevorteilen.

Contents

1 Introduction

From an outsider's perspective, the framework for making normative judgments in economics seems often simplistic. Standard results from Economics such as welfare theorems in competitive economies (e.g. Arrow, 1951; Debreu, 1959) solely rely on the preferences of individuals. One aspect where the contrast between economic practice and the moral intuitions outside of economics is especially stark, is the value of freedom. As Sen (1988) puts it:

> *The foundational importance of freedom may well be the most far-reaching substantive problem neglected in standard economics.* (Sen, 1988, p. 294)

There may be a diverse set of reasons for this. First and foremost is the ability of Economics to derive policy recommendations based on welfare criteria. Welfare, whether it is measured in the ordinal or cardinal sense, provides a simple and mathematically tractable way for making normative judgments in many economic settings. A measure that can easily be applied in general economic models has not yet been provided for freedom. Second, the concept of freedom has ever been an elusive one, with conflicting opinions held in what Berlin (1958) called an "open war" (p. 6) of conflicting ideals about freedom. This discussion has not only been limited to the philosophical literature. Indeed, attempts to measure freedom in the freedom of choice literature (e.g. Dowding & van Hees, 2009) have brought up a wide range of aspects that should (or should not) be included in a measure of freedom. Among them are the number of options an individual can choose from (Pattanaik & Xu, 1990), their degree of reasonableness (Jones & Sugden, 1982), their diversity (van Hees, 2004), their welfare implications (Sen, 1991), and the role of other individuals (Braham, 2006). Third, it is unclear whether individuals actually intrinsically care for freedom. While it is easy to observe that individuals care for their well-being, the case may be less clear for freedom. Even though in a recent contribution Fehr, Herz, and Wilkening (2013) have shown that individuals care about decision rights, it is unclear whether this preference for decision rights originates in preference for power over others or preference for freedom, or other reasons.

The first part of this dissertation strikes out against all three of these problems. In Chapter 2 a class of freedom measures is provided for game-theoretic models. Since virtually all eonomic analysis today can be expressed with the tools of game theory, this bridges the gap between freedom as an abstract concept and its implementation in normative economics. It is shown that this class of measures can account for the above mentioned intuitions of the freedom of choice literature and even different philosophical concepts of freedom. In an application of this measure, in Chapter 3, it is shown by an experiment that individuals show significant preference for freedom. More precisely, the results suggest that in our experiment individuals strongly disliked interferences of others in their outcomes and had no significant preference for power over other individuals.

The second part of the dissertation analyzes group contests. There are numerous settings where individuals compete in groups against each other. Among these settings there are lobbying contests, research and development contests, sports competitions, litigation, and warfare. The literature on group contests (e.g. Corchón, 2007; Garfinkel & Skaperdas, 2007; Konrad, 2009) has tried to examine the properties of such contests under the assumption that individuals behave rationally. With few exceptions, however, the focus has been almost exclusively on contests where the efforts of individuals are aggregated by summation. In Chapter 4 the Nash equilibrium behavior for contests is determined, in which efforts are aggregated via functions with a constant elasticity of substitution. It is shown that if individuals have heterogeneous valuations within the group, changing the elasticity of substitution from the case of additive impact functions indeed has an impact on behavior. Moreover, it is shown that the winning probability of a group is increasing in the elasticity of substitution of their members' efforts.

An important question in the literature on group contests has been the question whether larger or smaller groups have an advantage in such settings. Olson (1965) described what was later coined the "group-size paradox" by Esteban and Ray (2001). His claim was that due to free-rider problems larger groups are inherently at a disadvantage in collective action problems such as contests. While Esteban and Ray (2001) showed that this must not necessarily be the case, their analysis was limited in two important ways: First, though allowing for convex costs of effort, the analysis is still limited to the case where efforts of group members are summed over the group members. Second, all group members are assumed to have identical valuations of winning the contest. Chapter 5 generalizes their analysis by allowing for more general ways to aggregate efforts and allowing for heterogeneous valuations within groups.

I hope this dissertation will be interesting to the reader and will help shed some light on the issues mentioned above.

2 Measuring Freedom in Games [1]

Hendrik Rommeswinkel

This paper provides freedom measures for game theoretic settings. Three core philosophical concepts of freedom are measured: positive, negative, and republican liberty. The measures solve two fundamental problems in the literature on freedom of choice: The integration of freedom and welfare into a single measure and the measurement of freedom in situations where agents interact. Since most economic models contain interactions between agents, the measures open a wide range of economic applications for policy evaluation based on freedom.

[1]This paper is based on an earlier version titled "A Causal Measure of Freedom". I thank presentation participants at the Public Choice Society 2011 meeting, the Royal Economic Society 2011 meeting, the European Economic Association 2011 meeting, the LSE Choice Group, the California Institute of Technology, the University of Groningen, the Karlsruhe Institute of Technology and the University of St. Gallen for their helpful comments. Special thanks go to Dirk Burghardt, Philipp Denter, Martin van Hees, Chris Hitchcock, Martin Kolmar, Philip Pettit, and Clemens Puppe for more detailed comments on earlier versions of this paper.

2.1 Introduction

Presumably the largest difference in normative reasoning between an economist's model and a layperson's appraisal is the value of freedom. In standard economic models, the value of freedom is purely instrumental for utility satisfaction. However, philosophers have stressed the intrinsic importance of freedom (e.g. Berlin, 1958). Therefore, Sen (1988) has argued for the inclusion of freedom into economic analysis. To this end, the freedom of choice literature,[2] following the seminal contributions of Pattanaik and Xu (1990) and Jones and Sugden (1982), has attempted to provide measures which can be used to determine the freedom offered by an opportunity set. These contributions greatly enhanced our conceptual understanding of freedom but Pattanaik (1994) showed that these measures encountered problems when being applied to situations in which agents interact. The difficulty arises because in situations in which agents interact, opportunity sets from which agents can freely choose are no longer clearly defined: The choice of one agent may influence the available opportunities of another agent and vice versa. This problem has prevented the literature to provide measures even for a simple exchange economy as Pattanaik (1994) showed. Yet, it is exactly these cases when agents depend on each other to achieve their goals, when they exhibit power over each other, or when they are coerced by others that the measurement of freedom becomes interesting. The lack of freedom measures for situations where agents interact therefore creates an undesirable wedge between the normative analysis performed by economists and normative perceptions outside economic theory.

Also from a positive perspective, a microeconomic measure of freedom for interacting agents is desirable. In the macroeconomic literature on the relation of growth and freedom (e.g. Easton & Walker, 1997; de Haan & Sturm, 2000), proxies have been used such as the size of the government, price stability, or the security of property rights (Gwartney & Lawson, 2003; Gwartney, Hall, & Lawson, 2010). The development of such indices of economic freedom for

[2]For surveys of the literature, see Barberà, Bossert, and Pattanaik (2004), Baujard (2007), or Dowding and van Hees (2009).

cross-country comparisons and the contemporaneous development of microeconomic measures in the freedom of choice literature have been remarkably disconnected from each other. A microeconomic measure of freedom which can be applied in economic models may help bridge the gap between the economic freedom indices and the freedom of choice literature by providing microfoundations for the former.

The present paper attempts to provide a measure of freedom for interactive situations and therefore breaks with the opportunity-set based approach in favor of a game theoretic setting. The goal is to maintain the intuitions gained by the opportunity-set based measures from the freedom of choice literature and generalize them to interactive situations.

The main issue when measuring freedom in interactive situations is the imperfect control agents have over the outcomes, which a measure must accommodate. Additionally to the number of different outcomes and their value, the degree to which an agent controls each outcome becomes relevant. It makes a difference whether an individual brings about an outcome by her own actions or whether the outcome is determined by the actions of another person. This control over outcomes is not only relevant from a normative perspective: Fehr et al. (2013) have shown experimentally that individuals value the control over outcomes even if this control comes at a monetary cost. The idea of the measure is therefore that the better an agent can influence which outcome occurs, the larger the freedom. Meanwhile, it maintains the idea from the freedom of choice literature that freedom increases in the number of outcomes an agent can achieve.

An issue complicating the measurement of freedom are the diverse normative ideals people have about freedom. The paper therefore provides distinct measures for the most common philosophical concepts of freedom: Positive liberty as the degree of rational self-determination, negative liberty as the absence of interference by others, and republican liberty as the absence of subjugation by others. Moreover, the measures are only defined up to two functions which may contain additional normative considerations of the measurer. Aside from weighting the importance of freedom over the outcomes, they can be used in two interesting ways:

First, they may integrate a measure of well-being and freedom into a single measure. This problem has been extensively discussed in the literature: Sen (1985, 1988) argued that a measure of well-being should be included in a measure of freedom. Others have presented impossibility results on integrating freedom and welfare into a single measure (Puppe, 1995, 1996; Nehring & Puppe, 1996; Pattanaik & Xu, 1998; Gravel, 1998; Baharad & Nitzan, 2000), drawing a rather negative picture on the possibility to integrate welfare and freedom of choice. If one introduces utility into the measure via the normative inputs, it turns out that the freedom measure is equal to weighted expected utility, where the weights are causal influence measures representing the control an agent has over each outcome. Expected utility is only reached if the agent has perfect control over all outcomes. The less control the agent has over an outcome, the less the utility of the outcome matters for the measure. This strikes a balance between care for the well-being of an agent and her possibilities to influence her surroundings.

Second, the functions can be used for the freedom measure to capture the qualitative diversity of the outcomes of the game. For the freedom of an individual, it may not only matter to have control over a large number of outcomes, but also that these outcomes are qualitatively dissimilar. If one uses the qualitative diversity weights of Nehring and Puppe (2002) as normative inputs, the positive freedom measure is a generalization of a diversity measure proposed by Nehring and Puppe (2009) which captures both quantitative and qualitative diversity.

The properties of the freedom measures are shown in various examples. An example on labor market discrimination shows how the positive, negative, and republican liberty measures differ in their judgments of how discrimination affects freedom. The crucial difference aspect of the positive liberty measure is that it allows for positive discrimination as a way to improve the freedom of a group. In contrast, the negative and republican freedom measures regard both positive and negative discrimination as an interference and thus as a threat to freedom. The difference between negative and republican liberty is that a decrease in the republican liberty measure does not require actual discrimination from occuring. The republican liberty measure instead focuses on whether the institutional setting potentially allows for discrimination or not.

As an example of a combination of utility and freedom into a measure, an experiment by Fehr et al. (2013) is considered. In this experiment, individuals showed a strong preference against delegating a decision right. This preference could not fully be accounted for by expected utility maximization. In the present paper, a theory of preference for freedom is sketched, which combines the positive freedom measure with utility. Individuals with preference for freedom value outcomes of a game higher, if they have actively caused them. An individual is therefore less willing to forego a decision right if this transfers some causal influence over the outcomes to another person. In this way, preference for freedom explains the low delegation rates in Fehr et al. (2013).

Two more examples are given: In a simple voting model where the median voter theorem holds the freedom-maximizing distance between candidates is determined. The optimal candidate distance has to account for three factors: first, the degree of influence each voter has on the outcomes which roughly represents voting power. Second, the quantitative diversity of the outcomes. A set of candidates where one character is unlikely to be chosen provides less freedom than one where all candidates are plausible outcomes. Third, the qualitative difference in the outcomes, since if all candidates are identical, the voters are not left with a meaningful choice. It is shown that the freedom measure accounts for all three factors. Finally, the example of competitive equilibrium models from Pattanaik (1994) is considered. Though not a complete analysis, the example suggests that price stability in an economy contributes to positive freedom, which gives microfoundations for the use of price stability as an component of an index of freedom as in Gwartney et al. (2010).

The paper continues as follows. First, in Section 2.2 a review of the literature on various measures of freedom is given, with a focus on the ones related to the measures developed here. Section 2.3 provides the formal framework and an interpretation of the elements of the measure. The three philosophical concepts of liberty are introduced in Section 2.4 and a measure of freedom is given for each of them. Section 2.5 states convergence theorems of the measure to measures of the freedom of choice literature and the diversity measurement literature. The convergence results reveal that the measure is a generalization of a diverse set of measures from the literature examined in Section 2.2. To show the differences between the three concepts of freedom and how this translates into the measures an example of labor market discrimination is given in Section 2.6. The problem of integrating utility or other measures of well-being into a freedom measure is addressed in Section 2.7. A case where it seems intuitive for utility to enter a freedom measure is the case where individuals have preference for freedom. Therefore, Section 2.8 provides a theory of preference for freedom and an application to an experiment by Fehr et al. (2013). Two further applications are provided as examples: Section 2.9 shows in a simple voting model how optimal diversity of candidates is affected by accounting for qualitative diversity in the freedom measure. An application of the positive freedom measure to a production economy is given in Section 2.10. By this example it is shown that the class of measures in this paper indeed solves the problem posed by Pattanaik (1994).

2.2 Freedom Measures

In the following, only few measures will be reviewed. For a more extensive survey, the reader may consider Barberà et al. (2004), Baujard (2007), or Dowding and van Hees (2009). Two very early contributions to the freedom of choice literature are Pattanaik and Xu (1990) and Jones and Sugden (1982) for which later on some convergence results will be stated. Other closely related measures to the one presented in this paper are given in Braham (2006), Suppes (1996), and Nehring and Puppe (2009). Therefore these measures will be presented here, while the rest of the literature will only be touched upon briefly.

All measures will be indexed by the authors' last names. Since the measures are based on opportunity sets C, some notation needs to be introduced to state these measures: Suppose X is a set of alternatives. A freedom relation $\succsim_F$ holds between subsets C of X. $C \succsim_F C'$ with $C, C' \subseteq X$ can be interpreted as 'the opportunity set C offers weakly more freedom than the opportunity set C''. The measure of Pattanaik and Xu (1990) states that the freedom offered by an opportunity set C is its cardinality $\sharp(C)$, that is:

Definition 2.1. Cardinality Measure (Pattanaik & Xu, 1990)
Suppose $C, C' \subseteq X$. Then $C \succsim_{F,PX} C' \Leftrightarrow \sharp(C) \geq \sharp(C')$.

Since the interest here lies not in the axiomatization of this measure, the axioms from which the measure can be derived will not be stated. While still considered the starting point of any measure of freedom, criticism of the measure has been abundant. For example Sen (1991) demands a more intricate relation between preference and freedom. However, as shown for example by Baharad and Nitzan (2000) the joint evaluation of the welfare and the freedom provided by an opportunity set often runs into difficulties or allows only for lexicographic comparisons (Romero-Medina, 2001).

A mild introduction of preferences into a freedom measure has been made by Jones and Sugden (1982). In their measure, which has been given a formal basis by Pattanaik and Xu (1998), a set $\mathcal{R}$ of so-called "reasonable" preference relations R is introduced and freedom is measured according to the set of reasonably chosen alternatives $\{x \in C : \exists R \in \mathcal{R} : \forall y : xRy\}$. While the precise definition of "reasonable" is left open, Jones and Sugden (1982) give as an example the choice of a prisoner, who can either "stay in the cell" or "get shot". Since it would be unreasonable to prefer getting shot to staying in the cell, the set of reasonably chosen alternatives is the singleton "stay in the cell". On the basis of the ideas developed by Jones and Sugden (1982), Pattanaik and Xu (1998) axiomatize the following measure:

Definition 2.2. Reasonable Preference Measure (Jones & Sugden, 1982; Pattanaik & Xu, 1998)

Suppose $C, C' \subseteq X$. Then $C \succsim_{F,JS} C'$ iff

$$\sharp(\{x \in C : \exists R \in \mathcal{R} : \forall y : xRy\}) \geq \sharp(\{x \in C' : \exists R \in \mathcal{R} : \forall y : xRy\}).$$

The measure thus states that the freedom an opportunity set offers can be measured by the cardinality of the set of reasonably chosen alternatives. This does not solve the critique of Sen (1991), since it may still occur that an opportunity set C is ranked higher by the measure than C', although in terms of preference each element in C' dominates all other elements in C. The difficulties are even increased: The set C' may now have even more elements than C and each element in C' may dominate all elements in C. Still, the measure may rank C higher than C'.

The closest relatives to the measure which will be developed in this paper are given by the following measures by Braham (2006), Suppes (1996), and Nehring and Puppe (2009). Unlike the above two measures, these measures are not solely based on opportunity sets but also include probability information.

The measure by Braham (2006) relies on game forms to account for interactions between agents. It is not necessary to further examine the formal structure of the measure, since it has a very intuitive interpretation: The measure tries to capture the degree to which an individual i can force a certain outcome x to come about in the game. With some abuse of notation the measure states:

Definition 2.3. Game Form Measure (Braham, 2006)
$F_B(x,i) = P(\text{outcome is } x|i \text{ chooses } x)$

where it may occur that $P(\text{outcome is } x|i \text{ chooses } x) < 1$ because the actions of the other agents may lead to another outcome, even if i chooses x. The measure therefore takes up the idea that an agent is free if he can force certain outcomes to occur. This will also be the core idea of the causal influence measure in this paper, though the degree to which an agent can force certain outcomes will be measured in a different way.

It has been argued that freedom of choice is strongly connected to diversity. Individuals are more free if they are able to make choices over a more diverse opportunity set. Two types of diversity have been identified: Quantitative diversity refers to the relative frequencies with which different objects are chosen and can for example be measured by the Shannon (1948) entropy. Suppes (1996) proposes to measure freedom as the entropy of the relative frequencies with which an agent chooses the alternatives of an opportunity set:

Definition 2.4. Entropy Freedom Measure (Suppes, 1996)
$F_S(C,P) = -\sum_{x\in C} P(x)\ln P(x)$ where $P(x)$ refers to the probability with which an agent chooses element x of the opportunity set C.

The entropy measure results as one special case in the measure axiomatized in this paper. More specifically, if agents have perfect control with their actions over outcomes and no additional normative information is relevant, the measures converge. The precise relation between the two will be given in Proposition 2.2 in Section 2.5.

Qualitative diversity instead measures how different the elements of the opportunity set are. An example is Nehring and Puppe (2002). A model of qualitative diversity has been given by Nehring and Puppe (2002). Their model supposes that objects $x \in X$ have attributes A which can be defined via the subset of objects of X which also have this attribute. Therefore, $A \in X$ is the attribute that all elements in A share. To measure the diversity value of a set, each attribute is given a weight $\lambda(A)$. Diversity of a distribution of objects P can then be measured for example by $v(\lambda,P) = \sum_{A\subseteq X:\exists x\in A:P(x)>0} \lambda_A$. The intuition is very simple: the more attributes are represented in a distribution (and the more diversity weight they have), the higher the diversity. In case of an opportunity set C one could set $P(x) = 1/|C|$ if $x \in C$ and 0 otherwise to obtain the aggregate qualitative diversity of the opportunity set.

Building on this framework in Nehring and Puppe (2009) a generalized measure is proposed which captures both quantitative and qualitative aspects of diversity at once:

Definition 2.5. Diversity Measure (Nehring & Puppe, 2009)

$$D_{NP}(C,P,\lambda) = -\sum_{A:A\cap C\neq\varnothing} \lambda(A) \sum_{x\in A} P(x) \ln \sum_{y\in A} P(y)$$

which is the $\lambda(A)$ weighted entropy of the attributes. While the entropy is maximized if as many objects as possible have a distribution as even as possible, the measure $D_{NP}(\lambda,P)$ changes this in two ways: First, it considers the entropy over attributes, i.e. the attributes (and not the objects) need to have a distribution as even as possible. Second, there exists an additional tradeoff between a very even distribution and higher frequencies of objects with attributes that have a high weight $\lambda(A)$. While this diversity measure has not been explicitly formulated as a measure of freedom, it turns out that in the perfect control case the freedom measure in this paper is identical to D_{NP} if the weights $\lambda(A)$ are included as normative information in the measure.

The further literature can be divided into several branches. One branch deals with the aforementioned aspect of the diversity of an opportunity set (Bavetta and del Seta (2001), Bossert, Pattanaik, and Xu (2003), and van Hees (2004)). Another considers unstable preferences to be a source of preference for freedom (Koopmans (1964), Kreps (1979a), and Sugden (2007)). Freedom has been studied in game forms in Peleg (1997), Bervoets (2007), and Ahlert (2010). The idea of multiple preference relations as in Jones and Sugden (1982) has been further examined by Sugden (1998), Nehring and Puppe (1999), and Bavetta and Peragine (2006). Rosenbaum (2000) develops a measure of freedom based on underlying characteristics of the elements of the opportunity set. An important topic is also the distribution of freedom between individuals, for which a survey is given by Peragine (1999). Broader discussions are given by van Hees (1999), Bavetta (2004), Carter (2004), and Kolm (2010).

The paper is also related to the literature on axiomatizations for information theoretic measures, see Csiszár (2008) for a survey and Frankel and Volij (2011) for an application to segregation. Naturally very closely related are measures for power in voting systems, where mutual information between the cast vote and the outcome of the vote has been proposed as one dimension of power (Diskin & Koppel, 2010). The present paper uses a more general form of mutual information than in these papers in order to also account for normative information that becomes relevant when measuring freedom.

2.3 Theoretical Framework

The major assumption of the way freedom will be measured is the idea that freedom involves (a) the possibility of an agent *to do otherwise* and (b) to achieve other results by doing so. This means that freedom in this paper will always involve some counterfactual such as "if the agent had acted otherwise, he would have obtained a different outcome" containing an antecedent and a consequent. Most measures in the freedom of choice literature implicitly contain such a combination of antecedent and consequent: The above given measure by Jones and Sugden (1982) has reasonable preference relations as the antecedent and the chosen elements of the opportunity set as the consequent. Freedom then involves the counterfactual: "If the agent had had reasonable preferences $\mathcal{R}$, he would have chosen element x." Similarly, in the capabilities approach, freedom is accounted for by considering the set of functionings an individual can reach from the available commodity vectors. The counterfactual notion is again very clear: "If the agent had chosen commodity vector x, he would have obtained the function vector f".

The goal of this section is therefore to find a formal framework in which one can model interactions between individuals and which can account for both (a) and (b).

Assume an extensive game form $\eth = (N, A, \psi, \mathcal{P}, \mathcal{I}, \mathcal{C}, p, R)$, where $N = \{1, \ldots, n\}$ is a finite set of players, A is a finite set of nodes, and $\psi : A \backslash a_0 \to A$ is a predecessor function such that for node a, $\psi(a)$ is the immediate predecessor of a. $\mathcal{P}$ is the player partitioning of the nodes and $\mathcal{I} = \{I_0, \ldots, I_n\}$ the information partitioning with I_i being the set of information sets of player i. Let $A(I) = \{a \in A : \psi(a) \in I\}$ return the set of nodes following information set I. $\mathcal{C}$ is the set of choice sets C_I for each information set I. Further $\Delta(C_I)$ is the set of probability distributions over the choice set at I. For $b \in I$ and

$b = \psi(a)$ let $c(a|b) \in C_I$ be the choice that leads at node b to node a. p is the probability distribution for moves by nature. Finally, R is the set of result functions for each player, where a result function $r_i : A_\omega \to O_i$ maps the terminal nodes A_ω into the finite set of possible outcomes for player i, O_i. This last point deviates slightly from standard definitions of game forms, where result functions are not player specific. In case all result functions are identical, we are back in the standard case. The use of player-specific outcome functions is necessary to account for (b) in a meaningful way. The difference between some outcomes of the game may simply be irrelevant to all players except for player i. If player j then gets to make a choice between these outcomes, it would not be meaningful to call this a case where j is especially free. Rather, it is a case where j has power over the outcomes relevant for i.

Due to (a) the individual must possess some nontrivial form of agency, some capacity to make choices which are not completely prescribed by a single preference relation. For example, the measure of Jones and Sugden (1982) generates this capacity via the assumption of multiple preference relations which the individual may reasonably hold. In the model here, agency is introduced for all players in the game via a *preference expansion* $\dashv = (\mathcal{J}, \mathcal{U}, \hat{p})$. The preference expansion contains $\mathcal{U} = \{U_1, \ldots, U_n\}$ as the set of the sets of utility functions $U_i = \{u_{i,1}, \ldots, u_{i,\bar{u}}\}$ over outcomes for each player $u_{i,u} : O_i \to \mathbb{R}$. $\hat{p}$ assigns each of these utility functions a probability. It will be assumed that the preferences of players are independently distributed and the marginal distribution for each player i is $\hat{p}_i \in \Delta U_i$. This probability may be interpreted either as a degree of reasonableness as in Jones and Sugden (1982) or an ex-ante probability with which an individual holds a preference if randomly drawn from a population. A more precise definition on what is contained in $\mathcal{U}$ will be given when considering various concepts of freedom. $\mathcal{J}$ is the set of information partitions for each player over the reasonable utility functions. To simplify notation, it will at times be convenient to treat the sets U_i and O_i as discrete random variables instead, with realizations ranging over the elements of the original set. For example, we may want to write $P(O_i = o)$ as the probability of the outcome of the game being o but we may also want to write $\sum_{o \in O_i}$ as the sum over all possible outcomes. It will be clear from the context whether O_i refers to the set or the random variable.

A local strategy $s(I) \in \Delta(C_I)$ is a probability distribution over the elements of the choice set. Define a strategy profile S as a tuple of strategies specifying behavior at each information set $S = (s(I)|_{I \in I_i}|_{i \in N})$. Further, let θ^S be the joint probability distribution over nodes resulting from strategy profile S. Finally, we need to consider that strategies may depend on preferences in the preference expansion. Let $S(\hat{u})$ be the strategy profile resulting from preference profile $\hat{u} \in \times_{i \in N} U_i$. It makes sense to enrich the probability distribution θ by the outcomes and the preferences. For this, define:

$$\forall i \in N : \forall o \in O_i : \theta^{S(\hat{u})}(o|\hat{u}) = \sum_{a \in A_\omega : r_i(a) = o} \theta^{S(\hat{u})}(a) \tag{2.1}$$

$$\forall i \in N : \forall o \in O_i : \theta^{S(\hat{u})}(o) = \sum_{\hat{u} \in \times_{i \in N} U_i} \hat{p}(u)\theta^{S(\hat{u})}(o|\hat{u}) \tag{2.2}$$

$$\forall i \in N : \forall o \in O_i : \forall j \in N : \theta^{S(\hat{u})}(o|u_j) = \sum_{\hat{u} \in \times_{k \neq j} U_k \times u_j} \frac{\hat{p}(\hat{u})}{\hat{p}(u_j)} \theta^{S(\hat{u})}(o|\hat{u}) \tag{2.3}$$

These last definitions are the central elements of the measures in this paper as they express whether it is one's own preferences or the preferences of other players that determine which outcome occurs. For notational simplicity, where unambiguous the superscripts $S(\hat{u})$ will be omitted.

2.4 Concepts of Freedom

The concept of freedom has been heavily debated within Philosophy.[3] In a very influential article, Berlin (1958) attempted to categorize concepts of freedom in two categories: positive and negative freedom. Positive freedom refers to the actual ability to control one's own destiny while negative liberty refers to the absence of interferences of others in one's destiny. A common example

[3] For an overview, see Gaus and Courtland (2011).

(e.g. Carter, 2012) showing the difference is that of a smoker who due to his addiction does not have the ability to stop smoking. Due to this lack of control over whether he smokes or not, he does not have positive freedom. However, there is also nobody interfering with whether he smokes or not. Thus, he still has negative freedom. Only if another person was able to force him to smoke or prevent him from smoking would he lose negative freedom.

A third concept of freedom which according to its proponents falls out of the categorization by Berlin (1958) is republican liberty (e.g. Pettit, 1996). From the perspective of a republican liberal freedom is high when individuals are not subject to arbitrary power of other individuals. This idea is closely related to negative freedom. The main difference between the two is that republican freedom refers to the possibility of interference while negative freedom refers to actual interference. In the case of the smoker republican freedom is low if somebody has the possibility to prevent another from smoking (or force to smoke) even without actually preventing or forcing him.

All three concepts of liberty can be accomodated in the model. The following subsections will explore this in detail. For each concept, one measure will be derived. However, due to the large variety of conceptions of freedom by philosophers, it is unlikely that even when accounting for the three categories, the measures will fit all conceptions therein. Appendix 2.A provides an axiomatization of a more general class of measures than the three presented here. However, this axiomatization provides mainly a justification for the way causal relations can be measured and aggregated into a freedom measure and thus only applies to concepts of freedom that (implicitly) assume a causal relation between agents and outcomes as a primitive.

2.4.1 Positive Liberty

Positive freedom as the degree of rational self-control is the most straightforward to measure in the model. If U_i^h is a variable representing the preferences of the “higher” rational self which is unrestricted by addictions, irrationalities etc., then the degree to which an individual exhibits rational self-control can be measured by the causal influence from U_i^h to O_i.

Strategies for calculating this influence should be given by empirical behavior[4] S_e to account for deficits in self-control as in the example of the smoker. That is, an individual may behave according to a behavioral theory which may limit the influence of the preferences of the "higher" self on outcomes. Positive freedom is measured as:

$$\Phi^{pos}(\Game,\theta) = \sum_{u \in U_i^h} \hat{p}(u) \sum_{o \in O_i} \theta(o|u) \left(c(o,u) \ln \frac{\theta(o|u)}{\theta(o)} + d(o,u) \right) \tag{2.4}$$

where $c : O_i \times U_i \to \mathbb{R}$ and $d : O_i \times U_i \to \mathbb{R}$ are normative inputs into the measure. $c(o,u)$ and $d(o,u)$ are used to capture the value of having control over outcomes and to incorporate elements of freedom unrelated to the causal control, respectively. For example, Sen (1991) stressed the importance of freedom being increasing in well-being which can be captured using $c(o,u)$ and $d(o,u)$. This will be elaborated in more detail in Sections 2.7. The use of $c(o,u)$ to capture qualitative diversity as an important component of freedom will be shown in 2.5. For now, assuming $c(o,u) = 1$ and $d(o,u) = 0$, the focus rests on the causal influence measures $\ln \frac{\theta(o|u)}{\theta(o}$. The freedom measure is a weighted expectation of these $\ln \frac{\theta(o|u)}{\theta(o)}$ terms which are large if u makes o more likely and small if u makes o less likely. Going back to the example of the smoker without self control, let the outcomes be $O_i = \{s, ns\}$ and the preferences be $U_i^h = \{u_{ps}, u_{pns}\}$ where s and ns stand for the outcome of smoking or not smoking and u_{ps} is the utility function if the player prefers to smoke and u_{pns} is the utility function if the player prefers not to smoke. If the smoker has no self-control, we have $\theta(ns) = \theta(ns|u_{pns}) = 0$ and $\theta(s) = 1$. Therefore, $\Phi^{pos} = 0$ and the player has no freedom according to the measure since for each of the $\ln(\theta(o|u)/\theta(o)) = \ln(1) = 0$. If the smoker gains more self

[4]One can also use a descriptive model of behavior. The main point is that under positive freedom it is important that s_e reflects actual behavior (as opposed to theoretically optimal behavior), irrespective whether it is obtained from a descriptive model or field data.

control, $\theta(ns|pns) > \theta(ns) > \theta(ns|ps)$ and thus $\ln(\theta(ns|u_{pns})/\theta(ns))$ will increase while $\ln(\theta(s|u_{pns})/\theta(s))$ will decrease. Since each outcome-utility combination is weighted by their joint probability $\hat{p}(u)\theta(o|u)$, the former effect dominates and the overall effect will be an increase in the measure. While such self control cases are important for a proper measure of positive freedom, the present paper will mostly focus on limitations to freedom due to the structure of the game and not limits to rationality. Further research on freedom when agents are boundedly rational would be interesting, but are outside the scope of this paper.

2.4.2 Negative Liberty

Negative freedom is associated with two distinct aspects. The first is the idea of non-interference of others. Defining negative freedom, Berlin (1958) stated: "By being free in this sense I mean not being interfered with by others." (Berlin (1958), p.8). The second aspect is that of not being restricted by others: "Mere incapacity to attain a goal is not a lack of political freedom. [...] It is only because I believe that my inability to get a given thing is due to the fact that other human beings have made arrangements whereby I am, whereas others are not, prevented from having enough money with which to pay for it, that I think myself a victim of coercion or slavery" (Berlin (1958), p.7). These two aspects are quite distinct, since one can be restricted by others without being interfered with and vice versa.[5]

When measuring negative freedom, these two aspects open two possibilities. One is that negative freedom decreases in the extent to which the preferences of other players determine the outcomes of a player. This emphasizes the non-interference aspect of negative freedom. The other possibility is to emphasize the aspect of not being restricted by others. This would be achieved by some maximized version of the positive freedom measure, which does not depend on the actual ability of an agent to influence his outcomes but on the

[5]For example a group of the population who's vote does not count in an election is restricted in their political influence but not interfered with. A person convincing another not to go to vote may interfere with the other person's choice but does not restrict the other person's political influence.

potential ability given others' behavior. Since the latter can be interpreted as a variant of positive freedom, the proposed measure of negative freedom follows the idea of non-interference.

Thus, negative liberty of individual i is measured by the degree of causal influence of other individuals on the outcomes of individual i. Since negative liberty refers to actual interferences, both preferences of other players $\{U_j : j \neq i\}$ and strategy profiles s should equal their empirical counterparts. It is important to note that in such cases, all relevant institutions which limit an individual's freedom must be part of the model. That is, in order to measure limitations of freedom from the government, the government itself must be a player in the game.

$$\Phi^{neg}(\circlearrowleft,\theta) = -\sum_{j \neq i} \sum_{u_j \in \mathcal{U}_j^e} \hat{p}(u_j) \sum_{o \in O_i} \theta(o|u_j) \left(c(o,u_j) \ln \frac{\theta(o|u_j)}{\theta(o)} + d(o,u_j) \right) \tag{2.5}$$

Again the measure is a weighted expectation over the (negative) logarithmic terms measuring influence. In negative freedom we are interested however in the degree to which other agents' preferences determine one's outcomes which is why U_j is the variable causing or preventing o.

In the case of the smoker, nobody actually interferes with the decision of the smoker. Just as the smoker exercises no control over O_i, so does nobody else interfere, i.e. $\theta(ns) = \theta(ns|u_j)$. Suppose now i lives in a dictatorship where j may be a smoking-averse dictator $u_{j,pns}$ or a smoking-friendly dictator $u_{j,ps}$. If via the game played the dicator manages to influence the outcome of whether i smokes, freedom will be lower: $\theta(ns|u_{pns}) > \theta(ns) > \theta(ns|ps)$ and thus the measure decreases relative to the case where O_i and U_j are independent.

2.4.3 Republicanism

Republican liberty is closely related to negative liberty but does not only consider actual interventions of others, but also potential interventions. The most prominent conception of liberty in this class has been given by Pettit (1996), "taking the antonym of freedom to be subjugation, defenseless susceptibility to interference, rather than actual interference" (p.577). When measuring potential interferences of others, one can obviously no longer rely on empirical behavior S^e. A further difficulty in measuring this is the qualification that an agent must be "defenseless". We may consider a player i not to be defenseless if she has some means by which she may deter another player j from taking certain actions against her. This however requires some notion of rationality or at least responsiveness of j to the payoff threats that i can make against j. It therefore makes sense to use some equilibrium concept to solve the game and obtain $S^*(\hat{u})$ as the strategy profile with which to calculate $\theta^{S^*(\hat{u})}$ and the remaining probabilities defined in Section 2.3. Republican liberty[6] is measured as follows:

$$\Phi^{rep}(\Game,\theta) = -\max_{\hat{p}} \sum_{j \neq i} \sum_{u_j \in \mathcal{U}_j} \hat{p}(u_j) \sum_{o \in O_i} \theta(o|u_j) \cdot \left(c(o,u_j) \ln \frac{\theta(o|u_j)}{\hat{p}(u_j)\theta(o|u_j)} + d(o,u_j) \right) \tag{2.6}$$

By maximizing with respect to probability distributions over the preference relations of each individual $j \neq i$, the measure returns the maximal interference of others into i's affairs given that individuals still act rationally. The central difference to the negative freedom measure is that even if the actual distribution $\hat{p}$ is such that nobody would interfere with i, republican freedom may still be low if other individuals (in virtue of the structure of the game and thus the conditional probabilities $\theta(O_i|U_j)$) have potential influence on the outcomes of i. For example, if a dictator can decide whether to allow smoking or ban smoking, republican freedom is low even if the dictator in equilibrium decides

[6]One has to note though that Pettit (1996) acknowledges that for republican liberty "there are two subgoals involved. [...] One involves the reduction of subjugation, [...] the other involves the maximization of the domain of individual choice" (p. 593). This suggests that Pettit's conception of power should be measured as some aggregate of the above republican measure and a measure that is closer to the measure of positive freedom.

with probability 1 to allow smoking. In comparison, republican freedom is high if either there is no possibility to ban smoking or if many actors need to jointly decide to ban smoking for it to be banned. It is important to note that the measure strongly depends on the specification of U_j: a too narrow specification would bring the measure closer to negative freedom. A too wide specification may yield a very low freedom measure just by some absurd behavior which empirically never occurs.

2.5 Relation to Previous Measures

The positive freedom measure is a generalization of several measures in the freedom of choice literature and the literature on diversity measures. This section will explore these relations. Since all the measures are based on opportunity sets, it will be useful to define $\eth^T(C)$ as the trivial game where one player faces an opportunity set C as the outcomes of the game. The positive freedom measure turns out to be a generalization of the freedom measure by Jones and Sugden (1982):

Proposition 2.1. *Suppose for all trivial games $\eth^T(C)$ normative inputs $\forall x \in C, u \in U_i$: $c(x,u) = 1$, $d(x,u) = 0$ and rationality of player i:*

$$\forall u \in U_i : \exists x \in C : \theta(x|u) = 1 \Leftrightarrow x \in \arg\max_{x \in C} u(x). \tag{2.7}$$

Moreover, assume that U_i are utility representations of the reasonable preferences:

$$\forall R \in \mathcal{R} : \exists u \in U_i : \forall x, x' \in C : xRx' \Leftrightarrow u(x) \geq u(x'). \tag{2.8}$$

Let all outcomes with positive probability be equally probable:

$$\theta(x) > 0 \wedge \theta(x') > 0 \Rightarrow \theta(x) = \theta(x'). \tag{2.9}$$

Then positive freedom is a representation of the measure by Jones and Sugden

(1982):

$$\Phi^{pos}(\supsetneq^T(C),\theta) \geq \Phi^{pos}(\supsetneq^T(C'),\theta) \Leftrightarrow C \succsim_{F,JS} C' \qquad (2.10)$$

For the positive freedom measure to converge to the reasonable preference measure three steps are needed: First, no additional normative information such as utility or qualitative diversity may be included. Second, individuals must be rational and the utility functions representation of the reasonable preference relations. Third, quantitative diversity must be maximal such that each outcome is equally likely. If the set of reasonable preferences is the set of rational preferences, the measures of Jones and Sugden (1982) and Pattanaik and Xu (1990) are equivalent and positive freedom represents also the cardinality measure. Therefore, Φ^{pos} generalizes the measures of Jones and Sugden (1982) and Pattanaik and Xu (1990) by allowing for the game theoretic setting, normative inputs $c(o,U)$ and $d(o,U)$, and the quantitative diversity of outcomes.

The connection to the measure of the quantitative diversity of choices by Suppes (1996) is as follows:

Proposition 2.2. *Suppose for all trivial games that $\supsetneq^T(C)$ normative inputs are $\forall x \in C, u \in U_i$: $c(x,u) = 1$, $d(x,u) = 0$ and that the individual has full control over the outcomes:*

$$\forall u \in U_i : \exists x \in C : \theta(x|u) = 1. \qquad (2.11)$$

Then positive freedom is equal to the entropy freedom measure:

$$\Phi^{pos}(\supsetneq^T(C),\theta) = F_S(C,\theta). \qquad (2.12)$$

The measure can also include qualitative aspects of diversity. It turns out that by adding up the positive freedom of all attributes, one can obtain a generalization of $D_{NP}(C,P,\lambda)$. For this, define $\supsetneq^T(C,A)$ as the trivial game $\supsetneq^T(C)$ with C being the set of terminal nodes and the set of outcomes O_i replaced by a partitioning of nodes into members with attribute A and members without,

$O_{i,A} = \{A, A^C\}$.

Proposition 2.3. *Suppose for all trivial games* $\Game^T(C,A)$ *normative inputs are* $c(A,u) = \lambda(A)$, $c(A^C,u) = 0$, *and* $d(o,U) = 0$. *Further, suppose full control over the outcomes*

$$\forall u \in U_i : \exists x \in X : \theta(x|u) = 1. \tag{2.13}$$

Then positive freedom is equal to the diversity measure by Nehring and Puppe (2009)

$$\sum_{A \subseteq X} \Phi_i^{pos}(\Game_A, \theta) = D_{NP}(C, \theta, \lambda) \tag{2.14}$$

This means that when considering a game where a player has full control over the terminal nodes, we can measure aggregate diversity as the sum of positive freedom over all attributes. In situations of imperfect control, Φ^{pos} additionally controls for whether the diversity of chosen objects is due to the preferences of the individual or due to force by others, since if the realization of the attribute A is independent of U_i, i.e $\forall u \in U_i : \theta(A|u) = \theta(A)$, then $\Phi_i^{pos}(\Game_A, \theta) = 0$ and the added diversity from A does not increase freedom.

2.6 Freedom and Labor Market Discrimination

In this section the central properties of the freedom measures are discussed via the example of discrimination in the labor market. A simple game is studied where wages are fixed before the employer knows the applicant and the main source of discrimination is the difference in acceptance rates of equally qualified men and women. An empirical study can be found in Oaxaca and Ransom (1994). Let there be two players, $N = \{1,2\}$ with 1 being an applicant to an employer 2. The applicant can be either female or male, $g \in \{f, m\}$, which is determined by nature with probability 1/2. Also, the productivity ζ of the applicant is distributed uniformly over the interval $[0,1]$. After gender and productivity has been determined, the applicant chooses whether to apply or not apply $x_1 \in \{ap, nap\}$. If the applicant does not apply, the game ends and the applicant has no job. If the applicant applies, the employer decides whether to accept or reject $x_2 \in \{ac, nac\}$ the applicant which results in the applicant having a job (job) or not $(nojob)$. The outcome space for the employer is $\{noworker\} \cup [0,1] \times \{male, female\}$.[7] The last part necessary for the specification of the game is the information partition. We may consider two variants: First, $\eth_1$ is the perfect information game where each information set is a singleton. Therefore, the employer knows the gender and productivity of the applicant. Second, with $\eth_2$ we may consider a game where the gender of the applicant is not revealed before the application.

Given the extensive game form, we can measure the freedom if we supplement it with a preference expansion. Let the utility functions of the applicant be $U_1 = \{u_1^A, u_1^B\}$ with

$$u_1^A(job) = 1 > 0 = u_1^A(nojob) \tag{2.15}$$

$$u_1^B(job) = 0 < 1 = u_1^B(nojob). \tag{2.16}$$

Therefore, type A likes to take the job while type B does not. Both types occur with probability 1/2. The employer can either be discriminatory or nondiscriminatory: $U_2 = \{u_2^A, u_2^B\}$ with $u_2^A(noworker) = u_2^B(noworker) = 1$, $u_2^A(\zeta) = 2\zeta$, $u_2^B(\zeta, male) = 2(\zeta + d)$, and $u_2^B(\zeta, female) = 2(\zeta - d)$. The extent to which the discriminatory employer has preferences against women is therefore measured by $d \in [0, 0.5]$. Let the fraction of discriminatory types B be δ. Finally, $\mathcal{J} = \{\{\{u_1^A, u_2^A, u_2^B\}, \{u_1^B, u_2^A, u_2^B\}\}, \{\{u_2^A, u_1^A, u_1^B\}, \{u_2^B, u_1^A, u_1^B\}\}\}$ is the information partitioning over the preferences such that each individual knows his own preferences.

It is straightforward to calculate the Bayesian Nash equilibrium of each game:

[7] One may hold the normative conviction that the employer's freedom on whom to employ is also normatively relevant. In this case, one would need to find a way to aggregate the two. Since this would require a further analysis of interpersonal comparisons of freedom, this is outside the scope of this paper.

$$x_1^{\eth_1} = x_1^{\eth_2} = \begin{cases} ap, & U_1 = u_1^A \\ nap, & \text{else} \end{cases} \tag{2.17}$$

$$x_2^{\eth_1} = \begin{cases} ac, & U_2 = u_2^A \ \wedge \ \zeta \geq 1/2 \\ ac, & U_2 = u_2^B \ \wedge \ \zeta + d \geq 1/2 \ \wedge \ g = m \\ ac, & U_2 = u_2^B \ \wedge \ \zeta - d \geq 1/2 \ \wedge \ g = f \\ nac, & \text{else} \end{cases} \tag{2.18}$$

$$x_2^{\eth_2} = \begin{cases} ac, & \zeta \geq 1/2 \\ nac, & \text{else} \end{cases} \tag{2.19}$$

Assume that empirical strategies are identical with the above given Bayesian Nash equilibrium strategies. Then the above strategies give us the probability distribution $\theta(O_1|U_1)$ for the positive measure as shown in Tables 2.1 and 2.2.

$\eth_1$	job	$nojob$
u_1^A	$1/2 + \delta d(1 - 2 \cdot \mathbb{1}(g = f))$	$1/2 - \delta d(1 - 2 \cdot \mathbb{1}(g = f))$
u_1^B	0	1
u_2^A	$1/4$	$3/4$
u_2^B	$1/4 + d(1 - 2 \cdot \mathbb{1}(g = f))/2$	$3/4 - d(1 - 2 \cdot \mathbb{1}(g = f))/2$
marginal	$1/4 + \delta d(1 - 2 \cdot \mathbb{1}(g = f))/2$	$3/4 - \delta d(1 - 2 \cdot \mathbb{1}(g = f))/2$

Table 2.1: $\theta(O|U)$ in $\eth_1$

$\eth_2$	job	$nojob$
u_1^A	$1/2$	$1/2$
u_1^B	0	1
u_2^A	$1/4$	$3/4$
u_2^B	$1/4$	$3/4$
marginal	$1/4$	$3/4$

Table 2.2: $\theta(O|U)$ in $\eth_2$

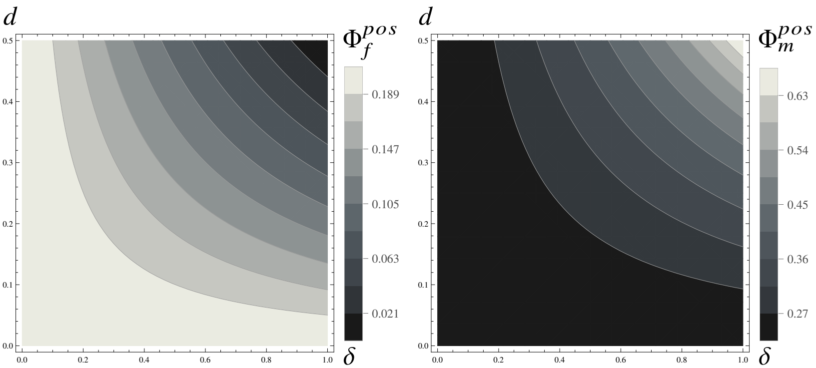
Positive freedom for each game is given by the following calculations:

$$\Phi_f^{pos}(\reflectbox{∂}_1,\theta) = \frac{1-2\delta d}{4}\ln(2) + \frac{1+2\delta d}{4}\ln\left(\frac{2+4\delta d}{3+2\delta d}\right) + \frac{1}{2}\ln\left(\frac{4}{3+2\delta d}\right) \tag{2.20}$$

$$\Phi_m^{pos}(\reflectbox{∂}_1,\theta) = \frac{1+2\delta d}{4}\ln(2) + \frac{1-2\delta d}{4}\ln\left(\frac{2-4\delta d}{3-2\delta d}\right) + \frac{1}{2}\ln\left(\frac{4}{3-2\delta d}\right) \tag{2.21}$$

$$\Phi^{pos}(\reflectbox{∂}_2,\theta) = (3/4)\ln(4/3), \tag{2.22}$$

where for simplicity $c(o,u) = 1$ and $d(o,u) = 0$. Moreover, in $\reflectbox{$\partial$}_2$ freedom does not depend on gender. The comparative statics are very intuitive as can be seen in Figure 2.1: Men's freedom increases in δ and d since the more likely it is that they will be employed if they apply, the greater their control over their employment status. For women the opposite is the case: If they apply, their employment chances are negatively affected by δ and d, which translates into lower freedom. If $\delta d = 0$ in $\reflectbox{$\partial$}_1$ or if $\reflectbox{$\partial$}_2$ is played, their freedom is maximal.



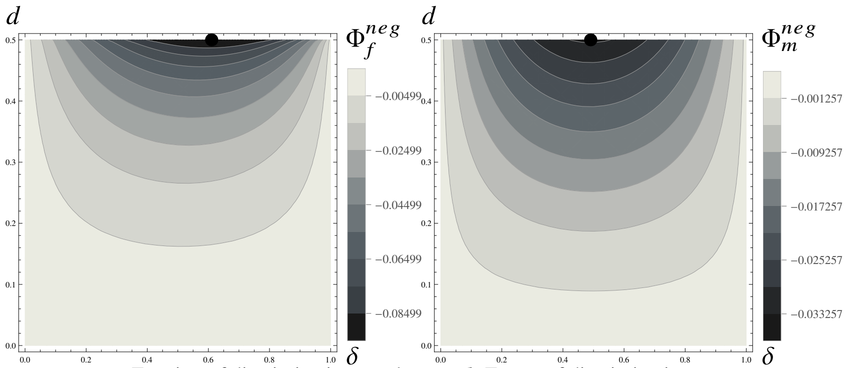
δ: Fraction of discriminating employers,
d: Extent of discrimination

Figure 2.1: Positive Freedom of Women and Men in $\reflectbox{$\partial$}_1$

$$
\begin{aligned}
\Phi_f^{neg}(\Game_1,\theta) =& -\frac{\delta(1-2d)}{4}\ln\left(\frac{1-2d}{1-2\delta d}\right) - \frac{\delta(3+2d)}{4}\ln\left(\frac{3+2d}{3+2\delta d}\right) \\
& -\frac{(1-\delta)}{4}\ln\left(\frac{1}{1-2\delta d}\right) - \frac{(1-\delta)3}{4}\ln\left(\frac{3}{3+2\delta d}\right) \qquad (2.23) \\
\Phi_m^{neg}(\Game_1,\theta) =& -\frac{\delta(1+2d)}{4}\ln\left(\frac{1+2d}{1+2\delta d}\right) - \frac{\delta(3-2d)}{4}\ln\left(\frac{3-2d}{3-2\delta d}\right) \\
& -\frac{(1-\delta)}{4}\ln\left(\frac{1}{1-2\delta d}\right) - \frac{(1-\delta)3}{4}\ln\left(\frac{3}{3+2\delta d}\right) \qquad (2.24) \\
\Phi^{neg}(\Game_2,\theta) =& 0, \qquad (2.25)
\end{aligned}
$$

where again the normative inputs have been set to $c(o,u) = 1$ and $d(o,u) = 0$. The central difference of negative freedom is that freedom for men is now also decreasing in d. This is because the positive discrimination of discriminating employers towards men constitutes just as much of an interference as their negative discrimination against women. This is a central conceptual difference: under negative freedom, affirmative action or other types of positive discrimination restrict freedom and are thus undesirable. No matter whether an individual is improved or worsened in its well-being by an interference of another player, in both cases negative freedom decreases in the extent and intensity of discrimination. What negative freedom shares with positive freedom is that $\delta d = 0$ yields the same freedom as $\Game_2$: The fact that nobody actually discriminates men and women is equivalent with a setting in which the employer cannot discriminate due to the structure of the game.

What appears counterintuitive at first is the fact that the negative freedom measure is not monotone in δ. This is related to the fact that it is a measure of actual interference but not restriction. If all employers discriminate women, the employment status of an individual no longer depends on the variations in the preferences of others. Naturally, one could move away from using the actual distribution of preferences to a hypothetical distribution to account for the fact that the employer still has power over the employee. However, by this one would move from actual to hypothetical interference and at a republican



δ: Fraction of discriminating employers; d: Extent of discrimination;
•: Republican liberty

Figure 2.2: Republican and Negative Freedom of Women and Men in $\eth_1$

liberty measure:

$$\Phi_f^{rep}(\eth_1,\theta) = \min_{\delta,d} \Phi_f^{neg}(\eth_1,\theta) \tag{2.26}$$

$$\Phi_m^{rep}(\eth_1,\theta) = \min_{\delta,d} \Phi_m^{neg}(\eth_1,\theta) \tag{2.27}$$

$$\Phi^{rep}(\eth_2,\theta) = 0 \tag{2.28}$$

Republican freedom in this model simply maximizes negative freedom with respect to the fraction of discriminating employers and the intensity of discrimination. The minimal freedom of women is reached at $d = 1$ and $\delta \approx 0.61$ as marked with the dots in Figure 2.2. Notice that in practice one may want to put restrictions on d for this measure, since extreme values for d may no longer be economically feasible for the employer. The republican freedom measure shares with the negative freedom measure the impossibility of positive discrimination increasing freedom. The main difference is that one does not need to observe actual interference to conclude that a person is unfree. Just by the information structure given in $\eth_1$ the applicant's freedom is jeopardized. This fits with current debates on privacy rights: From the perspective of a republican libertarian a corporation or government does not need to actually interfere with the lives of individuals to violate their freedom. It already constitutes a violation of freedom if it has acquired the information necessary to interfere. Having his information, even if it is not used to interfere in equilibrium, gives nontheless the potential to interfere which decreases republican freedom.

2.7 Freedom and Utility

An important issue in the discussion on measures of freedom has been the integration of utility into the measures such that freedom is increasing in well-being.[8] However, Puppe (1995), Nehring and Puppe (1996), Gravel (1998), Pattanaik and Xu (1998), and Baharad and Nitzan (2000) have provided impossibility results with respect to this endeavor. The freedom measures in this paper allows for the integration of utility in two different ways.[9] The first is simply additive via the function $d(\dots)$, while the second via $c(\dots)$ yields a more intricate connection between freedom and utility. For simplicity, only positive freedom will be considered. The combination utility with negative freedom or republican freedom is conceptually unclear, since in these measures it seems to intuitively make more sense to additionally include the qualitative impact other individuals have on one's life.

First, suppose $d(u,o) = u(o)$; then expected utility

$$EU_i(\ni,\theta) = \sum_{u \in U_i} \hat{p}(u) \sum_{o \in O_i} \theta(o|u) u(o) \tag{2.29}$$

[8] E.g. Sen (1991) or Sen (1985). Although the remarks here focus on the integration of utility into a freedom measure, they also hold for other measures of well-being, e.g. value functions over capabilities as long as they are at least defined on an ordinal scale.

[9] In principle there of course exist infinite possibilities to integrate freedom and utility. Therefore, mathematically the following results are trivial. They only gain relevance in conjunction with the axiomatization in Appendix 2.A which rules out other ways than $c(\dots)$ and $d(\dots)$ of integrating utility.

enters the positive freedom measure additively such that:

$$\Phi^{pos}(\backepsilon,\theta) = \sum_{u \in U_i^h} \hat{p}(u) \sum_{o \in O_i} \theta(o|u)c(o,u) \ln \frac{\theta(o|u)}{\theta(o)} + EU_i(\backepsilon,\theta) \tag{2.30}$$

This approach yields a simple, but plausible way to combine a measure of freedom and a measure of well-being. It allows for tradeoffs between utility gains and freedom gains and $c(o,u)$ controls how important freedom over outcome $o \in O_i$ is for type $u \in U_i$. Using $c(o,u)$ the measurer can make utility and freedom cardinally comparable such that one unit of causal influence on outcomes has the same value as $c(o,u)$ units of utility. The measure of course does not contradict the existing impossibility theorems. For example, Gravel (1994, p. 456, Axiom 3) requires for a measure defined over opportunity sets that if the highest ranked alternative of one set C is preferred to the highest ranked alternative of another set C', the set C is higher ranked than the set C'. This axiom would never allow a tradeoff of more freedom for less utility as the above measure does. The main way in which the present paper avoids these impossibility results is therefore to make causal influence over outcomes cardinally comparable to utility. This is done via the additional information provided by the strategies of the players. The opportunity-set based impossibility theorems cited above in comparison only use information about O_i.

The particular way in which freedom and utility are combined above, can be unsatisfactory however. One reason may be the perfect substitutability of freedom for utility and vice versa. Another reason may be the fact that the *direction* of the influence of the agent is not accounted for: Suppose by some lack of knowledge an agent always chooses the option *worst* for himself. As long as the $c(o,u)$ values are all positive, the measure still increases in the degree of control the agent has over the outcomes. While one can account for this explicitly via accounting for the agent's beliefs in $c(o,u)$, there is a more elegant way to solve this problem, which is the second way in which utility can enter the measure:

Suppose $c(u,o) = u(o)$ and $d(o,u) = 0$:

$$\Phi^{pos}(\eth,\theta) = \sum_{u \in U_i^h} \hat{p}(u) \sum_{o \in O_i} \theta(o|u)u(o) \ln \frac{\theta(o|u)}{\theta(o)} \tag{2.31}$$

The advantage of integrating freedom and well-being via the function $c(\ldots)$ is the complementarity between choice and well-being: There are now outcomes where without additional utility ($u(o) = 0$), more influence is irrelevant. Also, without any influence on outcomes ($\ln(\ldots) = 0$), additional utility is irrelevant. Finally, having control over the outcomes may be detrimental for one's freedom if the associated utility level is negative. It is important to realize that under this specification utility is no longer measured on a cardinal scale but instead on a ratio scale. The $u(o) = 0$ point becomes meaningful as the point where freedom does not matter.

The connection with expected utility is less obvious here: Suppose the individual has full control over the outcomes such that for each $u \in U_i$ there exists some $o \in O_i$ with $\theta(o|U) = 1$. Suppose further that all outcomes are equally likely: $\theta(o) = 1/|O_i|$. Then the measure becomes:

$$\Phi^{pos}(\eth,\theta) = \sum_{u \in U_i^h} \hat{p}(u) \sum_{o \in O_i} \theta(o|u)u(o) \ln(|O_i|) \tag{2.32}$$

Under the above given conditions, freedom is therefore expected utility times the logarithm of the number of alternatives. This captures the idea that freedom is both increasing in our well-being and the number of options we have. Moreover, one can neither be free without well-being nor without being able to influence one's outcomes.

2.8 Preference for Freedom

A natural application of the positive freedom measure accounting for utility is the case where agents prefer to be in situations where they have freedom with respect to some outcomes of a game. Such individuals may be thought of as maximizing a combination of freedom and utility as in (2.31) or (2.32). Examples of such behavior are given by an experiment in Fehr et al. (2013). In this study, individuals were given a choice between keeping or delegating a decision right and delegating was associated with some gain in expected payoffs. In the experiment, subjects chose to delegate to a lower extent than what could have been explained from utility maximization. This section shows that preference for freedom can explain the choices the subjects made in the experiment by Fehr et al. (2013).

Some further notation will be necessary. Let the subgame function denoted by $subg(\Game, a)$ return for any extensive form game $\Game$ the subgame starting at node a. Let θ_i be a joint probability distribution over nodes and outcomes representing the beliefs of player i. Let $\theta_i(b|I)$ denote the belief that the current node is $b \in I$ given that play has reached information set I. We can construct the belief of node a following the current information set given strategy s as $\tilde{\theta}(a|s) = \theta_i(\psi(a)|I)s(c(a|\psi(a)))$.

Define an equilibrium of $\Game$ as a strategy profile $S^* = (s^*(I,\theta_i)|_{I \in I_i}|_{i \in N})$ and beliefs such that $\forall i : \theta_i = \theta^{S^*}$ with:

$$s^*(I,\theta_i) = \arg\max_{s \in \Delta(C_I)} \sum_{a \in A(I)} \tilde{\theta}_i(a|s) V_i(subg(\Game, a), \theta_i) \tag{2.33}$$

Individuals therefore maximize the expected valuation V_i of the possible subgames following their choice s, given beliefs θ_i which are true in equilibrium. If V_i is the expected utility of the subgame, this definition corresponds to the Bayesian Nash equilibrium.

Definition 2.6. Freedom Adjusted Expected Utility:
Let the normative inputs in Φ_i^{pos} be:

$$c_i(u,o) = \alpha u(o) \tag{2.34}$$

$$d_i(u,o) = (1-\alpha)u(o) \tag{2.35}$$

then preference for positive freedom is represented by the valuation function:

$$
\begin{aligned}
V_i(\ni,\theta_i) =& \Phi_i^{pos}(\ni,\theta_i) \\
=& \alpha \sum_{u \in U_i} p(u) \sum_{o \in O_i} \theta_i(o|u)u(o) \ln\left(\frac{\theta_i(o|u)}{\theta_i(o)}\right) \\
&+ (1-\alpha) \sum_{u \in U_i} p(u) \sum_{o \in O_i} \theta_i(o|u)u(o)
\end{aligned} \tag{2.36}
$$

In the specified valuation function, α controls the degree to which positive[10] freedom is relevant for the agent. If $\alpha = 0$, then $V_i(\dots)$ equals expected utility. The higher α, the more relevant freedom becomes.

In the experiment by Fehr et al. (2013) subjects played a game about a card selection where nature randomly determined the player's preferences over 36 cards. One default card always had a fixed, known payoff $\bar{\pi}$, but the preferences over all other cards were unknown to both players at the beginning of the game. One of the other cards gave a high payoff $\hat{\pi}_1$ to player 1 and a lower payoff $\check{\pi}_2$ to player 2. Yet another card gave a high payoff $\hat{\pi}_2$ to player 2 and a lower payoff $\check{\pi}_1$ to player 1. All other cards gave an extremely low payoff $\not\pi$ to deter the player with the decision right to randomly choose a card. Payoffs were for each player i ordered as follows: $\hat{\pi}_i > \check{\pi}_i > \bar{\pi} > \not\pi$.

Player 1, the principal, could then choose to delegate a decision right to the agent player 2 or to keep it. Next, there was an investment stage at which both players could simultaneously invest effort (payoff) in order to raise the probability with which they learned about their preferences over the cards in the following stage. Let p_i be the probability of the player with the decision right and q_j be the probability of the player without the decision right. After players learned about their preferences, there was a card selection stage in which the player without the decision right could make a suggestion to the other player. Then the player with the decision right selected one of the cards.

Let $U_p \in \{1,\dots,35\}$ represent the possible preferences of the principal for

[10] A similar definition would also be possible for republican or negative freedom. In fact, with a different experimental setup one could determine the preference for each concept of freedom separately. See Chapter 3 for such an experiment.

her favorite card. Similarly, let $U_a \in \{1,...,35\}$ represent the possible preferences of the agent for his favorite card. In calculating this example, players are assumed to be risk neutral.[11]

The game can be solved using backward induction. Obviously, the last stage is uninfluenced by preference for freedom: If the player i with the decision right knows his preferences, he will choose the card giving payoff $\hat{\pi}_i$. If he does not know his preferences, but the other player has made a suggestion, he will choose the suggested card if he believes it is the card giving him payoff $\check{\pi}_i$ (in equilibrium, this is the case). In all other cases, the player will choose the default card giving payoff $\bar{\pi}$.

For the player j without the decision right, strategies are similarly simple: In case he knows his preferences, he will recommend the card giving payoff $\hat{\pi}_j$ to himself and $\check{\pi}_i$ to the other player. In case he does not know his preferences, he will recommend the card giving payoff $\bar{\pi}$.

In the previous stage, nature determines randomly whether the players learn about their preferences over cards. This happens with the previously chosen probabilities p_i for the player with the decision right and q_j for the player without the decision right.

The stage at which preference for freedom influences the decision making of the agents is the bidding stage. Under Nash equilibrium behavior with risk aversion, we should observe the following optimal efforts:

$$p_i^{*NE} = \arg\max_{p_i} p_i\hat{\pi}_i + (1-p_i)(q_j^*\check{\pi}_i + (1-q_j^*)\bar{\pi}) - c(p_i) \tag{2.37}$$

$$q_j^{*NE} = \arg\max_{q_j} p_i^*\check{\pi}_j + (1-p_i^*)(q_j\hat{\pi}_i + (1-q_j)\bar{\pi}) - c(q_j) \tag{2.38}$$

To determine efforts given preference for freedom, we need to measure freedom in the subgame after effort has been invested. Let $a(p_i,q_j,D)$ be the node in the game where the player with the decision right has invested p_i and the player without the decision right has invested q_j and where delegation

[11] It has already been verified by Fehr et al. (2013) that the players' measured risk/loss aversion cannot explain the behavior in the game.

decision $D \in \{0,1\}$ has been made. The freedom of player i with the decision right is:

$$\begin{aligned}\Phi_{i,dr}(subg(\eth,a(p_i,q_j,D)),\theta) = \\ p_i(\hat{\pi}_i - c(p_i))\left(\alpha\ln\left(\frac{35p_i}{p_i+(1-p_i)q_j}\right)+1-\alpha\right) \\ +(1-p_i)q_j(\check{\pi}_i - c(p_i))\left(\alpha\ln\left(\frac{35(1-p_i)q_j}{34(p_i+(1-p_i)q_j)}\right)+1-\alpha\right) \\ +(1-p_i)(1-q_j)(\bar{\pi}-c(p_i))(1-\alpha)\end{aligned} \quad (2.39)$$

where θ are equilibrium beliefs which contain the above described behavior in subsequent stages. The freedom of the player j who does not have the decision right is:

$$\begin{aligned}\Phi_{i,ndr}(subg(\eth,a(p_i,q_j,D)),\theta) = \\ p_i(\check{\pi}_j - c(q_j))\left(\alpha\ln\left(\frac{35p_i}{34(p_i+(1-p_i)q_j)}\right)+1-\alpha\right) \\ +(1-p_i)q_j(\hat{\pi}_j - c(q_j))\left(\alpha\ln\left(\frac{35(1-p_i)q_j}{p_i+(1-p_i)q_j}\right)+1-\alpha\right) \\ +(1-p_i)(1-q_j)(\bar{\pi}-c(q_j))(1-\alpha)\end{aligned} \quad (2.40)$$

Under preference for freedom, the optimal efforts are given by the system of equations:

$$p_i^{*\Phi} = \arg\max_{p_i} \quad \Phi_{i,dr}(subg(\eth,a(p_i,q_j^*,D),\theta) \quad (2.41)$$

$$q_j^{*\Phi} = \arg\max_{q_j} \quad \Phi_{j,ndr}(subg(\eth,a(p_i^*,q_j,D),\theta) \quad (2.42)$$

For both players, marginal utility from effort has increased, but even more so for player i. Player j will only gain valuation from freedom with probability $(1-p_i)$, i.e. if player i does not learn about his preferred card. Therefore, we should expect p_i^* to be larger for players who have $\alpha > 0$ than players who maximize expected utility ($\alpha = 0$).

In the delegation stage, given expected payoff maximization we have:

$$D^{*NE} = \mathbb{1}\Big(p_1^*\hat{\pi}_1 + (1 - p_1^*)(q_2^*\check{\pi}_1 + (1 - q_2^*)\bar{\pi}) - c(p_1^*) \\ < p_2^*\check{\pi}_1 + (1 - p_2^*)(q_1^*\hat{\pi}_1 + (1 - q_1^*)\bar{\pi}) - c(q_1^*)\Big) \quad (2.43)$$

The principal, player 1, simply compares the situation in which he has control and plays the optimal p_1^* to the situation in which he does not have control and plays the optimal q_1^*, subject to the other player also playing the optimal p_2^* and q_2^*. Under preference for freedom, player 1 compares the freedom in the situation with or without delegation:

$$D^{*\Phi} = \mathbb{1}\left(\Phi_{i,dr}\left(subg(\eth, a(p_1^*, q_2^*, 0), \theta\right) < \Phi_{i,ndr}\left(subg(\eth, a(p_2^*, q_1^*, 1), \theta)\right)\right) \quad (2.44)$$

where use has been made of the fact that p_i^* and q_i^* do not depend on U_i and U_j and therefore Φ does not depend on whether it is measured at $a(p_i^*, q_j^*, D)$ or at the node following the delegation decision. The above condition can be interpreted as follows. Player 1 compares the freedom he has if he has the decision right ($\Phi_{1,dr}$) after he has not delegated ($D = 0$) and has played p_1^* and the other player has played q_2^* to the freedom he has if he does not have the decision right ($\Phi_{1,ndr}$) after he has delegated ($D = 1$) and has played q_1^* while the other player has played p_2^*. Looking back at (2.39) and (2.40), compared to a player maximizing expected utility, a player with a large α will require much larger gains in expected payoffs to delegate. Under preference for freedom, we should therefore see lower delegation rates.

In Table 2.3 the payoffs used in the experiment by Fehr et al. (2013) are given. The players had identical cost functions with $c(p) = 25 \cdot p^2$. Based on this information, predictions of preference for freedom for a representative player can be made.

From Table 2.4 we can see that preference for freedom improves the Nash

Treatment	$\hat{\pi}_1$	$\check{\pi}_2$	$\check{\pi}_1$	$\hat{\pi}_2$	$\bar{\pi}$	$\not{\pi}$
PLOW	40	35	20	40	10	0
LOW	40	20	20	40	10	0
HIGH	40	35	35	40	10	0
PHIGH	40	20	35	40	10	0

Table 2.3: Payoffs in each treatment

	q_j			p_i			D		
Treatment	Obs.	$\alpha = 0.1$	$\alpha = 0$	Obs.	$\alpha = 0.1$	$\alpha = 0$	Obs.	$\alpha = 0.1$	$\alpha = 0$
PLOW	19.6	25.3	27.3	61.9	65.1	54.5	16.3	0	0
LOW	15.2	26.5	27.3	67.2	65.1	54.5	13.9	0	0
HIGH	23.0	30.1	34.3	53.4	59.8	42.9	35.5	0	100
PHIGH	19.0	31.6	34.3	61.6	59.4	42.9	42.7	0	100

Table 2.4: Observed average strategies and predicted strategies with $\alpha = 0.1$ and $\alpha = 0$

equilibrium predictions in several ways: first, preference for freedom explains the overprovision of effort[12] by the person with the decision right. Since freedom increases in p_i, the individual with the decision right is more willing to contribute effort. Second, preference for freedom explains the low delegation rates in treatments HIGH and PHIGH. For a principal with $\alpha = 0.1$ the loss of freedom in the last stage outweights the expected payoff gains associated with delegation. While $\alpha = 0.1$ was chosen to roughly optimize the match between observed and predicted q_j and p_i, one could naturally improve the fit further by considering heterogeneous agents which vary in α. For example, below $\alpha \approx 0.08$ an individual would delegate in PHIGH and below $\alpha \approx 0.06$ also in HIGH. With heterogeneous α, the observed delegation rates would naturally arise out of the model.

[12]Observed p_i (q_j) is the arithmetic mean of the strategies of the principal and the agent in case they did (not) have the decision right. The numbers were originally reported separately for principal and agent.

2.9 Freedom and Voting Systems

This section makes use of and exemplifies the capacity of the measure to incorporate diversity information. In a democracy, it is not only important that votes can be cast freely and have an influence on the election result, but also that the candidates are sufficiently distinct to make the vote meaningful.

Let the political spectrum be $S = [0,1]$. There are three politicians competing with exogeneous positions $o_1, o_2, o_3 \in S$. Assume a point line diversity measure (Nehring & Puppe, 2002) such that the qualitative diversity of the outcomes is given by: $v(O) = v(o_1) + d(o_1,o_2) + d(o_2,o_3)$ with $d(o_1,o_2) = (o_2 - o_1)^{1/2}$ and $v(o_1) = 0$. Then $\lambda_1 = d(o_1,o_2)$, $\lambda_2 = d(o_1,o_2) + d(o_2,o_3) - d(o_1,o_3)$ and $\lambda_3 = d(o_2,o_3)$. The maximum diversity candidate composition is then given by $(0, 0.5, 1)$.

Define the following notation for the binomial distribution:

$$f_k^n(x) = x^k (1-x)^{n-k} \frac{n!}{k!(n-k)!} \tag{2.45}$$

Consider now a voting model with $2k - 1$ voters where $k \in \mathbb{N}$. Suppose voters have preferences $u_i(o) = \|o - \alpha_i\|$ where α_i is distributed over S with density $p(\alpha_i) = 1$. If the median voter theorem holds, we have for the distribution $\theta(o)$:

$$\theta(o_1) = \int_0^{\frac{o_1+o_2}{2}} f_{k-1}^{2k-2}(x)(2k-1)dx \tag{2.46}$$

$$\theta(o_2) = \int_{\frac{o_1+o_2}{2}}^{\frac{o_2+o_3}{2}} f_{k-1}^{2k-2}(x)(2k-1)dx \tag{2.47}$$

$$\theta(o_3) = \int_{\frac{o_2+o_3}{2}}^{1} f_{k-1}^{2k-2}(x)(2k-1)dx \tag{2.48}$$

and for the conditional distribution $\theta(o|u_i)$:

$$\theta(o_1|\alpha_i) = \begin{cases} \int_0^{\frac{o_1+o_2}{2}} f_{k-1}^{2k-3}(x)(2k-2)dx, & \frac{o_1+o_2}{2} < \alpha_i \\ \frac{1}{2} f_{k-1}^{2k-2}(\alpha_i) + \int_0^{\frac{o_1+o_2}{2}} f_{k-1}^{2k-3}(x)(2k-2)dx, & \frac{o_1+o_2}{2} = \alpha_i \\ f_{k-1}^{2k-2}(\alpha_i) + \int_0^{\alpha_i} f_{k-1}^{2k-3}(x)(2k-2)dx & \\ \qquad + \int_{\alpha_i}^{\frac{o_1+o_2}{2}} f_{k-2}^{2k-3}(x)(2k-2)dx, & \frac{o_1+o_2}{2} > \alpha_i \end{cases} \tag{2.49}$$

$$\theta(o_2|\alpha_i) = \begin{cases} \int_{\frac{o_1+o_2}{2}}^{\frac{o_2+o_3}{2}} f_{k-1}^{2k-3}(x)(2k-2)dx, & \frac{o_2+o_3}{2} < \alpha_i \\ \frac{1}{2} f_{k-1}^{2k-2}(\alpha_i) + \int_{\frac{o_1+o_2}{2}}^{\frac{o_2+o_3}{2}} f_{k-2}^{2k-3}(x)(2k-2)dx, & \frac{o_1+o_2}{2} = \alpha_i \\ f_{k-1}^{2k-2}(\alpha_i) + \int_{\frac{o_1+o_2}{2}}^{\alpha_i} f_{k-1}^{2k-3}(x)(2k-2)dx & \\ \qquad + \int_{\alpha_i}^{o_2+o_3} f_{k-2}^{2k-3}(x)(2k-2)dx, & \frac{o_1+o_2}{2} < \alpha_i < \frac{o_2+o_3}{2} \\ \frac{1}{2} f_{k-1}^{2k-2}(\alpha_i) + \int_{\frac{o_1+o_2}{2}}^{\frac{o_2+o_3}{2}} f_{k-1}^{2k-3}(x)(2k-2)dx, & \frac{o_2+o_3}{2} = \alpha_i \\ \int_{\frac{o_2+o_3}{2}}^{\frac{o_1+o_2}{2}} f_{k-2}^{2k-3}(x)(2k-2)dx, & \frac{o_1+o_2}{2} > \alpha_i \end{cases} \tag{2.50}$$

$$\theta(o_3|\alpha_i) = \begin{cases} f_{k-1}^{2k-2}(\alpha_i) + \int_{\frac{o_2+o_3}{2}}^{\alpha_i} f_{k-1}^{2k-3}(x)(2k-2)dx & \\ \qquad + \int_{\alpha_i}^{1} f_{k-2}^{2k-3}(x)(2k-2)dx, & \frac{o_2+o_3}{2} < \alpha_i \\ \frac{1}{2} f_{k-1}^{2k-2}(\alpha_i) + \int_{\frac{o_2+o_3}{2}}^{1} f_{k-2}^{2k-3}(x)(2k-2)dx, & \frac{o_2+o_3}{2} = \alpha_i \\ \int_{\frac{o_2+o_3}{2}}^{1} f_{k-2}^{2k-3}(x)(2k-2)dx, & \frac{o_2+o_3}{2} > \alpha_i \end{cases} \tag{2.51}$$

We can now measure quantitative, qualitative, and joint quantitative and qualitative diversity in the model as well as freedom with $c(o,u) = 1$ and $c(o,u) = \lambda(o)$. For comparative statics purposes, let $(o_1,o_2,o_3) = (1/2 - d, 1/2, 1/2 + d)$ with $d \in [0,1/2]$.

There is a large range of possibilities for measuring qualitative diversity as shown in Nehring and Puppe (2002). However, the differences between these measures usually only appear when considering more than three elements. In our example, all reasonable measures should agree that larger d lead to larger qualitative diversity. Here, a square root distance metric is used for qualitative diversity:

$$v(o_1,o_2,o_3) = (o_3 - o_2)^{1/2} + (o_2 - o_1)^{1/2} = 2d^{1/2} \tag{2.52}$$

Notice that from this diversity function one can generate the diversity weights $\lambda(o)$:

$$\lambda(o_1) = (d)^{1/2} \tag{2.53}$$

$$\lambda(o_2) = 2(d)^{1/2} - 1 \tag{2.54}$$

$$\lambda(o_3) = (d)^{1/2} \tag{2.55}$$

Quantitative diversity of chosen candidates can be measured by the freedom measure of Suppes (1996) yielding the entropy over the chosen candidates:

$$F_S(O,\theta) = -\sum_{x=1}^{3} \theta(o_x) \ln \theta(o_x) \tag{2.56}$$

The combined quantitative and qualitative diversity as suggested in Nehring and Puppe (2009) is:

$$D_{NP}(O,\theta) = -\sum_{x=1}^{3} \lambda_x \theta(o_x) \ln \theta(o_x) \tag{2.57}$$

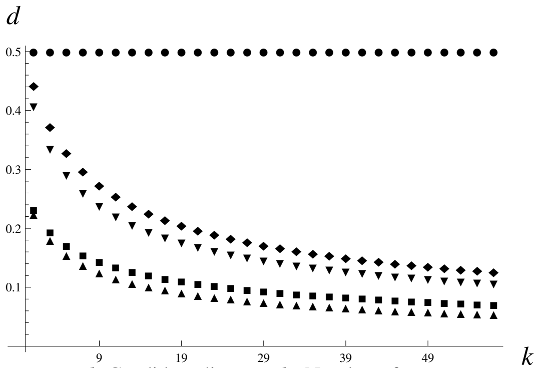
So far the freedom measure has only been used in settings with a finite number of outcomes and a finite number of preferences. Since in this example the preference variable is continuous, the freedom measure needs to accommodate this, which is done here by replacing the summations with integrals:

$$\Phi^{pos}(\eth) = \int_0^1 \sum_{o \in O_i} \theta(o|\alpha_i) \ln \left(\frac{\theta(o|\alpha_i)}{\theta(o)} \right) d\alpha_i \tag{2.58}$$

As discussed in the previous section, it is often normatively desirable for freedom to measure both quantitative and qualitative diversity. The following measure attempts this by replacing the normative weights $c(o,u) = \lambda(o)$:

$$\Phi^{pos,\lambda}(\eth) = \int_0^1 \sum_{o \in O_i} \lambda(o)\theta(o|\alpha_i) \ln\left(\frac{\theta(o|\alpha_i)}{\theta(o)}\right) d\alpha_i \tag{2.59}$$

Using the above specifications for diversity and freedom, the optimal diversity values d were obtained by numerical optimization over each measure.



d: Candidate distance, k: Number of voters,
●: arg max$_d$ v, ◆: arg max$_d$ $\Phi^{pos,\lambda}$, ▼: arg max$_d$ D_{NP}
■: arg max$_d$ Φ^{pos}, ▲: arg max$_d$ F_S

Figure 2.3: Optimal Candidate Distance

As can be seen in Figure 2.3, a voting system maximizing the qualitative diversity v of candidates will naturally maximize d. A voting system maximizing quantitative diversity will try to make all three alternatives equally likely to be chosen. As the number of voters increases, extreme candidates are less likely to be chosen and the quantitative diversity gain from these extreme candidates decreases. Φ^{pos} captures these effects and its optimal candidate distance therefore decreases in k. Additionally, it accounts for the expected influence a voter will have on the outcomes creating a tradeoff between greater quantitative diversity and causal influence. The direction and extent of this effect depends on the voting system analyzed. The freedom measure $\Phi^{pos,\lambda}$ accounting for diversity weights $\lambda(A)$ and the diversity measure D_{NP} strike a balance between optimal quantitative diversity Φ^{pos} and maximal qualitative diversity $\nu(\ldots)$.

Concluding, the example shows that the measure can account for both quantitative and qualitative diversity of choices in an intuitive way. In a voting game, agents have overall very low influence on the outcomes of the game. Therefore, the freedom measure naturally does not give radically different answers than measures which do not incorporate the causal influence agents have on outcomes.

2.10 Freedom in a Production Economy

To further illustrate the measure for positive freedom, it will be employed to analyze freedom in a production economy, a similar problem to the one posted in Pattanaik (1994). According to Pattanaik (1994), the problem of measuring freedom in any interactive model of an exchange economy is that prices and therefore also opportunity sets change both with one's own preferences and preferences of the other agents. Since most measures of freedom are based on opportunity sets, they fail to give a satisfying answer to the problem, as Pattanaik (1994) concludes.

For easier exposition, some simplifying assumptions on the production economy will be made. Suppose there is a single[13] consumer and a single firm, which produces an output good (food, x_1) from an input good (labor, l). The firm is maximizing its payoff $\pi = f(l)p - lw$ and the consumer is maximizing his utility $U_\alpha = \alpha \ln(x_1) + (1-\alpha)\ln x_2$ over food consumption and leisure ($x_2 = 1 - l$) and completely owns the company. The budget constraint of the agent is then $px_1 + w(1-l) = \pi + w$. The production function of the firm is linear with an efficiency parameter γ: $f(l) = \gamma l$.

It can be shown that in this economy the firm earns zero profit and the

[13] The author is aware of the fact that the common interpretation of this model is that there are many consumers and a change in an individual's preferences has no impact on prices. This does not affect the point made by this section, since the setting could be changed to a game theoretic one and instabilities in prices or technology would still have the same effect on freedom.

equilibrium consumption of food is $x_1^* = \alpha\gamma$ while equilibrium leisure is $x_2^* = 1 - \alpha$ with an equilibrium relative price $w^*/p^* = \gamma$. Although this is not a game, the measure can still be employed in exactly the same manner. Positive freedom is the effect of the preferences on certain relevant outcomes, where an obvious choice here is the consumption of food.[14].

To measure freedom, a preference expansion is needed, which specifies the set of utility functions and the probability with which each utility function occurs. Let us assume $U = \{U_\alpha : \alpha \in [(1-c)/2, (1+c)/2]$ and $\hat{p}(U_\alpha) = 1/(c)$, i.e. the preference parameter is uniformly distributed over an interval of size c. This can be interpreted as the reasonable preferences an individual may have. If there is no further uncertainty added, preferences completely determine consumption and freedom should be high. However, suppose some random weather conditions may affect the efficiency of the firm. γ is therefore random with $\gamma \in \{(1-d)/2, (1+d)/2\}$ and $\hat{p}(\gamma) = 1/d$, i.e. γ is distributed uniformly around $1/2$ with the support determined by $d < 1$.

Since both variables are continuous, the measure needs to accommodate this, which is done here by replacing the summations with integrals:

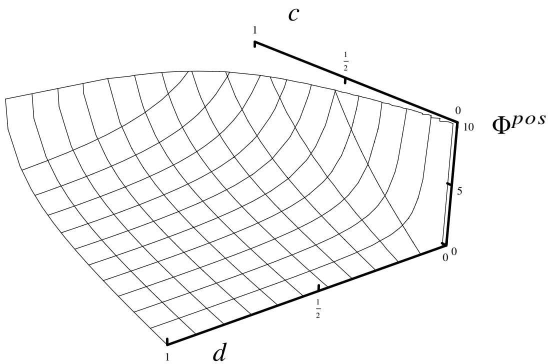
$$\Phi^{pos}(\eth) = \int_{(1-c)/2}^{(1+c)/2} \hat{p}(u_\alpha) \int_{\underline{x}(\alpha)}^{\overline{x}(\alpha)} \theta(x_1|u_\alpha) \ln\left(\frac{\theta(o|u_\alpha)}{\theta(o)}\right) d\alpha dx_1 \tag{2.60}$$

where $\underline{x}(\alpha)$ and $\overline{x}(\alpha)$ are the minimal and maximal values x_1^* can take, given u_α. Φ^{pos} measures how much the preference parameter α influences the equilibrium consumption x_1^*. For simplicity, $c(o,u) = 1$ and $d(o,u) = 0$.

The derivations to obtain the conditional distribution $\theta(x_1|u_\alpha)$ and the marginal distribution $\theta(x_1)$ are provided in the appendix. Given these distributions, one can plot Φ^{pos} as a function of the parameters c and d. The graph is given in Figure 2.4 and shows two main effects:

First, the wider the support of the reasonable preferences, the higher the freedom. This is due to the fact that more preferences become reasonable: For

[14]In principle, it would of course be possible to also consider combinations of food consumption and leisure as the outcome variable. However, it would not provide much further insight, since leisure is already a deterministic function of the preference parameter α only.



c: Support of preference parameter α, d: Support of productivity parameter γ, Φ: Freedom Measure

Figure 2.4: Freedom in a Production Economy

example a person who can also live with less sleep or with less food than others draws from a wider array of reasonable preferences, since she may also prefer to eat very little and have lots of leisure or have little leisure and lots of food. This effect could also be observed in the measure of Jones and Sugden (1982).

Second, the more random the productivity, the smaller the freedom measure. This is intuitive, since food consumption depends more on tomorrow's weather conditions than on the consumer's preferences. An individual is less free, if her consumption of food is strongly dependent on fluctuating weather conditions and does not face stable food production. Since the technology directly determines prices here, this argument extends to price stability: The more stable the prices are, the greater the freedom of an agent. The simple model of this section therefore provides microfoundations for the use of price stability as an indicator of freedom as for example done in Gwartney et al. (2010).[15]

[15]It has of course to be noted that the price stability in this model refers to relative price stability and not absolute price level stability. However, it is clear how an extension to an intertemporal framework with real imperfections would yield the same result for absolute price level stability over time. If unstable prices influence consumption, freedom will decrease as there is less room for individual preferences to influence consumption.

The example application shows how the measure can be employed in standard economic models. It also yields some intuitive comparative statics results. This opens the door for policy evaluation not solely based on welfarist or utilitarian evaluations, but also based on freedom.

2.11 Concluding Remarks

The paper has proposed a class of freedom measures for extensive form games. It has shown that such measures can be applied in a wide range of economic models, where freedom is normatively relevant. In many of these applications there already exist measures for normatively relevant phenomena such as option diversity, discrimination, voting power or price stability. The measure presented in this paper however provides a unified normative framework according to which one can evaluate all these cases.

The paper has also shown that aside from its use in normative economics, the measure has the potential to explain behavior in situations where individuals can influence their freedom with their actions. It may well be the case that the abstract philosophical idea that freedom is normatively desirable has behavioral roots in the fact that individuals have non-instrumental preference for freedom.

In both positive and normative economics, there is a large potential for further research. An axiomatic foundation of preference for freedom may be interesting, but also further investigations in variants of the authority game may explore to what degree individuals indeed have a preference for freedom. Possible normative applications include more interesting variants of the simple models used in this paper and other topics such as taxation, trade policy, and political rights.

Appendix

2.A Axiomatization of the measure

This section axiomatizes a measure of causal influence between two variables A and O that is general enough (i.e. allows for normative inputs) to measure freedom in the ways it is measured in this paper. Φ^{pos} and Φ^{neg} are instances of the measure that will be axiomatized while Φ^{rep} is a maximized version of it. It should be noted that the axiomatization is not based on game-theoretic primitives but on a probability distribution P over A and O. A is to be interpreted as either the actions of an agent or the preferences. O is to be interpreted as the outcomes like in the main body of the paper.

The measure will be developed in two steps: In the first, a given background state b_x is assumed. For this state, a measure Ψ is developed which states the degree to which a certain action a_i influences a certain outcome o_s. Since Ψ will be a probabilistic measure of influence without any interpretation of freedom, no further information beyond the probabilities is admitted. Ψ will be called a causal influence measure. In a second step, the values for Ψ are aggregated over i, s, x into an aggregate freedom measure Φ which describes the general influence of A on O under background conditions B. This is in line with the idea that the freedom measure should be based on the causal influence of individuals on outcomes. The aggregate freedom measure Φ however must also permit additional information which is relevant to distinguish freedom from blunt causal influence.

2.A.1 Causal Influence Measure

The first step in the axiomatization is to find a measure of causal influence $\Psi(a_i, o_s|b_x)$. The state b_x can be interpreted as a state that is conditioned on, such that the setting in which causation is measured is causally homogenous (Cartwright, 1979). This ensures that the causal relation between a_i and o_s is measured and not the influence of a third factor on both states. A causal influence measure is thus defined in the following way:

Definition 2.7. Causal Influence Measure:
A function $\Psi(a_i, o_s|b_x) = f(P(A \cap O \cap B))$ where $A \hookrightarrow O$ and $B \hookrightarrow A$ holds, is called a causal influence measure of a state a_i on a state o_s in a causally homogenous state b_x.

Various causal strength measures have been presented in the literature, over which Fitelson and Hitchcock (2011) gives an overview. A first axiom seems immediately plausible:

Axiom 2.1. Independence:
$\forall j : P(a_j \cap o_s|b_x) = P(a'_j \cap o'_s|b_x)$ and
$\forall t : P(a_i \cap o_t|b_x) = P(a'_i \cap o'_t|b_x)$ jointly imply:
$\Psi(a_i, o_s|b_x) = \Psi(a'_i, o'_s|b_x)$

The independence axiom states that the measure is independent of the probabilities of antecedents and outcomes unrelated to the antecedent and the outcome that is being considered. It is very intuitive to assume that the probability of an unrelated cause-effect combination should have no impact on the measure. In the example of the Bayesian game, one may want to know the influence of player 2 playing L after $A_1 = U$ on obtaining outcome o_2. Then the probability of playing R and obtaining o_3 should be irrelevant to the measure.

Next, we may want the measure to be independent of homogenous transformations among the probabilities of antecedents and outcomes. If an antecedent is simply less likely, the measure should not change:

Axiom 2.2. Homogeneity in Antecedents:
$P(A \cap O|b_x) = P(A' \cap O'|b_x)$ except:
$\forall t : P(a_i \cap o_t|b_x) = \alpha P(a_i' \cap o_t'|b_x)$ and
$\forall t : P(a_{\bar{i}+1} \cap o_t|b_x) = (1-\alpha)P(a_i' \cap o_t'|b_x)$ jointly imply:
$\Psi(a_i, o_s|b_x) = \Psi(a_i', o_s'|b_x)$ for $\alpha \in (0,1]$.

The somewhat complicated form is due to the fact that probabilities must always sum to one and thus the probability mass needs to be reallocated to other states. Here this happens by introducing a new action state and allocating the probability mass between the action state a_i and the new state $a_{\bar{i}+1}$ according to a factor α. Consider the above example again: In background state $A_1 = U$, player 2 choosing L completely determines that o_1 will come about. Then no matter how likely it is that player 2 chooses L, the measure of causal influence of L on o_1 should return a high value.

The next axiom states the same for outcomes. The degree to which an outcome is influenced by an action is independent of how probable the outcome is in general:

Axiom 2.3. Homogeneity in Outcomes:
$P(A \cap O|b_x) = P(A' \cap O'|b_x)$ except:
$\forall j : P(a_j \cap o_s|b_x) = \beta P(a_j' \cap o_s'|b_x)$ and
$\forall j : P(a_j \cap o_{\bar{s}+1}|b_x) = (1-\beta)P(a_j' \cap o_s'|b_x)$ jointly imply:
$\Psi(a_i, o_s|b_x) = \Psi(a_i', o_s'|b_x)$ for $\beta \in (0,1]$.

This axiom expresses that the measure is about how much the *relative* effect of the antecedent a_i is on the occurrence of the outcome o_s. The measure should be independent of the probability with which the outcome comes about in general. Homogeneity in Outcomes and Homogeneity in Antecedents are thus two sides of the same coin, stating that absolute probability levels do not matter, but only relative influence. This is the main difference to the causal influence measure of Braham (2006), where the absolute probability level matters. Unfortunately, one link to the probability of an outcome is unavoidable: If the outcome occurs with certainty $P(o_s|b_x) = 1$ or never occurs $P(o_s|b_x) = 0$, then the antecedents have no control over whether the outcome occurs or not. Also, if the probability of o_s converges to zero or one, the probability of a_i needs to converge to zero for the value of the measure to remain unchanged. These floor effects are unfortunately unavoidable.

Axiom 2.4. Irrelevance of Substitution among other Outcomes:
$P(A \cap O|b_x) = P(A' \cap O'|b_x)$ except:
$P(a_i \cap o_t|b_x) = P(a_i' \cap o_t'|b_x) + \epsilon$ with $j \neq i$ and
$P(a_i \cap o_u|b_x) = P(a_i' \cap o_u'|b_x) - \epsilon$ with $k \neq i$ jointly imply:
$\Psi(a_i, o_s|b_x) = \Psi(a_i', o_s'|b_x)$ for $0 \leq \epsilon \leq \min(P(a_k' \cap o_s|b_x), 1 - P(a_j' \cap o_s|b_x))$.

This axiom states that the probabilities $P(a_i \cap o_t|b_x)$ with $t \neq s$ are perfect substitutes. This may be seen as a sharpening of the independence axiom. Not only do unrelated probabilities not matter as we have seen above, but even the probabilities which to some extent are related (the other outcomes which may occur after the antecedent) matter only in their sum. This can again be exemplified by the above game. Suppose one is interested in the causal effect of player 2 choosing L on whether o_1 occurs. Then the probability of choosing R and obtaining o_2 influences the measure the same way as the probability of choosing R and obtaining o_3. This is natural as long as one assumes that all alternatives are equally distinct. If o_1 and o_3 are qualitatively very similar, one may disagree with treating both equally.

Axiom 2.5. Irrelevance of Substitution among other Antecedents:
$P(A \cap O|b_x) = P(A' \cap O'|b_x)$ except:
$P(a_j \cap o_s|b_x) = P(a_j' \cap o_s'|b_x) + \epsilon$ with $j \neq i$ and
$P(a_k \cap o_s|b_x) = P(a_k' \cap o_s'|b_x) - \epsilon$ with $k \neq i$ jointly imply:
$\Psi(a_i, o_s|b_x) = \Psi(a_i', o_s'|b_x)$ for $0 \leq \epsilon \leq \min(P(a_k' \cap o_s|b_x), 1 - P(a_j' \cap o_s|b_x))$.

The interpretation of this axiom is analogue to the interpretation of Axiom 2.4 above.

Axiom 2.6. Monotonicity[16]:
$\Psi(a_i, o_s|b_x)$ is strictly decreasing in $P(a_j \cap o_s|b_x)$ and $P(a_i \cap o_t|b_x)$ for all $j \neq i$ and $t \neq s$ unless $P(a_i \cap o_s|b_x) = 0$.

The monotonicity axiom ensures the compliance of the measure with the following intuition: Suppose we want to measure how much a_i causes o_s. If the probability of o_s is high even if a_i does not occur, a_i causes o_s to a lower degree than if the probability of o_s without a_i was lower. To put it more simply, the axiom follows the logic: 'How can a_i cause o_s if o_s is going to happen anyways?' Only if the joint probability of the antecedent and the outcome occurring is zero, this does not hold.

Given the above axioms, we can now characterize a general form of the measure Ψ:

Theorem 2.1. *The axioms Independence, Homogeneity in Antecedents, Homogeneity in Outcomes, Irrelevance of Substitution among other Antecedents, Irrelevance of Substitution among other Outcomes, and Monotonicity imply* $\Psi(a_i, o_s|b_x) = \psi(\frac{P(a_i \cap o_s|b_x)}{P(a_i|b_x)P(o_s|b_x)})$ *where* $\psi(\dots)$ *can be any strictly increasing function.*

Theorem 2.1 gives a way to measure the causal strength of a_i on o_s given b_x. Fitelson and Hitchcock (2011) gives an overview and discussion of other measures, which however all fail at least one of the above axioms.

2.A.2 Aggregate Freedom Measure

What we are interested in, is the overall freedom of the agent, which can of course not be reduced to a single antecedent-outcome-background state combination. For example, in different states of B, there may be more or less influence of the antecedent on the outcomes. Also, there may be different actions with which the agent can influence the outcomes. Therefore, the next step

[16]An alternative, and equally intuitive Monotonicity axiom would be that $\Psi(a_i, o_s|b_x)$ is strictly increasing in the conditional probability $P(o_s|a_i \cap b_x)$ under redistributions such that $\sum_t P(o_t|a_i \cap b_x) = 1$ holds. Both versions are equivalent given the other axioms. Preference is given here to the one with the simpler exposition.

is to find a reasonable way to aggregate the causal strength measures of each combination a_i, o_s, and b_x into an overall measure of freedom. It seems plausible to say that the overall measure $\Phi_Z(A,O,B)$ is a function of the statewise measures $\Psi(a_i,o_s|b_x)$ and some further information Z, which one may consider relevant for the measure. The further information Z is crucial here, since it allows the measure not only to capture more than just the causal relationship between the variables: In the capabilities framework, an agent may have a high influence on his functioning vector but only through illegal actions. In the reasonable preference framework an agent may always obtain the lowest ranked alternative instead of the highest one. In a game of incomplete information, a player may not be aware in which way her actions influence the outcomes. In all these cases the causal influence of the antecedents on the outcomes are high, but intuitively freedom still remains low. By adding further information, such as whether an action is legal, how a preference relation ranks the outcome or what the beliefs of a player are, one can control for such cases.

It is assumed that the additional relevant information $Z = \{z_{1,1,1},\dots,z_{\bar{i},\bar{s},\bar{x}}\}$ is partitioned such that $z_{i,s,x}$ contains all the relevant information for the state $a_i \cap o_s \cap b_x$.

Definition 2.8. Aggregate Freedom Measure:
$\Phi_Z(A,O,B) = G(\Psi(A,O|B), P(A \cap O \cap B), Z)$, where Z contains all further information we may consider relevant for our freedom measure, and $\Psi(A,O|B) = \{\Psi(a_1,o_1|b_1),\dots,\Psi(a_{\bar{i}},o_{\bar{s}}|b_{\bar{x}})\}$.

This definition rests at the very heart of the freedom measure, since it states that the aggregate freedom measure is a function of the statewise causal influence measures. To simplify notation, define $\xi_{i,s,x} \equiv \frac{P(a_i \cap o_s|b_x)}{P(a_i|b_x)P(o_s|b_x)} = \psi^{-1}(\Psi(a_i,o_s|b_x))$. Note that $\psi(\dots)$ has not yet been further specified. Since ψ is a strictly increasing function, any function $h(\psi(a),\psi(b),\dots)$ can be directly expressed as $h_\psi(a,b,\dots)$ without imposing further assumptions. In particular,

$$\begin{aligned} G(\Psi(A,O|B),P(A\cap O\cap B),Z) &\equiv \\ G_\psi(\psi^{-1}(\Psi(a_1,o_1|b_1)),\dots,\psi^{-1}(\Psi(a_{\bar{i}},o_{\bar{s}}|b_{\bar{x}})),P(A\cap O\cap B),Z) \\ &= G_\psi(\xi_{1,1,1},\dots,\xi_{\bar{i},\bar{s},\bar{x}},P(A\cap O\cap B),Z). \end{aligned} \quad (2.61)$$

This is useful for the following axioms.

Axiom 2.7. Unbiasedness:
$\Phi_Z(A,O,B) = H(h_\psi(\xi_{1,1,1},P(a_1\cap o_1\cap b_1),z_{1,1,1}),\dots,h_\psi(\xi_{\bar{i},\bar{s},\bar{x}},P(a_{\bar{i}}\cap o_{\bar{s}}\cap b_{\bar{x}}),z_{\bar{i},\bar{s},\bar{x}}))$ and $H(\dots)$ is a symmetric function.

The axiom states that the aggregate is a symmetric function H of identical functions h which each take the value of the statewise measure $\Phi(a_i,o_s|b_x)$, the probability of this state $P(a_i\cap o_s\cap b_x)$, and the related information $z_{i,s,x}$ into account.

Note that the arguments $\xi_{i,s,x}$ to the function h_ψ are measured on independent ratio scales[17] and therefore the admissible information preserving transformations are multiplications by constants for each value. From the mathematical theory of aggregation (Aczél & Roberts, 1989) we know that the scale type of both input and output variables is highly relevant for the correct specification of an aggregation function. For maximal generality, it is assumed that the final measure has at least ordinal scale[18]. According to Kim (1990), the measure then needs to fulfill the following functional equation:

[17]For each argument $\xi_{i,s,x}$ of the function we have a natural zero $P(a_i\cap o_s|b_x) = 0$ and neutral elements $P(a_i\cap o_s|b_x) = P(a_i|b_x)P(o_s|b_x)$.

[18]The remaining axioms indeed imply that the final measure has cardinal scale. A stronger scale type axiom requiring cardinal scale of the final measure however does not imply the remaining axioms.

Axiom 2.8. Scale Type:
The measure fulfills the functional equation:

$$G_\psi(\lambda_{1,1,1}\xi_{1,1,1},\ldots,\lambda_{\bar{\imath},\bar{s},\bar{x}}\xi_{\bar{\imath},\bar{s},\bar{x}},P(A\cap O\cap B),Z) = \hat{G}_\psi(\lambda_{1,1,1},\ldots,\lambda_{\bar{\imath},\bar{s},\bar{x}},G_\psi(\xi_{1,1,1},\ldots,\xi_{\bar{\imath},\bar{s},\bar{x}},P(A\cap O\cap B),Z)) \quad (2.62)$$

The interpretation of the scale type axiom is the following: If one performs linear transformations of the causal strength measures, then one obtains order-maintaining transformations of the aggregate measure. If an aggregate measure fails to fulfill this axiom, this means that it attempts to extract additional information from $\xi_{i,s,x}$, which are beyond its ratio-scale nature.

The following branching axioms deal with the question how the measure should react if the state descriptions become more detailed.

Axiom 2.9. Antecedent Branching:
$P(A\cap O\cap B) = P(A'\cap O'\cap B')$ and $Z = Z'$ except for some x,i:
$\forall s: P(a_i\cap o_s\cap b_x) = \alpha P(a'_i\cap o'_s\cap b'_x)$ and
$\forall s: P(a_{\bar{\imath}+1}\cap o_s\cap b_x) = (1-\alpha)P(a'_i\cap o'_s\cap b'_x)$ and
$\forall s: z_{\bar{\imath}+1,s,x} = z'_{i,s,x}$ jointly imply:
$\Phi_Z(A,O,B) = \Phi_{Z'}(A',O',B')$ for $\alpha\in[0,1]$.

The antecedent branching axiom is formally strongly related to the homogeneity in antecedents in the statewise causal strength measure. Its interpretation is different however for the aggregate: It states that the aggregate measure should not change if an antecedent is split into two actions that have the same probabilistic structure such that their probabilities are linearly dependent. Thus, the idea is here that if we specify the states in more detail, the measure does not change if these details yield linearly dependent probabilities.

Axiom 2.10. Outcome Branching:
$P(A \cap O \cap B) = P(A' \cap O' \cap B')$ and $Z = Z'$ except for some x, s:
$\forall i : P(a_i \cap o_s \cap b_x) = \beta P(a'_i \cap o'_s \cap b'_x)$ and
$\forall i : P(a_i \cap o_{\bar{s}+1} \cap b_x) = (1-\beta) P(a'_i \cap o'_s \cap b'_x)$
$\forall i : z_{i,\bar{s}+1,x} = z'_{i,s,x}$ jointly imply:
$\Phi_Z(A, O, B) = \Phi_{Z'}(A', O', B')$ for $\beta \in [0,1]$.

The Outcome Branching axiom has a similar interpretation as the Antecedent Branching axiom, except for that it applies to outcomes. One can thus understand it such that the distinction between two outcomes does not matter if in each situation the probabilities are linearly dependent of each other and thus the Ψ measure is identical in both.

Axiom 2.11. Background Branching:
$P(A \cap O \cap B) = P(A' \cap O' \cap B')$ except for some x:
$\forall i, t : P(a_i \cap o_t \cap b_x) = \gamma P(a'_i \cap o'_t \cap b'_x)$ and
$\forall i, t : P(a_i \cap o_t \cap b_{\bar{x}+1}) = (1-\gamma) P(a'_i \cap o'_t \cap b'_x)$
$\forall i, s : z_{i,s,\bar{x}+1} = z'_{i,s,x}$ jointly imply:
$\Phi_Z(A, O, B) = \Phi_{Z'}(A', O', B')$ for $\gamma \in [0,1]$.

The Background Branching axiom now applies the branching property to the restrictions: If we have two states of the background variable under which the conditional probabilities of the actions and outcomes are the same, then it does not change the measure if we integrate these two background states into a single state.

To obtain a full characterization, we need to add an axiom of how the values of different variables should be comparable.

Axiom 2.12. Additivity:

$$P(A \cap O \cap B)P(A' \cap O' \cap B') = P(A \cap O \cap B \cap A' \cap O' \cap B') \Rightarrow$$
$$\Phi_Z(A,O,B) + \Phi_{Z'}(A',O',B') = \Phi_{Z''}(A \cap A', O \cap O', B \cap B') \quad (2.63)$$

where Z'' is an appropriate union of the information contained in Z and Z'. The Additivity axiom regulates how independent situations add up in their freedom. Namely, if two sets of variables A,O,B and A',O',B' are independent, measuring the freedom via the joint variables $A \cap A'$, $O \cap O'$, $B \cap B'$ is equivalent to adding up the freedom values separately. The Additivity axiom makes no restrictions if the probabilities are not independent, for example if the antecedent in one situation also affects the outcome of the other situation. It is also possible to consider non-additive measures. For example $e^{\Phi^{pos}}$ is a measure which is multiplicative. However, all measures not fulfilling the Additivity axiom will be monotone transformations thereof, as evident from (2.103) of the axiomatization proof.

Finally, some technical issues also need to be addressed by the following regularity axiom:

Axiom 2.13. Regularity:
Continuity: For given variables A,O,B, the aggregate measure $\Phi_Z(A,O,B)$ is a continuous function of $P(A \cap O \cap B)$ and $\Psi(A \cap O \cap B)$.
Boundedness: $\exists \overline{\underline{\Phi}} \in \mathbb{R} : |\Phi_Z(A,O,B)| \leq \overline{\underline{\Phi}}$ for a given number of antecedents and outcomes and for some A,O,B it has to hold that $\overline{\underline{\Phi}} > 0$.
Responsiveness: $\exists Z : \frac{\Delta \Phi_Z(A,O,B)}{\Delta \Psi(A \cap O \cap B)} \neq 0$.

The boundedness requirement states that for a given number of states of antecedents, outcomes, and restrictions the measure has a maximum (minimum) value it either reaches or converges to. Responsiveness ensures that the measure is in fact a function of the statewise measures of influence and does not only depend on Z.

Given these axioms, the measure can be characterized in the following way:

Theorem 2.2. *The axioms Unbiasedness, Scale Type, Action Branching, Outcome Branching, Restriction Branching, Additivity, and Regularity imply that the aggregate measure takes the form:*

$$\Phi_Z(A,O,B) = \sum_{i,s,x} P(a_i \cap o_s \cap b_x) \cdot \Bigg(c(z_{i,s,x}) \cdot \ln\left(\frac{P(a_i \cap o_s | b_x)}{P(a_i|b_x)P(o_s|b_x)}\right) + \ln d(z_{i,s,x}) \Bigg) \quad (2.64)$$

with for all A', O', B' independent of A, O, B:

$$c(z_{i,s,x}) = P(a_i \cap o_s \cap b_x) \sum_{i',s',x'} P(a'_{i'} \cap o'_{s'} \cap b'_{x'}) c(z''_{i,s,x,i',s',x'}) \quad (2.65)$$

$$\prod_{i,s,x} d(z_{i,s,x})^{P(a_i \cap o_s \cap b_x)} = \prod_{i,s,x,i',s',x'} \frac{d(z''_{i,s,x,i',s',x'})^{P(a_i \cap o_s \cap b_x) P(a'_{i'} \cap o'_{s'} \cap b'_{x'})}}{d(z'_{i',s',x'})^{P(a'_{i'} \cap o'_{s'} \cap b'_{x'})}}. \quad (2.66)$$

Note that all the non-probabilistic information is aggregated into c and d. The former determines the weight of a certain action-outcome-restriction state combination, while the latter is a scalar of the whole measure. Also, the base of the logarithm may be changed via a rescaling of $c(\dots)$. To clarify the features of the measure it may be useful to return to the example from Section 2.3:

2.A.3 Proof of Theorem 2.1

Proof. We start with the definition of a causal influence measure:

$$\Psi(a_i, o_s | b_x) = f\,(P(A \cap O \cap B)) \quad (2.67)$$

By independence,

$$f(P(A \cap O \cap B)) = \\ \hat{f}(P(a_1 \cap o_s|b_x),\dots,P(a_{\bar{i}} \cap o_s|b_x),P(a_i \cap o_1|b_x),\dots,P(a_i \cap o_{\bar{s}}|b_x)). \tag{2.68}$$

By the two substitution axioms,

$$\begin{aligned} \hat{f}(\dots) &= g(P(a_i \cap o_s|b_x),\sum_{j\neq i} P(a_j \cap o_s|b_x),\sum_{t\neq s} P(a_i \cap o_t|b_x)) \\ &= g\left(P(a_i \cap o_s|b_x),P(a_i|b_x) - P(a_i \cap o_s|b_x),P(o_s|b_x) - P(a_i \cap o_s|b_x)\right) \\ &\equiv h\left(P(a_i \cap o_s|b_x),P(a_i|b_x),P(o_s|b_x)\right). \end{aligned} \tag{2.69}$$

The homogeneity axioms now imply that h has to fulfill the following functional equation:

$$h\left(\alpha\beta P(a_i \cap o_s|b_x),\beta P(a_i|b_x),\alpha P(o_s|b_x)\right) = \\ h\left(P(a_i \cap o_s|b_x),P(a_i|b_x),P(o_s|b_x)\right). \tag{2.70}$$

Since the functional equation has to hold for all possible values of β, we can assume for a moment $\beta = 1$. This implies that the function h is homogeneous of degree zero in the two variables $P(a_i \cap o_s|b_x)$ and $P(o_s|b_x)$. By Aczél and Dhombres (1989) it then has the general solution:

$$h\left(P(a_i \cap o_s|b_x),P(a_i|b_x),P(o_s|b_x)\right) = \hat{h}\left(\frac{P(a_i \cap o_s|b_x)}{P(o_s|b_x)},P(a_i|b_x)\right) \tag{2.71}$$

where $\hat{h}$ can be any arbitrary function. $\hat{h}$ however is a solution to the functional equation (2.70) if and only if for all β

$$\hat{h}\left(\beta\frac{P(a_i \cap o_s|b_x)}{P(o_s|b_x)},\beta P(a_i|b_x)\right) = \hat{h}\left(\frac{P(a_i \cap o_s|b_x)}{P(o_s|b_x)},P(a_i|b_x)\right). \tag{2.72}$$

We again have a functional equation by which $\hat{h}$ is homogeneous of degree zero in two variables, $\frac{P(a_i \cap o_s|b_x)}{P(o_s|b_x)}$ and $P(a_i|b_x)$. Applying the solution from Aczél and Dhombres (1989) again yields as the solution to the functional equation (2.70):

$$\Psi(a_i, c_s|b_x) = \psi\left(\frac{P(a_i \cap o_s|b_x)}{P(a_i|b_x)P(o_s|b_x)}\right). \tag{2.73}$$

The monotonicity axiom now implies that $\psi(\dots)$ needs to be a strictly increasing function since its argument is decreasing in $P(a_j \cap o_s|b_x)$ and $P(a_i \cap o_t|b_x)$ for all $j \neq i$ and $t \neq s$. □

2.A.4 Proof of some useful Lemmas

For the proof of Theorem 2.2, some Lemmas will be necessary. Since the measure assumes Responsiveness, all Lemmas will also focus on responsive solutions.

Proof. Some lemmas will be useful for the proof and are thus stated first:

Lemma 2.1. *(Kim, 1990) The class of continuous functions $u : \mathbb{R}^n \to \mathbb{R}$ fulfilling the functional equation $u(\gamma_1 x_1, \dots, \gamma_n x_n) = f(\gamma_1, \dots, \gamma_n, u(x_1, \dots x_n))$ with $\forall i : \gamma_i > 0$ and $x_i > 0$ and where f is strictly increasing in $u(\dots)$, is:*
u is constant or $u(x_1, \dots, x_n) = g(\prod_i^n x_i^{d_i})$ where d_i can be arbitrary constants and g is continuous and strictly increasing.

The proof is given by Kim (1990). The next two lemmas are useful functional equations:

Lemma 2.2. *$\forall a, b : g(a) + h(b) = i(v(a) \cdot w(b))$ implies:*
$g(x) = c_1 + c_2 \cdot \ln v(x)$, $h(x) = c_3 + c_2 \ln w(x)$, $i(x) = c_1 + c_3 + c_2 \cdot \ln x$

Proof. Set $i^*(\ln x) = i(x)$ to obtain $g(a) + h(b) = i^*(\ln v(a) + \ln w(b))$ from the assumption $g(a) + h(b) = i(v(a) \cdot w(b))$. The solution to this functional equation is (Polyanin & Zaitsev, 2004): $g(x) = c_1 + c_2 \cdot \ln v(x)$, $h(x) = c_3 + c_2 \cdot \ln w(x)$ and $i^*(x) = c_1 + c_3 + c_2 \cdot x$. From $i^*(\ln x) = i(x)$ we then know that $i(x) = c_1 + c_3 + c_2 \cdot \ln x$ as desired. □

Lemma 2.3.

$$\forall A,B,x,y : g(A\cdot x) + g(B\cdot y) = g(C\cdot x\cdot y) \tag{2.74}$$

implies: $g(x) = c\cdot\ln(x)$, $C = A\cdot B$.

Proof. Holding A,B,C constant, we define:

$$g_A(x) = g(A\cdot x) \tag{2.75}$$

$$g_B(y) = g(B\cdot y) \tag{2.76}$$

$$g_C(x\cdot y) = g(C\cdot x\cdot y) \tag{2.77}$$

and thus obtain the functional equation:

$$g_A(x) + g_B(y) = g_C(x\cdot y) \tag{2.78}$$

By Lemma 2.2, we have as a solution:

$$g_A(x) = \hat{c}_1 + \hat{c}_2\ln(x) \tag{2.79}$$

$$g_B(y) = \hat{c}_3 + \hat{c}_2\ln(y) \tag{2.80}$$

$$g_C(xy) = \hat{c}_4 + \hat{c}_2\ln(xy) \tag{2.81}$$

where $c_4 = c_1 + c_3$ must hold. Holding A,B,C no longer constant, we notice that $\hat{c}_2$ cannot be a function of any of these arguments, since it appears in all three equations. However, $\hat{c}_1$ can still be a function of A, so we have:

$$g(A\cdot x) = \hat{c}_1(A) + \hat{c}_2\ln(x) \tag{2.82}$$

which is a special form of the functional equation $i(a\cdot b) = g(a) + h(b)$ from Lemma 2.2, which we can therefore apply:

$$g(A\cdot x) = c_1 + c_3 + c_2\ln(A\cdot x) \tag{2.83}$$

One can quickly verify that this solution is in line with equations (2.79) to (2.81). However, we still need to verify (2.74). Inserting the solution yields:

$$2c_1 + 2c_3 + c_2 \ln(A \cdot x) + c_2 \ln(B \cdot y) = c_1 + c_3 + c_2 \ln(C \cdot x \cdot y) \tag{2.84}$$

Simplifying yields:

$$c_1 + c_3 + c_2 \ln(A \cdot B) = c_2 \ln(C) \tag{2.85}$$

From which follows that $c_1 = c_3 = 0$ and focussing on the responsive solution with $c_2 \neq 0$ we also need $A \cdot B = C$. □

Lemma 2.4. $\forall A, B, a_i, x_i, b_j, y_j$:

$$g\left(A \cdot \prod_i a_i^{x_i}\right) + g\left(B \cdot \prod_j b_j^{y_j}\right) = g\left(C \cdot \prod_i a_i^{x_i'} \cdot \prod_j b_j^{y_j'}\right) \tag{2.86}$$

implies: $g(x) = c \cdot \ln x$ *and* $A \cdot B = C$ *and* $\forall i : x_i = x_i'$ *and* $\forall j : y_j = y_j'$.

Proof. The proof proceeds in two steps. First it will be shown that $x_i = x_i'$ and $y_j = y_j'$ for all i, j. The second step is then a direct application of Lemma 2.3. Further, define:

$$\breve{g}_A\left(a_i^{x_i}\right) \equiv g\left(A \cdot \prod_i a_i^{x_i}\right) \tag{2.87}$$

holding all factors except $a_i^{x_i}$ constant and

$$\breve{g}_B\left(b_j^{y_j}\right) \equiv g\left(B \cdot \prod_j b_j^{y_j}\right) \tag{2.88}$$

holding all factors except $b_j^{y_j}$ constant and

$$\breve{g}_C\left(a_i^{x_i'} \cdot b_j^{y_j'}\right) \equiv g\left(C \cdot \prod_i a_i^{x_i'} \cdot \prod_j b_j^{y_j'}\right) \tag{2.89}$$

holding all factors except $a_i^{x_i'}$ and $b_j^{y_j'}$ constant.

From equation (2.86) we now obtain:

$$\breve{g}_A\left(a_i^{x_i}\right) + \breve{g}_B\left(b_j^{y_j}\right) = \breve{g}_C\left(a_i^{x_i'} \cdot b_j^{y_j'}\right). \tag{2.90}$$

if we hold everything except $a_i, x_i, x_i', b_j, y_j, y_j'$ constant. This equation we can rewrite:

$$\breve{g}_A\left(a_i^{x_i}\right) + \breve{g}_B\left(b_j^{y_j}\right) = \breve{g}_C\left(\left(a_i^{x_i}\right)^{x_i'/x_i} \cdot \left(b_j^{y_j}\right)^{y_j'/y_i}\right) \tag{2.91}$$

which allows us to use Lemma 2.2 yielding:

$$\breve{g}_A\left(a_i^{x_i}\right) = \breve{d}_A + \breve{c} \cdot \ln\left(a_i^{x_i'}\right) \tag{2.92}$$

$$\breve{g}_B\left(b_j^{y_j}\right) = \breve{d}_B + \breve{c} \cdot \ln\left(b_j^{y_j'}\right) \tag{2.93}$$

Note that the constants $\breve{d}_A$, $\breve{d}_B$, and $\breve{c}$ become functions of the variables held constant before, once these are no longer held fixed:

$$g\left(A \cdot \prod_i a_i^{x_i}\right) = \breve{d}\left(A \prod_{k \neq i} a_k^{x_k}\right) + \breve{c}\left(A \prod_{k \neq i} a_k^{x_k}\right) \cdot \ln\left(a_i^{x_i'}\right) \tag{2.94}$$

$$g\left(B \cdot \prod_j b_j^{y_j}\right) = \breve{d}\left(B, \prod_{l \neq j} b_l^{y_l}\right) + \breve{c}\left(B, \prod_{l \neq j} b_l^{y_l}\right) \cdot \ln\left(b_j^{y_j'}\right) \tag{2.95}$$

But since by equations (2.92) and (2.93) we have that

$$\breve{c}\left(A, \prod_{k \neq i} a_k^{x_k}\right) = \breve{c}\left(B, \prod_{l \neq j} b_l^{y_l}\right), \tag{2.96}$$

we know that $\breve{c}$ is invariant in its arguments and thus a constant. Further, since in equation (2.94) a_i only appears in the logarithm on the RHS and to the power of x_i', but x_i' does not enter the LHS, we know that $\forall i : x_i = x_i'$ and symmetrically for equation (2.95) that $\forall j : y_j = y_j'$. Having obtained $x_i = x_i'$

and $y_j = y'_j$ for all i, j, we can rewrite (2.86):

$$g\left(A \cdot \prod_i a_i^{x_i}\right) + g\left(B \cdot \prod_j b_j^{y_j}\right) = g\left(C \cdot \prod_i a_i^{x_i} \cdot \prod_j b_j^{y_j}\right) \tag{2.97}$$

We apply Lemma 2.3 to equation (2.97) from which follows $C = A \cdot B$ and $g(x) = c \ln(x)$

□

2.A.5 Proof of Theorem 2.2

For the main proof of Theorem 2.2, we start out with the definition of the aggregate freedom measure:

$$\begin{aligned}\Phi_Z(A,O,B) &= G\left(\Psi(A,O|B), P(A \cap O \cap B), Z\right) \\ &= G_\psi\left(\xi_{1,1,1}, \ldots, \xi_{\bar{i},\bar{s},\bar{x}}, P(A \cap O \cap B), Z\right).\end{aligned} \tag{2.98}$$

By Lemma 2.1, the scale type axiom and the regularity axiom (continuity, responsiveness) then imply that the measure takes the form

$$g\left(\prod_{i,s,x} (\xi_{i,s,x})^{\tilde{c}_{i,s,x}(P(A \cap O \cap B),Z)}, P(A \cap O \cap B), Z\right).$$

Further, the unbiasedness axiom implies: $\tilde{c}_{i,s,x}(P(A \cap O \cap B), Z) = \hat{c}(P(a_i \cap o_s \cap b_x), z_{i,s,x})$ and also:

$$\begin{aligned}&g\left(\prod_{i,s,x} (\xi_{i,s,x})^{\hat{c}(P(a_i \cap o_s \cap b_x), z_{i,s,x})}, P(A \cap O \cap B), Z\right) = \\ &\quad \hat{g}\left(\prod_{i,s,x} (\xi_{i,s,x})^{\hat{c}(P(a_i \cap o_s \cap b_x), z_{i,s,x})} \cdot \hat{d}\left(P(a_i \cap o_s \cap b_x), z_{i,s,x}\right)\right)\end{aligned} \tag{2.99}$$

where $\hat{g}$ is a strictly increasing function. The branching axioms now directly imply that $\hat{c}(P(a_i \cap o_s \cap b_x), z_{i,s,x})$ is homogenous of degree one in $P(a_i \cap$

$o_s \cap b_x)$. By the homogeneity equation (Aczél & Dhombres, 1989, p. 345) this implies

$$\hat{c}\left(P(a_i \cap o_s \cap b_x), z_{i,s,x}\right) = P(a_i \cap o_s \cap b_x)\check{c}(z_{i,s,x}). \tag{2.100}$$

The branching axioms also imply

$$\begin{aligned}\hat{d}\left(P(a_i \cap o_s \cap b_x), z_{i,s,x}\right) = \ & \hat{d}\left(\alpha P(a_i \cap o_s \cap b_x), z_{i,s,x}\right) \\ & \cdot \hat{d}\left((1-\alpha)P(a_i \cap o_s \cap b_x), z_{i,s,x}\right).\end{aligned} \tag{2.101}$$

Holding $z_{i,s,x}$ fixed, this is an exponential Cauchy equation and thus has the solution (Aczél & Dhombres, 1989, p. 28):

$$\hat{d}\left(P(a_i \cap o_s \cap b_x), z_{i,s,x}\right) = \check{d}(z_{i,s,x})^{P(a_i \cap o_s \cap b_x)}. \tag{2.102}$$

Implementing the above results into equation (2.98) yields:

$$\Phi_Z(A,O,B) = \hat{g}\left(\prod_{i,s,x} (\xi_{i,s,x})^{P(a_i \cap o_s \cap b_x)\check{c}(z_{i,s,x})} \cdot \check{d}(z_{i,s,x})^{P(a_i \cap o_s \cap b_x)}\right). \tag{2.103}$$

What remains to be specified is the function $\hat{g}$. For this, we can use the Additivity axiom, which now states:

$$\begin{aligned}&\hat{g}\left(\prod_{i,s,x} (\xi_{i,s,x})^{P(a_i \cap o_s \cap b_x)\check{c}(z_{i,s,x})} \cdot \check{d}(z_{i,s,x})^{P(a_i \cap o_s \cap b_x)}\right) \\ &+ \hat{g}\left(\prod_{i',s',x'} (\xi'_{i',s',x'})^{P(a'_{i'} \cap o'_{s'} \cap b'_{x'})\check{c}(z'_{i',s',x'})} \cdot \check{d}(z'_{i',s',x'})^{P(a'_{i'} \cap o'_{s'} \cap b'_{x'})}\right) = \\ &\quad \hat{g}\Big(\prod_{i,s,x,i',s',x'} (\xi_{i,s,x}\xi'_{i',s',x'})^{P(a_i \cap o_s \cap b_x)P(a'_{i'} \cap o'_{s'} \cap b'_{x'})\check{c}(z''_{i,s,x,i',s',x'})} \\ &\qquad \cdot \check{d}(z''_{i,s,x,i',s',x'})^{P(a_i \cap o_s \cap b_x)P(a'_{i'} \cap o'_{s'} \cap b'_{x'})}\Big)\end{aligned} \tag{2.104}$$

Rewriting the RHS yields:

$$\begin{aligned}\cdots =&\hat{g}\Big(\prod_{i,s,x}(\xi_{i,s,x})^{P(a_i\cap o_s\cap b_x)\sum_{i',s',x'}P(a'_{i'}\cap o'_{s'}\cap b'_{x'})\check{c}(z''_{i,s,x,i',s',x'})}\\ &\cdot\prod_{i',s',x'}(\xi'_{i',s',x'})^{P(a'_{i'}\cap o'_{s'}\cap b'_{x'})\sum_{i,s,x}P(a_i\cap o_s\cap b_x)\check{c}(z''_{i,s,x,i',s',x'})}\\ &\cdot\prod_{i,s,x,i',s',x'}\check{d}(z''_{i,s,x,i',s',x'})^{P(a_i\cap o_s\cap b_x)P(a'_{i'}\cap o'_{s'}\cap b'_{x'})}\Big)\end{aligned}$$

Therefore, equation (2.104) has the form needed to apply Lemma 2.4. Applying this lemma yields:

$$\hat{g}(x) = \dot{c}\cdot\ln(x) \tag{2.105}$$

$$\check{c}(z_{i,s,x}) = \sum_{i',s',x'} P(a'_{i'}\cap o'_{s'}\cap b'_{x'})\check{c}(z''_{i,s,x,i',s',x'}) \tag{2.106}$$

$$\check{c}(z'_{i',s',x'}) = \sum_{i,s,x} P(a_i\cap o_s\cap b_x)\check{c}(z''_{i,s,x,i',s',x'}) \tag{2.107}$$

$$\begin{aligned}\prod_{i,s,x,i',s',x'}\check{d}(z_{i,s,x})^{P(a_i\cap o_s\cap b_x)}\cdot\check{d}(z'_{i',s',x'})^{P(a'_{i'}\cap o'_{s'}\cap b'_{x'})} =\\ \prod_{i,s,x,i',s',x'}\check{d}(z''_{i,s,x,i',s',x'})^{P(a_i\cap o_s\cap b_x)P(a'_{i'}\cap o'_{s'}\cap b'_{x'})}\end{aligned} \tag{2.108}$$

Define $c(z_{i,s,x}) = \check{c}(z_{i,s,x})\cdot\dot{c}$ to obtain the aggregate measure:

$$\Phi_Z(A,O,B) = \prod_{i,s,x}(\xi_{i,s,x})^{P(a_i\cap o_s\cap b_x)c(z_{i,s,x})}\cdot d(z_{i,s,x})^{P(a_i\cap o_s\cap b_x)} \tag{2.109}$$

with $c(z_{i,s,x}) = \sum_{i',s',x'} P(a'_{i'}\cap o'_{s'}\cap b'_{x'})c(z''_{i,s,x,i',s',x'})$ and

$$\begin{aligned}\prod_{i,s,x,i',s',x'} d(z_{i,s,x})^{P(a_i\cap o_s\cap b_x)}\cdot d(z'_{i',s',x'})^{P(a'_{i'}\cap o'_{s'}\cap b'_{x'})} =\\ \prod_{i,s,x,i',s',x'} d(z''_{i,s,x,i',s',x'})^{P(a_i\cap o_s\cap b_x)P(a'_{i'}\cap o'_{s'}\cap b'_{x'})}\end{aligned} \tag{2.110}$$

for all A', O', B' independent of A, O, B. □

2.B Proof of Proposition 2.1

Proof. Setting $c(x,u) = 1$, $d(x,u) = 0$ gives:

$$\Phi^{pos}(\eth^T(C),\theta) = \sum_{u \in U_i} \hat{p}(u) \sum_{x \in C} \theta(x|u) \left(\ln \frac{\theta(x|u)}{\theta(x)} \right) \tag{2.111}$$

Since for each u there exists x: $\theta(x|u) = 1$ it holds for all other x' that $\theta(x'|u) = 0$ and thus $\theta(x'|u)\ln(\theta(x'|u)/\theta(x')) = 0$. Define $U(x) = \{u \in U_i : \theta(x|u) = 1\}$. We then get after rearranging terms:

$$\Phi^{pos}(\eth^T(C),\theta) = \sum_{x \in C} \left(\ln \frac{1}{\theta(x)} \right) \sum_{u \in U(x)} \hat{p}(u)\theta(x|u) \tag{2.112}$$

Moreover,

$$\theta(o) = \sum_{u \in U(x)} \hat{p}(u)\theta(x|u) + \sum_{u \in U_i \setminus U(x)} \hat{p}(u)\theta(x|u) = \sum_{u \in U(x)} \hat{p}(u)\theta(x|u) \tag{2.113}$$

and thus:

$$\Phi^{pos}(\eth^T(C),\theta) = \sum_{x \in C} \theta(o) \left(\ln \frac{1}{\theta(o)} \right) \tag{2.114}$$

By rationality and the fact that the members of U_i represent the members of $\mathcal{R}$:

$$\theta(x) > 0 \Leftrightarrow x \in \{x \in C : \exists R \in \mathcal{R} : \forall y : xRy\} \tag{2.115}$$

Since all outcomes with strictly positive probability have equal probability:

$$\theta(x) > 0 \Rightarrow \theta(x) = \frac{1}{\sharp\{x \in C : \exists R \in \mathcal{R} : \forall y : xRy\}} \tag{2.116}$$

Inserting this yields:

$$\Phi^{pos}(\eth^T(C),\theta) = \ln\left(\sharp\{x \in C : \exists R \in \mathcal{R} : \forall y : xRy\}\right) \tag{2.117}$$

Since $\ln()$ is monotonically increasing, it follows that Φ^{pos} represents $\succsim_{F,JS}$.

□

2.C Proof of Proposition 2.2

Proof. Setting $c(x,u) = 1$, $d(x,u) = 0$ gives:

$$\Phi^{pos}(\eth^T(C),\theta) = \sum_{u \in U_i} \hat{p}(u) \sum_{x \in C} \theta(x|u)\left(\ln \frac{\theta(x|u)}{\theta(x)}\right) \tag{2.118}$$

Since for each u there exists x: $\theta(x|u) = 1$ it holds for all other x' that $\theta(x'|u) = 0$ and thus $\theta(x'|u)\ln(\theta(x'|u)/\theta(x')) = 0$. Define $U(x) = \{u \in U_i : \theta(x|u) = 1\}$. We then get after rearranging terms:

$$\Phi^{pos}(\eth^T(C),\theta) = \sum_{x \in C}\left(\ln \frac{1}{\theta(x)}\right)\sum_{u \in U(x)} \hat{p}(u)\theta(x|u) \tag{2.119}$$

Finally $\theta(o) = \sum_{u \in U(x)} \hat{p}(u)\theta(x|u) = \sum_{u \in U_i} \hat{p}(u)\theta(o|u)$.

$$\Phi^{pos}(\eth^T(C),\theta) = \sum_{x \in C} \theta(o)\left(\ln \frac{1}{\theta(o)}\right) = F_S(C,\theta) \tag{2.120}$$

□

2.D Proof of Proposition 2.3

Proof. Setting $c(A,u) = \lambda(A)$, $c(A^C,u) = 0$, and $d(o,U) = 0$ gives:

$$\Phi^{pos}(\eth^T(C,A),\theta) = \sum_{u \in U_i} \hat{p}(u)\lambda(A)\theta(A|u)\ln\frac{\theta(A|u)}{\theta(A)} \tag{2.121}$$

Since for each u there exists x: $\theta(x|u) = 1$ it holds for outcome A that $\theta(A|u) = 1$ or $\theta(A|u) = 0$. Define $U(A) = \{u \in U_i : \theta(A|u) = 1\}$. It follows that:

$$\Phi^{pos}(\circlearrowleft^T(C,A),\theta) = \sum_{u \in U(A)} \hat{p}(u)\lambda(A) \ln \frac{1}{\theta(A)} \tag{2.122}$$

Since $\theta(A) = \sum_{u \in U(A)} \hat{p}(u)\theta(A|u) = \sum_{u \in U(A)} \hat{p}(u)$, we have:

$$\Phi^{pos}(\circlearrowleft^T(C,A),\theta) = \lambda(A)\theta(A) \ln \frac{1}{\theta(A)} \tag{2.123}$$

Summing over A:

$$\begin{aligned}\sum_{A \subseteq X} \Phi^{pos}(\circlearrowleft^T(C,A),\theta) &= \sum_{A: A \cap C \neq \emptyset} \Phi^{pos}(\circlearrowleft^T(C,A),\theta) \\ &= -\sum_{A: A \cap C \neq \emptyset} \lambda(A)\theta(A) \ln \theta(A)\end{aligned} \tag{2.124}$$

where the first step follows from the fact that if $A \cap C = \emptyset$ then $\theta(A) = 0$. Since with the aforementioned (Section 2.3) abuse of notation $\theta(A) = \sum_{x \in A} \theta(x)$, we get:

$$D_{NP}(C,\theta,\lambda) = -\sum_{A: A \cap C \neq \emptyset} \lambda(A) \sum_{x \in A} \theta(x) \ln \sum_{y \in A} \theta(y) = \sum_{A \subseteq X} \Phi^{pos}(\circlearrowleft^T(C,A),\theta) \tag{2.125}$$

$\square$

2.E Derivations of Distributions for Section 2.10

The probability density functions of α and γ are:

$$f_\alpha(a) = \begin{cases} \frac{1}{c} & (1-c)/2 \leq a \leq (1-c)/2 \\ 0 & else \end{cases} \tag{2.126}$$

$$f_\gamma(g) = \begin{cases} \frac{1}{d} & (1-d)/2 \leq b \leq (1+d)/2 \\ 0 & else \end{cases} \tag{2.127}$$

Due to independence their joint distribution is:

$$f_{\alpha,\gamma}(a,b) = f_\alpha(a) f_\gamma(g) = \frac{1}{cd}. \tag{2.128}$$

The measure requires the joint distribution of α and x_1^*, $f_{\alpha,x_1^*}(a,x)$ and the marginal distributions, $f_\alpha(a)$ and $f_{x_1^*}(x)$ of these variables. While the marginal distribution of α is given above, the other distributions are unknown, but can be found via the change of variables technique. The change of variables will be made from γ and α to x_1^* and α: Defining the vectors $\vec{w} = \begin{bmatrix} x_1^* & \alpha \end{bmatrix}'$ and $\vec{v} = \begin{bmatrix} \gamma & \alpha \end{bmatrix}'$, we have $\vec{w} = M(\vec{v})$. In our case M is such that:

$$\begin{bmatrix} x_1^* \\ \alpha \end{bmatrix} = \begin{bmatrix} \alpha\gamma \\ \alpha \end{bmatrix} \tag{2.129}$$

The change of variables yields the joint probability density function of $\vec{w}$:

$$f_{\vec{w}}(\vec{w}) = f_{\vec{v}}(\vec{v}) \left| \det\left(\frac{d\vec{v}}{d\vec{w}} \right) \right| \tag{2.130}$$

which is in this case:

$$f_{\alpha,x_1^*}(a,x) = f_\alpha(a) f_\gamma(g) \left| \det \begin{bmatrix} \frac{1}{\alpha} & -\frac{x_1^*}{\alpha^2} \\ 0 & 1 \end{bmatrix} \right| \tag{2.131}$$

$$= \begin{cases} \frac{1}{cda} & (1-c)/2 \le a \le (1+c)/2 \quad \wedge \quad \underline{x}(a) \le x \le \overline{x}(a) \\ 0 & else \end{cases} \tag{2.132}$$

where $\overline{x}(a) = a(1+d)/2$ and $\underline{x}(a) = a(1-d)/2$. Having obtained the joint distribution of x_1^* and α, what remains to be found is the marginal density of x_1^* by integration over a. For this, the boundaries of the distribution as a function of x need to be found first:

$$\underline{a}(x) = \max\left(\frac{2x}{1+d}, \frac{1-c}{2}\right) \tag{2.133}$$

$$\overline{a}(x) = \min\left(\frac{2x}{1-d}, \frac{1+c}{2}\right) \tag{2.134}$$

This is equivalent with:

$$\underline{a}(x) = \begin{cases} \frac{1-c}{2}, & \frac{1-c}{2}\frac{1-d}{2} \le x < \frac{1-c}{2}\frac{1+d}{2} \\ \frac{2x}{1+d}, & \frac{1-c}{2}\frac{1+d}{2} \le x \le \frac{1+c}{2}\frac{1+d}{2} \end{cases} \tag{2.135}$$

$$\overline{a}(x) = \begin{cases} \frac{2x}{1-d}, & \frac{1-c}{2}\frac{1-d}{2} \le x < \frac{1+c}{2}\frac{1-d}{2} \\ \frac{1+c}{2}, & \frac{1+c}{2}\frac{1-d}{2} \le x \le \frac{1+c}{2}\frac{1+d}{2} \end{cases} \tag{2.136}$$

Notice that for $d \lesseqgtr c$, we have that $\frac{1-c}{2}\frac{1+d}{2} \lesseqgtr \frac{1+c}{2}\frac{1-d}{2}$. Therefore it is necessary to differentiate between whether c is larger or smaller than d when calculating the marginal distribution of x_1^*. Having obtained the integration boundaries, the marginal distribution of x_1^* is for the case $c \leq d$:

$$
\begin{aligned}
f_{x_1^*}(x) &= \int_{\underline{a}(x)}^{\overline{a}(x)} f_{\alpha,x_1^*}(a,x)da \\
&= \begin{cases}
\frac{\ln\left(\frac{4x}{(1-c)(1-d)}\right)}{cd}, & \frac{1-c}{2}\frac{1-d}{2} \leq x < \frac{1+c}{2}\frac{1-d}{2} \\
\frac{\ln\left(\frac{1+c}{1-c}\right)}{cd}, & \frac{1+c}{2}\frac{1-d}{2} \leq x < \frac{1-c}{2}\frac{1+d}{2} \\
\frac{\ln\left(\frac{(1+c)(1+d)}{4x}\right)}{cd}, & \frac{1-c}{2}\frac{1+d}{2} \leq x \leq \frac{1+c}{2}\frac{1+d}{2}
\end{cases}
\end{aligned}
\tag{2.137}
$$

For the case $c > d$, the marginal distribution is:

$$
\begin{aligned}
f_{x_1^*}(x) &= \int_{\underline{a}(x)}^{\overline{a}(x)} f_{\alpha,x_1^*}(a,x)da \\
&= \begin{cases}
\frac{\ln\left(\frac{4x}{(1-c)(1-d)}\right)}{cd}, & \frac{1-c}{2}\frac{1-d}{2} \leq x < \frac{1-c}{2}\frac{1+d}{2} \\
\frac{\ln\left(\frac{1+d}{1-d}\right)}{cd}, & \frac{1-c}{2}\frac{1+d}{2} \leq x < \frac{1+c}{2}\frac{1-d}{2} \\
\frac{\ln\left(\frac{(1+c)(1+d)}{4x}\right)}{cd}, & \frac{1+c}{2}\frac{1-d}{2} \leq x \leq \frac{1+c}{2}\frac{1+d}{2}
\end{cases}
\end{aligned}
\tag{2.138}
$$

The remaining step to obtain the measure is to insert the marginal distributions into equation (2.60) and solving the integrals. Since the terms become complicated and add little intuition beyond what is shown in Figure 2.4, they are not shown here, but can be obtained from the author.

3 Freedom and Power: An Experiment [1]

Claudia Neri
Hendrik Rommeswinkel

We propose a theoretical foundation for preference for decision rights, driven by preference for positive freedom, power, and negative freedom, which can lead subjects to value decision rights intrinsically, i.e. beyond the expected utility associated with them. We conduct a novel laboratory experiment in which the effect of each preference can be disentangled. The experimental design combines a bidding stage in which a decision right is allocated between two players and a decision stage in which the player holding the decision right exercises it, generating payoff consequences for both players. Risk preferences are elicited via an additional lottery-choice questionnaire. We find evidence of a stronger role of preference for negative freedom than of preference for positive freedom or power. This result suggests that individuals value decision rights not because of the actual decision making process, but rather because they have preference against others interfering in their outcomes.

[1] We have benefitted from the opportunity to present earlier versions of this work at the 2014 International Meeting on Experimental and Behavioral Social Sciences, at the Social Choice and Welfare 2014 Meeting, at the 2014 International Meetings of the Economic Science Association and at the University of St.Gallen. We are grateful to seminar participants and to Martin Kolmar and Clemens Puppe for their comments. Financial support of the Deutsche Forschungsgemeinschaft (DFG) and the University of St. Gallen is gratefully acknowledged.

3.1 Introduction

Freedom and power are pervasive components in any social, political and economic interaction in our life. In any organization, from clubs to corporations and government bodies, individuals interact by taking decisions, affecting themselves to the extent that they have the freedom to do so, and affecting others to the extent that they have the power to do so. Thus, freedom and power are fundamentally related to the exercise of decision rights. Economics, which has traditionally considered decision rights solely for their instrumental value in achieving outcomes, has recently (e.g. Fehr et al. (2013) or Bartling, Fehr, and Herz (2013)) moved to consider decision rights also for their intrinsic value, i.e. the value beyond the expected utility associated with them.

In this paper we propose a theoretical foundation for preference for decision rights, driven by preference for positive freedom, negative freedom, and power and we conduct a novel laboratory experiment in which the effect of each preference can be disentangled. We employ the following terminology. An agent experiences *positive freedom* when his actions influence his own outcomes. An agent experiences *power* when his actions influence another agent's outcomes. An agent experiences *negative freedom*, when his outcomes are not influenced by another agent's actions, i.e. if no other player has power over him. Influence is intended as causal influence: an agent, by acting based on his preferences over the outcomes, determines the outcomes. In addition to preferences over outcomes, agents have preference for positive freedom, negative freedom, and power, which can lead them to value decision rights intrinsically.

For intuition, consider the following situation as an example. On Tuesday John and his siblings agree that they will watch a movie all together at the cinema the following Sunday and that John on Sunday will choose the movie to watch. On Tuesday it is already known that two movies will be available on Sunday: a drama and a comedy. What neither John nor any of his siblings knows yet on Tuesday is what movie they will each prefer on Sunday. Holding the decision right, John will be able to choose one movie or the other, depending on his preferences. If his preferences change, so would the movie he chooses. According to our terminology, John has freedom since his preferences determine which movie he watches. John has also power since his preferences determine which movie his siblings watch. Finally, John experiences negative freedom since his siblings' preferences do not influence which movie he watches. But what if only the comedy is available? Then, since John necessarily watches the comedy, neither his preferences nor his siblings' preferences determine which movie he watches. Thus, he does not have positive freedom but he does have negative freedom. Also, since his siblings necessarily watch the comedy, John's preferences do not determine which movie his siblings watch: he does not have power. Finally, what if John's preferences are fixed such that he cannot prefer anything else than comedy? Then, even if he has the decision right, John has neither positive freedom nor power: he cannot choose one movie or the other, depending on his preferences, but he necessarily watches the comedy and so do his siblings too.

We present a general theoretical model of decision-rights allocation and choice, which we formulate in the context of extensive form games. Within a Bayesian Nash equilibrium setting, the model can represent a player who may change his behavior at an earlier stage of the game in anticipation of greater positive freedom, power, and negative freedom at a later stage. Specifically, in an auction setting, where a player bids for the decision right, a bid may be influenced by the positive freedom, power and negative freedom the decision right conveys. The model has several key features. First, since players may at a point in time not yet know their preferences over outcomes (e.g., John does not yet know on Tuesday whether he will prefer drama or comedy on Sunday), information sets contain both nodes and preference profiles. Second, in order to distinguish positive freedom, which involves influencing one's own outcomes, from power, which involves influencing other players' outcomes, outcome functions associating each terminal node to an outcome, are player-specific. Third, the causal influence that preference profiles have on outcomes is measured by how far the joint distribution of outcomes and preference profiles is from the independent case.

We then implement a simplified version of the model in our experiment. In the experiment pairs of participants (Player 1 and Player 2) play a game which involves the allocation and the exercise of a decision right. First, Player 1 bids for the decision right. Second, if Player 1 receives the decision right, he exercises it, otherwise Player 2 exercises it. The exercise of the decision right consists of making a final choice, which generates payoff consequences for both players. Uncertainty regarding the payoff consequences is resolved before the final choice is made but only after the bid for the decision right is submitted by Player 1. Across treatments and rounds we vary the positive freedom, power, and negative freedom associated with the decision right. We estimate how Player 1's preference for positive freedom, power, and negative freedom affect his valuation of the decision right, as revealed by his bid. A higher bid has two effects. First, it increases the probability that Player 1 holds the decision right. Second, it decreases the payoff uncertainty for Player 1. Therefore, it is crucial to distinguish between two different motivations of a high bid: intrinsic valuation of the decision right or risk aversion. By eliciting individual risk preferences in an additional game, we compare the actual bids with the bids implied by the elicited risk preferences.

Evidence from our experiment confirms the existence of an intrinsic value of decision rights, as previously reported in Fehr et al. (2013) and Bartling et al. (2013), and extends it from a delegation setting to a willingness to pay/auction setting. Most importantly, our theoretical framework and experimental design allow us to disentangle the drivers behind this phenomenon.

We highlight two main findings. First, we find no evidence of preference for power. This result suggests that preference for power as casually observed in politics or other institutional settings may simply be instrumental to other components of well-being, such as status recognition.

Second, we find stronger evidence of preference for negative freedom than for positive freedom. This result suggests that individuals value decision rights not because of the actual decision making process, but rather because they have preference against others intervening in their outcomes. This result leads to a fundamental change of perspective on preference for decision rights. In contrast to the interpretation presented by Fehr et al. (2013) and Bartling et al. (2013), individuals like to have decision rights in virtue of the absence of decision rights of other individuals. An individual's evaluation of risks then depends on whether risks are generated by an objective process or by the behavior of other individuals.

We are aware of several limitations in our results. The weak evidence of preference for power may partly be driven by the experimental setting, in which each player learns his own preferences towards the final choice but never learns the preferences of the other player. Therefore, a Player 1 with preference for power may not find the exercise of power over Player 2 particularly satisfying because he does not know Player 2's preferences over outcomes. Experimental settings that relax such information constraints may shed further light on the role of preference for power. We consider this an interesting direction for further research.

Further, preference for negative freedom may be driven by ambiguity aversion. If a subject believes that other individuals, when having the decision right, will not choose with certainty the option in their best interest, then he will perceive ambiguity with respect to the type of individuals he is facing. However, evidence from our experiment seems not to support this conjecture. Almost all participants in our experiment chose the option in their best interest. Thus, in order to fully explain the extent of preference for negative freedom we would need to posit either very strong ambiguity aversion or beliefs about other players that are far off the equilibrium path.

This paper lies at the intersection of several literatures, both experimental and theoretical. The paper builds on previous experimental work documenting the intrinsic value of decision rights. In a principal-agent experiment, Fehr et al. (2013) find that principals often decide not to delegate a decision right to an agent, even when delegation would provide large expected utility gains. Bartling et al. (2013) report that two game-specific characteristics affect the intrinsic value of decision rights. The intrinsic value of decision rights is higher, the higher the stake size and the higher the alignment of interests between the principal and the agent. They find that the intrinsic value of decision rights cannot be explained by risk preferences, social preferences, ambiguity aversion, loss aversion, illusion of control, preference reversal, reciprocity or bounded rationality. Instead, they conclude that the intrinsic value of decision rights originates from a preference for decision rights. Our paper tackles the unanswered question of what are the ultimate drivers of a preference for decision rights. Our theoretical framework and experimental design allow us to distinguish three drivers: positive freedom, power, and negative freedom.

Our paper builds on concepts and measures originally developed in the freedom of choice literature (Barberà et al., 2004, Baujard, 2007, Dowding & van Hees, 2009) and the power index literature (Penrose, 1946, Shapley & Shubik, 1954, Banzhaf, 1965, Diskin & Koppel, 2010). The distinction between positive freedom and negative freedom dates back to Berlin (1958) though not in a game theoretic context.

In addition to the literatures mentioned above, our work can contribute to diverse literatures that analyze attitudes towards decision rights and their effect on behavior in applied settings, such as: the corporate governance literature on allocation and exercise of control (Dyck & Zingales, 2004), and the human resource management literature on workers' autonomy in the work place (Handel & Levine, 2004).

We highlight two concepts that are related to our main result (i.e., the intrinsic value of decision rights), but not to our framework: preference for flexibility (Kreps, 1979b) and betrayal aversion (Bohnet & Zeckhauser, 2004). First, preference for flexibility does not apply to our framework, nor to Fehr et al. (2013) and Bartling et al. (2013), since preference for flexibility is already captured in the behavior predicted by Nash equilibrium. In our experimental design players learn about their preferences over outcomes after the decision right is assigned. In the Nash equilibrium individuals anticipate at an earlier stage the value of being able at a later stage to make a final choice instead of receiving the outcome of a lottery. Thus, the value of flexibility is fully incorporated in the Nash equilibrium behavior. Our observed deviations from Nash equilibrium behavior cannot be explained by preference for flexibility.[2]

Second, Bohnet and Zeckhauser (2004) report experimental evidence suggesting that the decision not to trust another agent is driven by betrayal aversion. In their experimental design, the decision to trust someone (letting him make a final choice which has payoff consequences for both agents) entails an additional risk premium compared to the decision to let a random-device lottery determine the final choice and payoff consequences. They argue that the additional risk premium is required to balance the costs of trust betrayal.[3] However, as they acknowledge, their design cannot distinguish whether differences in behavior are due to different assessments of the outcomes, and thus they cannot rule out that their results are driven not by an aversion to betrayal but by an aversion to relinquishing control to another agent (preference for negative freedom in our framework).[4] Our results suggest that aversion to interference may be a driver of behavior in their experiment.

The paper proceeds as follows. In Section 3.2 we outline a behavioral model of preference for positive freedom, power, and negative freedom. Section 3.3 describes the experimental design. We present the theoretical predictions of the model in Section 3.4 and the empirical strategy in Section 3.5. The results are given in Section 3.6. Section 3.7 concludes.

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[2]In our movie example, preference for flexibility refers to the expected utility gain from being able to choose the movie that one likes best. This is captured by Nash equilibrium behavior. Preference for positive freedom is the procedural rather than consequentialist value of one's own preferences determining the outcomes.

[3]Bohnet and Zeckhauser (2004) compare behavior in a trust game and a risky dictator game. The trust game involves a binary choice by Player 1 (trust or not trust) followed by a binary choice by Player 2 conditional on Player 1's decision to trust. The risky dictator game differs only in that Player 1's decision to trust is followed by a random-device lottery, not by a choice by Player 2. In both games, a decision not to trust yields payoffs (S,S) to Player 1 and 2, respectively. Following a decision to trust, the payoff pairs can be either (B,C) or (G,H), with $G > S > B$ and $C > H > S$. In both games, participants with the role of Player 1 report their minimum acceptable probability (MAP) of getting G such that they prefer to trust instead of not to trust.

[4]'A MAP gives us information on how a Decision Maker assesses the risky-choice problem he is confronted with, but not on how he values each possible outcome. Based on our data, we are not able to distinguish whether differences in MAPs are due to different assessments of S or of B and G.'

3.2 Theoretical Framework

In this section we describe a model of decision-rights allocation and choice. In order to provide a general theoretical framework, we formulate the model in the context of extensive form games. We then implement a simplified version of the model in our experiment.

Consider an extensive-form game $\eth = (N, A, \psi, \mathcal{P}, \mathcal{I}, \mathcal{C}, O, \mathcal{U}, p)$. There is a finite set of players, $N = \{1, \ldots, n\}$, and A is a finite set of nodes. $\psi : A/a_0 \to A$ is a predecessor function such that for node a, $\psi(a)$ is the immediate predecessor of a. $\mathcal{P}$ is the player partitioning of the nodes. $\mathcal{I} = \{I_0, \ldots, I_n\}$ is the information partitioning, with I_i being the set of information sets of Player i, and $A(I) = \{a \in A : \psi(a) \in I\}$ is the set of nodes following information set I. $\mathcal{C}$ is the set of choice sets C_I for each information set I and $\Delta(C_I)$ is the set of probability distributions over the choice set at I. For $b \in I$ and $b = \psi(a)$ let $c(a|b) \in C_I$ be the choice that leads from node b to node a.

Our notation diverges from standard notation of game forms in two main aspects. First, $O = \{o_1, ..., o_n\}$ is the set of outcome functions, where $o_i : A_\omega \to O_i$ maps the terminal nodes $A_\omega = A \backslash \psi(A)$ into the finite set of possible outcomes for Player i, O_i. We require player-specific outcome functions to distinguish power and positive freedom. Having power means being able to influence another player's outcomes. Having positive freedom means being able to influence one's own outcomes. Outcome functions that are not player-specific would conflate power and positive freedom.

Second, $\mathcal{U} = \{U_1, \ldots, U_n\}$ is a set of sets $U_i = \{u_i^1, \ldots, u_i^J\}$ of utility functions for each Player i where $u_i^j : O_i \to \mathbb{R}$. Since positive freedom requires the possibility to act in one way or another, individuals need to potentially have more than one preference profile in order to have positive freedom. Since individuals may at a point in time not yet know their preferences, information sets contain both nodes and utility functions: $I \subseteq A \cup_{i \in N} U_i$ such that $I \cap A \neq \varnothing$ and $\forall i : I \cap U_i \neq \varnothing$. For example, at an information set $I \in I_1 = \{a_1, a_2, u_1^1, u_1^2, u_2^1\}$ Player 1 does not know whether he is at node a_1 or a_2 and whether he has preferences u_1^1 or u_1^2, but knows that Player 2 has preferences u_2^1.

A local strategy $s_I \in \Delta(C_I)$ is a probability distribution over the elements of the choice set at information set I. A strategy profile S is a tuple of local strategies specifying behavior at each information set $S = (s_I|_{I\in I_i}|_{i\in N})$. p is the probability distribution for moves by Nature at information sets in $\mathcal{I}_0$ and over utility functions for each player. Finally, θ^S denotes the joint probability distribution over nodes, outcomes, and preference profiles resulting from strategy profile S and moves by Nature according to p. The subgame function $subg(\eth,a)$ returns for any game $\eth$ the subgame starting at node a. Let θ_i be a joint probability distribution over nodes, outcomes and preference profiles representing the beliefs of Player i. Let $\theta_{i|I}$ ($\theta_{i|a}$) denote the beliefs of Player i given that play has reached information set I (node a) derived from Bayesian updating on θ_i. We can construct the belief of node a following the current information set given strategy s_I as $\tilde{\theta}_{i|s_I}(a) = \theta_{i|I}(\psi(a)) \cdot s_I(c(a|\psi(a)))$.

Finally, we define an equilibrium of game $\eth$ as a strategy profile $S^* = (s^*(I,\theta_i)|_{I\in I_i}|_{i\in N})$ and beliefs such that $\forall i : \theta_i = \theta^{S^*}$ with:

$$s^*(I,\theta_i) = \arg\max_{s\in\Delta(C_I)} \sum_{a\in A(I)} \tilde{\theta}_{i|s}(a) V_i(subg(\eth,a),\theta_{i|a}). \tag{3.1}$$

This definition corresponds to a standard Bayesian Nash equilibrium if $V_i(\eth,\theta)$ coincides with expected utility $EU_i(\eth,\theta)$:

$$EU_i(\eth,\theta) = \sum_{u\in U_i} \theta(u) \sum_{o\in O_i} \theta(o|u)u(o). \tag{3.2}$$

Instead, we define V_i as to also include the utility from positive freedom, power and negative freedom for each subgame. Thus, individuals may change their behavior at earlier stages of the game in anticipation of greater positive freedom, power and negative freedom at later stages. Note that in this framework there are two distinct notions of preferences. First, there are the non-procedural preferences over outcomes, $u \in U_i$. Second, there is the procedural preference for subgames, V_i, containing a player's preference for positive freedom, negative freedom, and power. To avoid confusion, we refer to the former in the plural and the latter in the singular. We use the following terminology.

Positive Freedom Player i has *positive freedom* if he causally influences his own outcomes. In our movie example, John has positive freedom if his preferences on Sunday determine which movie he watches. Thus, positive freedom is measured by the degree to which Player i's own preferences determine his own outcomes, as:

$$\Phi_i^{pf}(\Game,\theta) = \sum_{u \in U_i} \theta(u) \sum_{o \in O_i} g(o,u)\theta(o|u)\log_2 \frac{\theta(o|u)}{\theta(o)}, \tag{3.3}$$

where $\log_2 \frac{\theta(o|u)}{\theta(o)}$ is the causal influence measure capturing how far the joint probability of outcome o and preference profile u is from the independent case and the expectation is taken over all preference-outcome combinations. For example, take two outcomes A and B and an individual which either prefers A or B, i.e. has preferences u^A or u^B. If $\theta(A|u^A) = \theta(A) = 1 - \theta(B)$, the fact that an individual prefers A or B makes no difference on whether the outcome is A or B. This is captured by the causal influence measure via $\log_2 \frac{\theta(o|u)}{\theta(o)} = 0$, for all $o \in \{A,B\}$ and $u \in \{u^A,u^B\}$. However, if the individual has some influence, then $\theta(A|u^A) > \theta(A)$ and this will result in a positive causal influence measure. This measure captures Berlin's definition of positive freedom as "[t]he freedom which consists in being one's own master" (1958, p.8) and other intuitions from the freedom of choice literature.[5]

The function $g(o,u)$ is included to capture the value of the causal influence. For example, if two outcomes are qualitatively very similar, the value of having freedom to choose between the two may be very low. If in the cinema only one movie is playing and the only choice to make is whether to watch it in theater 1 or 2, the alternative outcomes may not be qualitatively distinct enough for the decision right to provide a high amount of positive freedom. The causal influence measure $\log_2 \frac{\theta(o|u)}{\theta(o)}$ between outcome o and preferences u is there-

[5]For details, see Section 2.5 of this Dissertation.

fore weighted by $g(o,u)$. Several specifications of $g(o,u)$ will be discussed in Section 3.4.

Negative Freedom Player i has *negative freedom* if other players do not causally influence his outcomes. In our movie example, John experiences negative freedom if he chooses the movie or if there is only one movie available. In both cases others' preferences do not influence which movie he watches. Interference is measured by the degree to which other players' preferences determine Player i's own outcomes. Thus negative freedom is measured by:

$$\Phi_i^{nf}(\circlearrowleft,\theta) = -\sum_{j\in N\setminus i}\sum_{v\in U_j}\theta(v)\sum_{u\in U_i}\theta(u|v)\sum_{o\in O_i}g(o,u)\theta(o|v)\log_2\frac{\theta(o|v)}{\theta(o)}. \tag{3.4}$$

The intuition for negative freedom is analogous to the one provided for positive freedom. The difference is that negative freedom captures not the causal influence that a player has on his own outcomes but the causal influence that other players have on his outcome. This measure captures Berlin's definition of negative freedom as "not being interfered with by others. The wider the area of non-interference, the wider my freedom"(1958, p.3). Again, $g(o,u)$ can be used to determine the value of not being interfered with. For example, interference may matter little to John, if his siblings only get to choose whether to watch the movie in theater 1 or 2, but do not choose the movie itself. Reducing the interference of another player may be less valuable when its qualitative impact on the outcome is small compared to the case in which it is large.

Power Player i has *power* if he causally influences the outcomes of other players. In our movie example, if John chooses the movie, then John has power, since his preferences determine which movie his siblings watch. However, if there is only one movie available at the cinema, John does not have power, since his preferences do not determine which movie his siblings watch: they simply watch the only available movie. Power is measured as:

$$\Phi_i^p(\Game,\theta) = \sum_{u\in U_i} \theta(u) \sum_{j\in N/i} \sum_{o\in O_j} g(o,u)\theta(o|u)\log_2 \frac{\theta(o|u)}{\theta(o)}. \tag{3.5}$$

This measure is similar to the voting power measure by Diskin and Koppel (2010) with the exceptions that we introduced player-specific outcomes, a weighting function $g(o,u)$ and the measure is generalized to extensive form games. The weighting function $g(o,u)$ measures the qualitative impact on the outcomes of those players over whom Player i has power.

The valuation function $V_i(\Game,\theta)$ of a Player i with preference for positive freedom, negative freedom and power includes all the above components as:

$$V_i(\Game,\theta) = \alpha_i\Phi_i^{pf}(\Game,\theta) + \beta_i\Phi_i^{nf}(\Game,\theta) + \gamma_i\Phi_i^{p}(\Game,\theta) + \delta_i EU_i(\Game,\theta), \tag{3.6}$$

where the coefficients α, β, γ and δ determine the intensity of each component. An individual with preference for positive freedom/negative freedom/power evaluates the choices not only by the expected utility of the subgame following the choice but also by the expected positive freedom/negative freedom/power offered by the subgame.

Measuring positive freedom, negative freedom and power requires determining not only what individuals can causally influence (i.e., their own or others' outcomes), but also what enables individuals to exercise such a causal influence (i.e. the source of agency). Agency is what allows an individual to behave in one way or another and to achieve an outcome or another by doing so. Outside of an experimental setting, the source of agency lies in an individual's preferences over the alternative outcomes.

In an experimental setting, it is standard practice to induce the value of each alternative via monetary payments.[6] Thus, the source of agency is introduced by the game structure by means of a payment structure. This is unproblematic in experiments which investigate how behavior changes if the values of the alternatives change: manipulating the monetary payments is suf-

[6]For an introduction to induced-value theory, see Smith (1976).

ficient. However, an experiment such as ours, which investigates how behavior changes if positive freedom/negative freedom/power change, requires making the formation of preferences part of the game, since manipulating positive freedom/negative freedom/power requires manipulating the relationship between preferences over outcomes and outcomes. We achieve this by having preferences over outcomes randomly determined by moves of Nature at the beginning of a subgame.

While we are aware that positive freedom in real-world situations may be qualitatively different from positive freedom induced by the game structure, we also believe that our framework makes preference for positive freedom more unlikely to be observed in the experiment. Therefore, evidence of preference for positive freedom in the experiment suggests that such preference for positive freedom is even more likely to arise in real-world settings, where preferences are not induced but formed internally. Analogous arguments can be made for preference for negative freedom and for power.

3.3 Experimental Design

The experiment implements a simplified version of the theoretical framework presented in Section 3.2. Two players, Player 1 and Player 2, play a game involving the selection of a card from one of either two boxes, Box L and Box R. Box L and Box R each contain two cards, Card A and Card B. Each card has two sides, Side 1 and Side 2.

The game consists of two stages: a bidding stage and a choice stage. The bidding stage serves to determine which player has the decision right in the choice stage. In the choice stage the player with the decision right makes the card selection. The decision right is allocated via a Becker-DeGroot-Marschak (BDM) mechanism (Becker, Degroot, & Marschak, 1964). Player 1 is required to bid for the decision right by choosing an integer between 0 and 100, $y \in \{0,\dots,100\}$. The computer then randomly determines an integer between 1 and 100 with uniform probability, $r \in \{1,\dots,100\}$. If $y \geq r$, Player 1 has the decision right: he will select a card from Box L in the choice stage and pay a fee equal to r. Otherwise, Player 2 has the decision right: he will select a card from Box R in the choice stage and no fee is paid by either player.

In each box independently, the colors of the sides of the cards are determined via a random draw from the four cases represented in Figure 3.1. Each case has a priori equal probability. The color of Side 1 is payoff-relevant for Player 1 and the color of Side 2 is payoff-relevant for Player 2. Green is associated with a higher payoff, i.e. $\pi_i^{high,K} > \pi_i^{low,K}$, where $\pi_i^{high,K}$ denotes Player i's payoff if Side i of the card selected from box K is Green, and $\pi_i^{low,K}$ denotes Player i's payoff if Side i of the card selected from box K is Red, and $K \in \{L,R\}$.

Each side of each card can be Green or Red with equal probability. Moreover, side i of Card A and side i of Card B are always of a different color, which guarantees that Player i either prefers Card A to be selected or Card B to be selected. If Side 1 and Side 2 of a given card are of the same color, then Players prefer the same card. Otherwise, Players prefer different cards.[7] We can interpret the random draw from the four cases in Figure 3.1 as a move by Nature, which randomly determines players' preferences over outcomes, $U_1 \in \{u_1^A, u_1^B\}$ and $U_2 \in \{u_2^A, u_2^B\}$, as discussed in Section 3.2.

The order of events is shown in Figure 3.2. As the bidding stage starts, players learn the values of $\pi_i^{high,K}$ and $\pi_i^{low,K}$ for $i = 1,2$ and $K \in \{L,R\}$. Thus, they learn, for each player and for each box, what the payoff associated with Green and the payoff associated with Red are. At this moment, neither player knows, for either box, whether he prefers Card A or B, or whether the other player prefers Card A or B.

As the choice stage starts, players receive additional information. The box, from which the card selection will occur, is opened and each player observes the colors on 'his' side of the two cards: Player 1 observes Side 1 of Card A and Side 1 of Card B, Player 2 observes Side 2 of Card A and Side 2 of Card

[7]As shown in Figure 1: in case 1 both players prefer Card B, in case 2 Player 1 prefers Card B and Player 2 prefers Card A, in case 3 Player 1 prefers Card A and Player 2 prefers Card B, in case 4 both players prefer Card A.

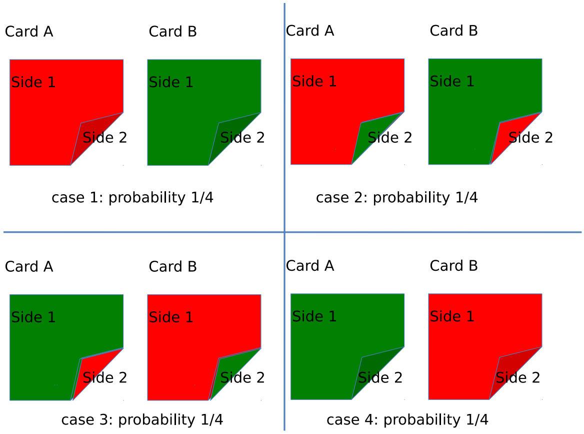

Figure 3.1: Card colors in Box $K = L, R$

B. Therefore, each player learns which card gives him the higher payoff, i.e. learns which card he prefers. However, no player observes the colors on 'the other' side of the two cards. Therefore, no player learns which card the other player prefers.

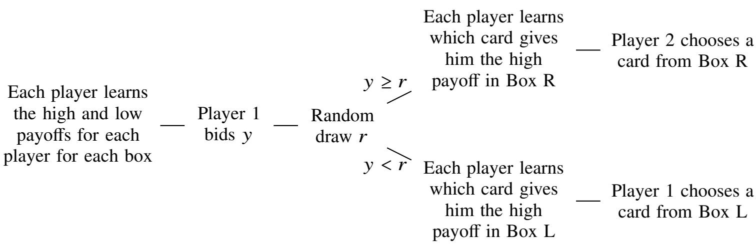

Figure 3.2: Order of events

To represent preference for positive freedom, negative freedom and power we need to define the set of outcomes. For Player 1, let $O_1 = \{0, \ldots, 100\} \times \{1, 2\} \times \{A, B\}$ with $o_1(r, i, c)$ denoting the outcome where the randomly drawn number is r, Player i has the decision right and chooses card c. For Player 2 the number r is never relevant, so let $O_2 = \{1,2\} \times \{A,B\}$ with $o_2(i,c)$ denoting the outcome where Player i has the decision right and chooses card c.

	$i = 1$	
	$c = A$	$c = B$
$\pi_1(o_1(r,i,c), u_1^A)$	$w_1 + \pi_1^{high,L} - r$	$w_1 + \pi_1^{low,L} - r$
$\pi_1(o_1(r,i,c), u_1^B)$	$w_1 + \pi_1^{low,L} - r$	$w_1 + \pi_1^{high,L} - r$
$\pi_2(o_1(i,c), u_2^A)$	$w_2 + \pi_2^{high,L}$	$w_2 + \pi_2^{low,L}$
$\pi_2(o_1(i,c), u_2^B)$	$w_2 + \pi_2^{low,L}$	$w_2 + \pi_2^{high,L}$
	$i = 2$	
	$c = A$	$c = B$
$\pi_1(o_1(r,i,c), u_1^A)$	$w_1 + \pi_1^{high,R}$	$w_1 + \pi_1^{low,R}$
$\pi_1(o_1(r,i,c), u_1^B)$	$w_1 + \pi_1^{low,R}$	$w_1 + \pi_1^{high,R}$
$\pi_2(o_1(i,c), u_2^A)$	$w_2 + \pi_2^{high,R}$	$w_2 + \pi_2^{low,R}$
$\pi_2(o_1(i,c), u_2^B)$	$w_2 + \pi_2^{low,R}$	$w_2 + \pi_2^{high,R}$

Table 3.1: Payoff structure

The payoff structure of the game is always common knowledge. Payoffs vary across rounds and treatments, as described in detail in Section 3.3.1-3.3.2. Table 3.1 provides the general payoff structure. Player 1's payoff is $\pi_1(o_1(r,i,c),u_1^A)$ if he prefers Card A and $\pi_1(o_1(r,i,c),u_1^B)$ if he prefers Card B. Analogously, Player 2's payoff is $\pi_2(o_2(i,c),u_2^A)$ if he prefers Card A and $\pi_2(o_2(i,c),u_2^B)$ if he prefers Card B. Moreover, Player 1 and Player 2 start the game holding endowments w_1 and w_2, respectively.

3.3.1 Rounds

The game is played repeatedly for 20 rounds. Across rounds, we vary the values for Player 2's payoffs $\pi_2^{high,L}$ and $\pi_2^{low,L}$ to account for situations in which the decision right gives Player 1 power or not. $\eth^{np}$ are games where $\pi_2^{high,L} = \pi_2^{low,L}$. Therefore, when Player 1 has the decision right and selects a card from Box L, he does not have power since he cannot influence Player 2's outcomes: Player 2 is indifferent between the cards since $\pi_2^{high,L} = \pi_2^{low,L}$. $\eth^{p}$ are games where $\pi_2^{high,L} > \pi_2^{low,L}$, and therefore the decision right gives Player 1 power.

Across the 20 rounds, participants play 10 $\Game^{np}$ games and 10 $\Game^{p}$ games. Within $\Game^{np}$ and $\Game^{p}$, the rounds differ in the expected payoff and the stake size for each player, as shown in Table 3.2. The order in which the rounds are played is random. Notice that in both $\Game^{np}$ and $\Game^{p}$ we have $\pi_2^{high,R} > \pi_2^{low,R}$: Player 2 is never indifferent between the cards when he has the decision right. Finally, Player 1's payoffs are $\pi_1^{high,L} = \pi_1^{high,R} = \pi_1^{high}$ and $\pi_1^{low,L} = \pi_1^{low,R} = \pi_1^{low}$.

		Box L		Box R	
		Player 1	Player 2	Player 1	Player 2
game	round	Green/Red π_1^{high}/π_1^{low}	Green/Red $\pi_2^{high,L}/\pi_2^{low,L}$	Green/Red π_1^{high}/π_1^{low}	Green/Red $\pi_2^{high,R}/\pi_2^{low,R}$
$\Game^{np}$	1	100/30	70/70	100/30	100/30
$\Game^{np}$	2	90/40	70/70	90/40	90/40
$\Game^{np}$	3	80/50	70/70	80/50	80/50
$\Game^{np}$	4	85/15	70/70	85/15	85/15
$\Game^{np}$	5	75/25	70/70	75/25	75/25
$\Game^{np}$	6	65/35	70/70	65/35	65/35
$\Game^{np}$	7	70/0	70/70	70/0	70/0
$\Game^{np}$	8	60/10	70/70	60/10	60/10
$\Game^{np}$	9	50/20	70/70	50/20	50/20
$\Game^{np}$	10	100/0	70/70	100/0	100/0
$\Game^{p}$	11	75/25	85/15	75/25	85/15
$\Game^{p}$	12	75/25	75/25	75/25	75/25
$\Game^{p}$	13	75/25	65/35	75/25	65/35
$\Game^{p}$	14	75/25	90/40	75/25	90/40
$\Game^{p}$	15	75/25	60/10	75/25	60/10
$\Game^{p}$	16	85/15	75/25	85/15	75/25
$\Game^{p}$	17	65/35	75/25	65/35	75/25
$\Game^{p}$	18	90/40	75/25	90/40	75/25
$\Game^{p}$	19	60/10	75/25	60/10	75/25
$\Game^{p}$	20	100/0	100/0	100/0	100/0

Table 3.2: Payoffs in each round

3.3.2 Treatments

We conduct the experiment under three treatments, in which we modify key features of the game. Games are denoted $\Game_1$, $\Game_2$ and $\Game_3$ in Treatment 1, 2 and 3, respectively. In the benchmark Treatment 1 both players receive an endowment of 100 points ($w_1 = w_2 = 100$). In Treatment 2 only Player 1 receives an endowment ($w_1 = 100$, $w_2 = 0$). The variation in endowments allows us to verify whether social preferences play a role. Specifically, Player

1 may prefer to bid higher or lower due to advantageous or disadvantageous inequality. We explore the role of inequality aversion in Appendix 3.D.

In Treatment 3 $w_1 = 100$ and $w_2 = 0$ as in Treatment 2, but Box L contains only one card (Card C) which is green on Side 1 and is either red or green on Side 2. Under this modified design, the decision right provides Player 1 only negative freedom, but neither positive freedom nor power. Similarly to the other treatments, if Player 1 has the decision right, he enjoys negative freedom since Player 2 cannot influence Player 1's outcomes. However, Player 1 does not have positive freedom since he cannot influence his own outcomes: there is no choice for him to make, since Box L contains only Card C. Moreover, Player 1 has no power since he cannot influence Player 2's outcomes. Treatment 3 allows us to distinguish negative freedom from positive freedom, which are not distinguishable in Treatment 1 and 2.

Treatment	Endowments	Cards	Games	decision right gives Player 1		
	w_1, w_2	in Box L		positive freedom	negative freedom	power
1	100,100	A, B	$\Game_1^{np}$	*yes*	*yes*	*no*
			$\Game_1^{p}$	*yes*	*yes*	*yes*
2	100,0	A, B	$\Game_2^{np}$	*yes*	*yes*	*no*
			$\Game_2^{p}$	*yes*	*yes*	*yes*
3	100,0	C	$\Game_3$	*no*	*yes*	*no*

Table 3.3: Treatments

Table 3.3 summaries the characteristics of each treatment. Notice that the distinction between games $\Game^{np}$ and games $\Game^{p}$ is relevant for Treatment 1 and 2, but not for Treatment 3, which does not involve power either in games $\Game^{np}$ or in games $\Game^{p}$.

3.3.3 Procedures

We conducted 8 sessions: 3 sessions of Treatment 1, 3 sessions of Treatment 2 and 2 sessions of Treatment 3. The sessions took place over two consecutive days in October 2013 at the University of Cologne. Each session lasted approximately 1.5 hours. In total 244 subjects participated: 86 in Treatment 1, 96 in Treatment 2 and 62 in Treatment 3.[8] Participants were recruited via ORSEE (Greiner, 2004) and consisted mostly of students at the University of Cologne. The experiment was implemented in zTree (Fischbacher, 1999). The experiment is divided into three parts. Participants receive instructions for each part only after completing the previous part.

In Part 1 subjects play the card game described above.[9] At the start, half of the subjects are randomly assigned the role of Player 1 and the other half of the subjects the role of Player 2. Each Player 1 is randomly matched with a Player 2. The roles and the matches are then fixed for the entire duration of Part 1. Subjects play a trial round of game $\Game^{np}$ (which does not count for their earnings) and then play 20 rounds (10 games $\Game^{np}$ and 10 games $\Game^{p}$). Rounds are played in random order and feedback regarding each round is given only at the end of the experiment (i.e. end of Part 3). At the end of the experiment one round is randomly selected and each subject is paid according to the payoff earned in that round only.

Part 2 and Part 3 involve individual decisions, with no interaction among subjects. In Part 2 subjects answer a lottery-choice questionnaire similar to Holt and Laury (2002). The lottery-choice questionnaire, which is reported in Table 3.A.3 in Appendix 3.A, allows us to elicit subjects' risk attitudes. Each question involves the choice between a safe lottery (Option A) which yields prize π^{A} with certainty and a risky lottery (Option B) yielding a high prize $\pi^{B,high}$ with probability 0.5 and a low prize $\pi^{B,low}$ with probability 0.5.

[8]One session had 22 participants, one session 30 participants and 6 sessions 32 participants.

[9]As Part 1 starts, subjects receive written instructions. In order to have participants focus on the key features of the game, we present them with four comprehension questions. The questions are reported in Appendix 3.A. When participants submitted an incorrect answer, they were provided with a correction and a short explanation. In general, subjects understood the experiment well. Questions 1, 2 and 3 are answered correctly by 96, 98 and 97 percent of the subjects, respectively. Question 4 is presented to highlight the fact that, if Player 1's bid is successful, Player 1 has to pay not his own bid but the number randomly drawn by the computer. Question 4, which is clearly the most difficult question, is answered correctly by 58 percent of the subjects. Individuals were thereby reminded, in a non-technical way, of the second-price nature of the bidding mechanism. Despite the lower fraction of initial correct answers, we believe that the provided correction and explanation are instrumental in achieving subjects' understanding.

The lotteries of Part 2 are designed to resemble the implicit lotteries faced by the players in the games of Part 1. Prize π^A resembles the certain payoff that a player receives when he has the decision right, while prizes $\pi^{B,high}$ and $\pi^{B,low}$ resemble the payoffs that a player may receive when the other player has the decision right. As discussed in Section 3.4, an expected-utility-maximizer Player 1 who chooses bid y^* in a game of Part 1 should choose the safe Option A in the corresponding lottery-choice question of Part 2 (with $\pi^{B,high} = \pi_1^{high}$, $\pi^{B,low} = \pi_1^{low}$) if and only if $\pi^A \geq \pi^{B,high} - y^*$. At the end of the experiment one lottery-choice question is randomly selected. Each subject has his chosen option played out and is paid accordingly.

Finally, in Part 3, subjects answer a Locus of Control Test (Rotter, 1966, Levenson, 1981, Krampen, 1981).[10] In personality psychology, locus of control refers to the extent to which individuals believe that they can control events that affect them. A person's 'locus' is either internal (if he believes that events in his life derive primarily from his own actions) or external (if he believes that events in his life derive primarily from external factors, such as chance and other people's actions, which he cannot influence). There may be several reasons why attitudes towards locus of control may be related to attitudes towards positive freedom and negative freedom. For example, subjects who believe that other individuals control their lives may have a greater preference for positive freedom and negative freedom. However, as reported in Appendix 3.C, we do not find strong evidence that attitudes towards locus of control are correlated with preference for positive freedom or negative freedom.

At the end of the experiment participants answer a socio-demographic questionnaire. All payoffs in the experiment were expressed in points. The conversion rate was €1 = 12 points. Individuals earned on average €10.97 in Part 1 and €4.90 in Part 2. In addition, subjects received €2.50 for participation.

[10]The questionnaire is reported in Appendix 3.C.

3.4 Theoretical Predictions

The Bayesian Nash equilibrium predictions, assuming $V_i(\backepsilon,\theta) = EU_i(\backepsilon,\theta)$ and a utility function u linear in payoffs, are straightforward. In the choice stage Player i with the decision right chooses $c^{*RNNE} = A \Leftrightarrow U_i = u_i^A$ and $c^{*RNNE} = B \Leftrightarrow U_i = u_i^B$. In the bidding stage it is optimal for Player 1 to bid his true valuation of the decision right. The continuation payoff from the subgame where Player 1 has the decision right is π_1^{high} and the continuation payoff from the subgame where he does not have the decision right is $(\pi_1^{high} + \pi_1^{low})/2$. Therefore, the optimal bid of a risk-neutral Player 1 is $y^{*RNNE} = (\pi_1^{high} - \pi_1^{low})/2$.

Allowing for risk aversion, while keeping $V_i(\backepsilon,\theta) = EU_i(\backepsilon,\theta)$, does not affect behavior in the choice stage: Player i with the decision right chooses $c^{*NE} = A \Leftrightarrow U_i = u_i^A$ and $c^{*NE} = B \Leftrightarrow U_i = u_i^B$. However, in the bidding stage Player 1 is influenced by the fact that Box R involves the risky lottery $(\frac{1}{2},\pi_1^{high};\frac{1}{2},\pi_1^{low})$ while Box L involves the safe lottery $(1,\pi_1^{high})$.[11] Therefore, the optimal bid y^{*NE} satisfies the following condition:

$$u(w_1 - y^{*NE} + \pi_1^{high}) = \frac{1}{2}u(w_1 + \pi_1^{high}) + \frac{1}{2}u(w_1 + \pi_1^{low}). \tag{3.7}$$

Defining the certainty equivalent CE of the risky lottery as:

$$CE\left(\frac{1}{2},\pi_1^{high};\frac{1}{2},\pi_1^{low}\right) = c : u(c) = \frac{1}{2}u(\pi_1^{high}) + \frac{1}{2}u(\pi_1^{low}), \tag{3.8}$$

we can rewrite equation (3.7) in terms of certainty equivalent as:

$$w_1 - y^{*NE} + \pi_1^{high} = CE\left(\frac{1}{2},w_1 + \pi_1^{high};\frac{1}{2},w_1 + \pi_1^{low}\right). \tag{3.9}$$

To predict the behavior of a participant with preference for positive free-

[11] $(\frac{1}{2},\pi_1^{high};\frac{1}{2},\pi_1^{low})$ is the lottery yielding π_1^{high} with probability 0.5 and π_1^{low} with probability 0.5. $(1,\pi_1^{high})$ is the lottery yielding π_1^{high} with probability 1.

dom, negative freedom and power, we need to determine positive freedom, negative freedom and power at each subgame following the bid of Player 1: the measures Φ_1^{pf}, Φ_1^{nf}, and Φ_1^{p} introduced in Section 3.2. Before doing so, we have to determine the functional form of $g(o,u)$ in equations (3.3)-(3.5).

We consider two specifications. First, and most simply, we can set $g(o,u) = 1$, assuming that the value of positive freedom, negative freedom or power is independent of the outcome and the utility of the outcome. According to this first specification, we index the measures as $\Phi_1^{pf,c}$, $\Phi_1^{nf,c}$, and $\Phi_1^{p,c}$. Second, we can set $g(o,u) = \Delta\pi_i = |\pi_i^{high} - \pi_i^{low}|$. While the logarithmic terms in equations (3.3)-(3.5) account for the probabilistic causal influence of preferences on outcomes, the distance in payoffs $\Delta\pi_i$ measures the qualitative effect of such causal influence. For example, the decision between two outcomes yielding very similar payoffs may be seen as having a smaller qualitative effect than a decision between two outcomes yielding very different payoffs. Thus, positive freedom, negative freedom and power may become more important as the alternative outcomes differ more in terms of the payoffs they yield. We need to use $\Delta\pi_1$, the qualitative impact on Player 1's payoffs, for positive freedom and negative freedom, and $\Delta\pi_2$, the qualitative impact on Player 2's payoffs, for power. According to this second specification, we index the measures as $\Phi_1^{pf,d}$, $\Phi_1^{nf,d}$, and $\Phi_1^{p,d}$.[12]

Decisions in the choice stage are unaffected by preference for positive freedom, negative freedom and power. Since the subgame following each choice is a terminal node a_ω, we have $\theta(o(a_\omega)) = 1$ and thus the causal influence measures $\log_2 \frac{\theta(o|u)}{\theta(o)}$ are equal to zero. This is intuitive: while the individual has control over the outcome at the moment of making the decision, he loses the control by exercising it. Since the terminal nodes do not offer any positive freedom, negative freedom or power, the choice over terminal nodes is therefore unaffected by preference for them. Thus, an individual i with $\delta_i > 0$ in

[12]We are aware that this is a very crude way of comparing the qualitative difference of an element to a set. For the purposes of this experiment with essentially only two outcomes, such a simple metric will be sufficient. More sophisticated measures of qualitative diversity and their relation to difference metrics are given in Nehring and Puppe (2002). It may be interesting to consider experiments where outcomes have a qualitative difference aside from payoffs.

equation (3.6) chooses $c^* = A \Leftrightarrow U_i = u_i^A$ and $c^* = B \Leftrightarrow U_i = u_i^B$, just as in the Bayesian Nash equilibrium. In the bidding stage, instead, the bid of Player 1 is affected by preference for positive freedom, negative freedom and power. Derivations of all measures ($\Phi_1^{pf,c}$, $\Phi_1^{pf,d}$, $\Phi_1^{nf,c}$, $\Phi_1^{nf,d}$, $\Phi_1^{p,d}$) for Treatment 1, 2 and 3 are given in Appendix 3.B and a summary is presented in Table 3.1.[13] With a slight abuse of notation, let $subg(\Game, y)$ refer to the subgame following a bid y by Player 1.

Game	Specification	Measure
$\Game_1, \Game_2$	$\Phi^{pf,c}$	$\frac{y}{100}$
$\Game_3$	$\Phi^{pf,c}$	0
$\Game_1, \Game_2$	$\Phi^{pf,d}$	$\frac{y}{100}\left(\pi_1^{high} - \pi_1^{low}\right)$
$\Game_3$	$\Phi^{pf,d}$	0
$\Game_1, \Game_2, \Game_3$	$\Phi^{nf,c}$	$-\frac{100-y}{100}$
$\Game_1, \Game_2, \Game_3$	$\Phi^{nf,d}$	$-\frac{100-y}{100}\left(\pi_1^{high} - \pi_1^{low}\right)$
$\Game_1^p, \Game_2^p$	$\Phi^{p,d}$	$\frac{y}{100}\left(\pi_2^{high} - \pi_2^{low}\right)$
$\Game_1^{np}, \Game_2^{np}, \Game_3$	$\Phi^{p,d}$	0

Table 3.1: Positive Freedom, Power and Negative Freedom measures

As an example, lets analyze the decision problem in Treatment 1 of a Player 1 with preference for positive freedom under the $\Phi^{pf,c}$ specification. Intuitively, positive freedom under such specification is equal to the probability of having the decision right. This is because if Player 1 has the decision right, then $g(A, u_1^A)\log_2 \frac{\theta(A|u_1^A)}{\theta(A)} = g(B, u_1^B)\log_2 \frac{\theta(B|u_1^B)}{\theta(B)} = \log_2 \frac{1}{1/2} = 1$. If Player 1 does not have the decision right, then $g(o,u)\log_2 \frac{\theta(o|u)}{\theta(u)} = 0 \ \forall o,u$. Thus, a Player 1 with preference for positive freedom chooses his bid so as to solve:

$$\max_y V_1 = \max_y \alpha_1 \frac{y}{100} + \delta_1 EU_1(subg(\Game_1, \theta_{1|y})). \tag{3.10}$$

The optimal bid condition corresponding to (3.7) then becomes:

[13]Since games $\Game^p$ differ from games $\Game^{np}$ uniquely because of a positive payoff difference for Player 2, $\Delta\pi_2 = \pi_2^{high,L} - \pi_2^{low,L}$, we consider only the specification $\Phi^{p,d}$ for power.

$$\alpha_1 + u(w_1 - y^{*F} + \pi_1^{high}) = \frac{1}{2}u(w_1 + \pi_1^{high}) + \frac{1}{2}u(w_1 + \pi_1^{low}). \quad (3.11)$$

This means that the utility from having the decision right is increased by a constant α_1. In Treatment 3, instead, in which by design Card C is the outcome of the game if Player 1 has the decision right, it would be $g(C,u_1^C)\log_2 \frac{\theta(C|u_1^C)}{\theta(C)} = g(C,u_1^C)\log_2 \frac{1}{1} = 0$ and thus positive freedom would be zero.

3.5 Empirical Strategy

Equation (3.11) gives an especially simple way of measuring Player 1's preference for positive freedom in a game of Treatment 1. The parameter α_1 can be inferred from a regression of the difference in estimated utilities from Box L and Box R, $\Delta EU_1 = u(w_1 - y + \pi_1^{high}) - \frac{1}{2}u(w_1 + \pi_1^{high}) - \frac{1}{2}u(w_1 + \pi_1^{low})$, on a constant.[14] A similar approach can be also applied to measuring Player 1's preference for negative freedom and preference for power. For simplicity, since we consider exclusively Player 1's behavior, we introduce a subscript denoting each subject in the sample who plays as Player 1. For each subject k playing as Player 1, we consider the following estimation equation:

$$\Delta EU_{k,t} = \alpha_k V_{k,t}^f + \beta_k V_{k,t}^{ni} + \gamma_k V_{k,t}^p + \epsilon_{k,t} \quad (3.12)$$

where k stands for the subject, t for the round of play, V^f, V^{ni}, V^p for the positive freedom, negative freedom and power variable, respectively, and where we normalized $\delta_k = 1$ of equation (3.6), in order to achieve identification of α_k, β_k, and γ_k. Table 3.1 gives an overview of the measures and their empirical implementation.

[14]The estimated utility from Box L in ΔEU_1 is computed setting $r = y$.

Measure	Variable	Value
$\Phi^{pf,c}$	$V^{f,c}$	$-1_{[\Game_1,\Game_2]}$
$\Phi^{pf,d}$	$V^{f,d}$	$-1_{[\Game_1,\Game_2]}\Delta\pi_1$
$\Phi^{nf,c}$	$V^{ni,c}$	-1
$\Phi^{nf,d}$	$V^{ni,d}$	$-\Delta\pi_1$
$\Phi^{p,d}$	$V^{p,d}$	$-1_{[\Game_1^p,\Game_2^p]}\Delta\pi_2$

$1_{[\Game,\Game']}$ = 1 if game is $\Game$ or $\Game'$, = 0 otherwise.

Table 3.1: Empirical Implementation of Measures

As discussed above, in Treatment 1 the positive freedom measure $\Phi^{pf,c}$ corresponds to a constant. The same holds in Treatment 2. In Treatment 3, instead, positive freedom is excluded by design.[15] Therefore, estimating preference for positive freedom under the specification $\Phi^{pf,c}$ corresponds to running a regression on a dummy variable which equals 1 in Treatments 1 and 2 and equals 0 in Treatment 3, denoted $1_{[\Game_1,\Game_2]}$. Under the specification $\Phi^{pf,d}$, the dummy is interacted with the payoff distance $\Delta\pi_1 = \pi_1^{high} - \pi_1^{low}$.

Differently from positive freedom, negative freedom is present in all treatments.[16] Therefore, estimating preference for negative freedom under the specification $\Phi^{nf,c}$ corresponds to running a regression on a constant. The specification $\Phi^{nf,d}$ takes into account the difference in payoffs $\Delta\pi_1$.

Power is present only in games $\Game^p$ in Treatment 1 and 2, denoted $\Game_1^p$ and $\Game_2^p$.[17] We focus on the specification $\Phi^{p,d}$ since games $\Game^p$ differ from $\Game^{np}$ uniquely because of a positive payoff distance for Player 2, $\Delta\pi_2 = \pi_2^{high,L} - \pi_2^{low,L}$. Thus, estimating preference for power under the specification $\Phi^{p,d}$ corresponds to running a regression on $\Delta\pi_2$ times a dummy variable which

[15]In Treatment 3 Box L contains only 1 card, so even if his bid is successful Player 1 does not select a card and thus has no positive freedom.

[16]In Treatment 3 Player 2 affects the outcomes of Player 1 if the bid is not successful, therefore a successful bid yields negative freedom for Player 1.

[17]In Treatment 3 Box L contains only 1 card, so even if his bid is successful Player 1 does not select a card and thus has no power on Player 2. In games $\Game^{np}$ in Treatment 1 and 2, Player 2's payoffs in box L are equal, $\pi_2^{high,L} = \pi_2^{low,L}$, and thus Player 1 has no power on Player 2. In games $\Game^p$ in Treatment 1 and 2, instead Player 2's payoffs in box L differ, $\pi_2^{high,L} > \pi_2^{low,L}$, and thus Player 1 has power on Player 2.

equals 1 in games $\Game^{p}$ in Treatment 1 and 2, and equals zero otherwise.

3.6 Results

3.6.1 Allocation and Exercise of Decision Rights

Before turning to the results obtained via the empirical strategy described in the previous section, we briefly present descriptive results on how Players 1 bid for the decision right, and on how the players with the decision right (Players 1 or 2) make the card selection.

First, we inspect whether bids differ across treatments. Table 3.A.4 in Appendix 3.A reports the median bids submitted by Players 1 for each treatment and each game. For most games, bids in Treatment 3, in which the decision right gives Player 1 only negative freedom (in all rounds 1-20), are significantly higher than in Treatment 1, in which the decision right gives positive freedom and negative freedom (rounds 1-10, i.e. games $\Game^{np}$), or power and positive freedom and negative freedom (rounds 11-20, i.e. games $\Game^{p}$). This evidence suggests the key role of negative freedom, which we further investigate later in this section.

Second, we inspect whether bids in games that do not involve power ($\Game^{np}$) differ from those in games that involve power ($\Game^{p}$). We make pair-wise comparisons across rounds in which Player 1 faces the same stake size and the same expected payoff. We compare round 5 versus round 12, and round 10 versus round 20.[18] We find no significant differences between $\Game^{np}$ and $\Game^{p}$ in either pair of comparisons.[19] This evidence suggests that considerations regarding power may be less relevant than considerations regarding positive freedom and

[18]Player 1 faces a stake size of 25 and an expected payoff of 50 in rounds 5 and 12, and a stake size of 50 and an expected payoff of 50 in rounds 10 and 20.

[19]We perform a Wilcoxon signed rank sum test on observations paired at the participant level. For round 5 versus round 12, we have $z = 0.658$ ($p = 0.5102$) in Treatment 1 and $z = 1.339$ ($p = 0.1806$) in Treatment 2. For round 10 versus round 20, we have $z = -1.143$ ($p = 0.2531$) in Treatment 1 and $z = -1.356$ ($p = 0.1750$) in Treatment 2. In Treatment 3, as highlighted in Section 3.3.2, all rounds involve negative freedom, but do not involve either positive freedom or power. Therefore, distinguishing $\Game^{np}$ and $\Game^{p}$ in Treatment 3 is not meaningful.

negative freedom. We investigate further this aspect later in this section.

Once the decision right is allocated, the player with the decision right makes the card selection. Recall from Section 3.3 that if Player 1 has the decision right, he chooses a card from Box L, knowing which card yields him the highest payoff.[20] Similarly, if Player 2 has the decision right, he chooses a card from Box R, knowing which card yields him the highest payoff.[21] Do agents with the decision right use it in their favor, selecting the card that yields them the highest payoff? Pooling all data together, we find that in more than 98 percent of the observations, the decision right is exercised by selecting the card that yields the decision-maker his highest payoff.[22]

3.6.2 Certainty Equivalents

To verify whether subjects playing as Player 1 behave according to expected utility maximization, we compare the certainty equivalent in each lottery-choice in Part 2, $CE_{lottery}(L)$ with $L = (\frac{1}{2}, \pi_1^{high}; \frac{1}{2}, \pi_1^{low})$, to the certainty equivalent implied in the bidding choice in the corresponding situation in Part 1, i.e. involving the same π_1^{high} and π_1^{low}:

$$\pi_1^{high} - y = CE\left(\frac{1}{2}, \pi_1^{high}; \frac{1}{2}, \pi_1^{low}\right). \tag{3.13}$$

Denote ΔCE as:

$$\Delta CE = \pi_1^{high} - y - CE_{lottery}\left(\frac{1}{2}, \pi_1^{high}; \frac{1}{2}, \pi_1^{low}\right). \tag{3.14}$$

Overbidding occurs if ΔCE is negative: the subject behaves more risk averse in the bidding-choice than in the lottery-choice. Underbidding occurs if ΔCE is positive: the subject behaves more risk averse in the lottery-choice

[20] In Treatment 1 and 2, the highest payoff for Player 1 is generated by Card B in case 1 and 2, by Card A in case 3 and 4, as shown in Figure 3.1. In Treatment 3, Box L contains only Card C, making the choice of Player 1 trivial.

[21] In Treatment 1 and 2, the highest payoff for Player 2 is generated by Card B in case 1 and 3, by Card A in case 2 and 4, as shown in Figure 3.1.

[22] See Table 3.A.1 in Appendix 3.A for details.

than in the bidding-choice.[23]

If the only error in ΔCE is due to the imprecise measurement of the certainty equivalent (which are measured on intervals of 5 payoff units), we should expect ΔCE to be distributed uniformly with mean 0 and standard deviation $(25/12)^{1/2} \approx 1.44$. We find instead that the mean is too low (-14.11) and the standard deviation is too high (25.41).[24] Both deviations are significant at the 1% level. We can therefore reject the hypothesis of expected utility maximizing behavior.

3.6.3 Risk Preferences

Among the variables defined in Section 3.5, ΔEU requires knowledge of an individual's utility function over payoffs, $u(\pi)$. We approximate $u(\pi)$ by a CRRA utility function $u(\pi) = \frac{\pi^{1-\rho}}{1-\rho}$. For each subject, we estimate his risk aversion coefficient via maximum likelihood estimation from his responses in the lottery-choice questionnaire in Part 2, using a random utility model with:

$$u_k\left(\frac{1}{2}, \pi^{high,q}; \frac{1}{2}, \pi^{low,q}\right) = \frac{(\pi^{high,q})^{1-\rho_k}}{2(1-\rho_k)} + \frac{(\pi^{low,q})^{1-\rho_k}}{2(1-\rho_k)} + \epsilon_{q,k} \quad (3.15)$$

where $\epsilon_{q,k} \sim_{iid} N(0, \sigma_k^2)$, k indicates the subject and q the lottery in question.

We are able to estimate the risk aversion coefficients for 235 out of 244 subjects: 9 subjects exhibit such extreme risk preferences in the lottery-choice questionnaire that we are unable to fit a CRRA model. In general, risk preferences range from slightly risk loving to strongly risk averse.[25] Based on the risk aversion coefficients, we calculate the expected utility values of the payoffs

[23] We are aware of a caveat. When subjects answer the lottery-choice questionnaire in Part 2, they already know their endowment in Part 1 (w_1), but they do not know their earnings in Part 1 yet. Therefore, if there are significant income effects on risk aversion, we cannot expect (3.9) to be identical to (3.13).

[24] The empirical distribution of ΔCE over 1132 observations has mean -14.11, median -12.50, 25% percentile -27.5, 75% percentile 2.5, standard deviation 25.41.

[25] The empirical distribution of $\hat{\rho}$ over 235 observations has mean 0.59, median 0.37, 25% percentile .28, 75% percentile .46, standard deviation 2.58.

from Box L and Box R.

3.6.4 Preference for Positive Freedom, Negative Freedom and Power

As a preliminary analysis, we perform a linear regression on the whole dataset for different combinations and specifications of V^f, V^{ni} and V^p. We assume that, for each individual k, $\alpha_k = \alpha$, $\beta_k = \beta$ and $\gamma_k = \gamma$, i.e. all individuals have the same preference for positive freedom, negative freedom and power. Thus, equation (3.12) simplifies to:

$$\Delta EU_{k,t} = \alpha V^f_{k,t} + \beta V^{ni}_{k,t} + \gamma V^p_{k,t} + \epsilon_{k,t} \tag{3.16}$$

Results are reported in Table 6. Among the alternatives (1)-(4), the best fit is provided by (1), i.e. the model where both the positive freedom variable and the negative freedom variable are specified as a constant. We find no conclusive evidence of preference for positive freedom. Instead, we find that the effect of preference for negative freedom is both economically and statistically significant. Negative freedom parameters in regressions (1) and (2) for example mean that the control of Player 2 of Box R comes with a utility loss of 6.711 and 6.082, respectively, which corresponds to an endowment loss of 35 payoff units in rounds 10 and 20, for example. Due to the nature of the CRRA function, this amount will be higher for individuals with higher risk aversion and lower for individuals with lower risk aversion. Finally, preference for power is neither statistically nor economically significant.

A limitation of the population regression is that it tries to estimate a single parameter for all individuals, even though after the risk aversion regression their ΔEU will differ in scale and standard deviation. It therefore makes sense to estimate the preferences for each individual separately with the more general model:

$$\Delta EU_{k,t} = \alpha_k V^f_{k,t} + \beta_k V^{ni}_{k,t} + \gamma_k V^p_{k,t} + \epsilon_{k,t}, \tag{3.17}$$

which we interpret as a random coefficient model with $\alpha_k = \alpha + \epsilon_{\alpha,k}$, $\beta_k =$

	model (16)				model (17)-(22)	
	(1)	(2)	(3)	(4)	(I)	(II)
$V^{f,c}$	-1.748		0.565		0.1895	
	(1.935)		(1.427)		(0.7423)	
$V^{f,d}$		-0.029		-0.059		-0.0077
		(0.053)		(0.065)		(0.0192)
$V^{ni,c}$	6.711***	6.082***				5.6507***
	(1.520)	(1.163)				(0.4032)
$V^{ni,d}$			0.171***	0.218***	0.2081***	
			(0.039)	(0.049)	(0.0291)	
$V^{p,d}$	0.004	-0.0004	-0.007	0.008		
	(0.005)	(0.005)	(0.005)	(0.005)		
obs	2360	2360	2360	2360	2360	2360
subjects	118	118	118	118	117	117
F-test	12.7	12.71	11.11	11.65		
R-squared	0.1556	0.1539	0.1405	0.1425		
J test $\chi^2(1)$					0.0319	0.0591

Table 3.1: Freedom and Power Estimation Results[26]

$\beta + \epsilon_{\beta,k}$, $\gamma_k = \gamma + \epsilon_{\gamma,k}$. Since power is not a statistically significant explanatory variable in the estimation of (3.16), we include only positive freedom and negative freedom as explanatory variables in the estimation of (3.17). We estimate (3.17) using the following moment conditions:

[26]Column (1)-(4) report estimation results of model (3.16). Standard errors are clustered at the individual level and are shown in parenthesis: * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$. Column (I)-(II) report estimation results of model (3.17)-(3.22). We used simulated annealing with 1000 search points. The estimation of parameters and weighting matrix was iterated 5 times to achieve better finite-sample properties. To avoid misspecification, we excluded one individual who perfectly maximized expected payoffs. This does not impact the statistical or economical significance of the results. Standard errors are shown in parenthesis: * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$. J test $\chi^2(1)$ is the Hansen test of over-identifying restrictions. Since $\chi^2(1)_{.05} = 3.841$, we do not reject the null hypothesis of a correctly specified model in either (I) or (II).

$$E[\epsilon_{k,t} V_{k,t}^{f}] = 0 \tag{3.18}$$

$$E[\epsilon_{k,t} V_{k,t}^{ni}] = 0 \tag{3.19}$$

$$E[\epsilon_{\alpha,k} - \alpha] = 0 \tag{3.20}$$

$$E[\epsilon_{\beta,k} - \beta] = 0 \tag{3.21}$$

$$E[\epsilon_{\beta,k} 1_{k,[\ni 3]}] = 0 \tag{3.22}$$

Conditions (3.18)-(3.19) state that errors $\epsilon_{k,t}$ are independent of the regressors, the positive freedom variable $V_{k,t}^{f}$ and the negative freedom variable $V_{k,t}^{ni}$, respectively. Conditions (3.20)-(3.21) identify the population parameters α and β. Condition (3.22) states that the mean of individual negative freedom parameters in Treatment 3 is equal to the one in the other treatments. Since treatment assignment was random, individuals' preference for positive freedom or negative freedom should be independent across treatments. This allows identification of the positive freedom parameters α_k for individuals in Treatment 1 and 2. Without condition (3.22) we cannot distinguish whether their bidding behavior was motivated by preference for positive freedom or preference for negative freedom. However, assuming that the mean preference parameters are identical across treatments, we can identify the mean α via the difference in behavior between Treatment 3 and the other treatments.

The random coefficient model (3.17)-(3.22) confirms the previous results. Preference for negative freedom is the driving force for preference for decision rights. The median preference for negative freedom parameters in (I) and (II) were 0.04 and 1.70. For median risk aversion (.37) losing the decision right in rounds 10 and 20 was therefore equivalent to an endowment loss of 10.37 and 12.38 points. The results therefore remain statistically and economically highly significant.

Additionally, in Appendix 3.C we used the obtained estimates on individual-level α_k and β_k to examine whether preference for positive freedom and negative freedom can be explained by individuals' locus of control, which is measured in Part 3 of the experiment. We find that one of the three separate scales used to measure locus of control, the P-scale, which measures the degree to which individuals believe that other persons control their lives, explains preference for positive freedom and negative freedom in model (I), but not in (II). Thus evidence suggests that preference for positive freedom and negative freedom cannot fully be explained by locus of control.[27]

3.7 Conclusions

In this paper we present theoretical foundations for preference for decision rights, driven by preference for positive freedom, power and negative freedom. We conduct a laboratory experiment in which the role of each preference can be distinguished.

Our results confirm the existence of an intrinsic value of decision rights and extend these from delegation settings to a willingness to pay/auction setting. Evidence from our experiment highlights two main results. First, we find no evidence of preference for power. Thus, preference for power as casually observed in politics or other institutional settings may simply be instrumental to other components of well-being, such as status recognition. This result, however, may partly depend on the experimental setting, in which each player learns his own preferences towards the final choice but never learns the preferences of the other player. Therefore, a Player 1 with preference for power may not find the exercise of power over Player 2 particularly satisfying because he does not know Player 2's preferences, and thus does not know in which way he can influence him. We consider experimental settings that relax such information constraints an interesting direction for further research.

Second, we find stronger evidence of preference for negative freedom than for positive freedom. This result suggests that individuals value the decision right not because of the actual decision-making process, but rather because they have preference against others intervening in their outcomes. This result

[27]For details on the scales used to measure locus of control, see Appendix 3.C.

leads to a fundamental change of perspective on preference for decision rights. In contrast to the interpretation presented by Fehr et al. (2013) and Bartling et al. (2013), individuals like to have decision rights in virtue of the absence of decision rights of other individuals. An individual's evaluation of risks then depends on whether the risks are generated by an objective process or by the behavior of other individuals.

Appendix

3.A Additional Tables and Figures

Treatment	Player 1		Player 2	
	has decision right	chooses preferred card	has decision right	chooses preferred card
1	0.41	1	0.59	0.98
2	0.4	0.99	0.6	0.99
3	0.55	1	0.45	0.94
all	0.44	1	0.56	0.98

Table 3.A.1: Decision rights and choice behavior conditional on having the decision right. Fraction of observations.

Statement	Correct Answer	Correct %
If participant 1 has the decision right, box R is opened.	Not True	96.3
It is in the best interest of participant 1, to bid equal to his/her true valuation for the decision right.	True	98.0
The participants receive payments for each round of part 1.	Not True	96.7
If the bid of participant 1 is higher than the randomly determined number, participant 1 has to pay a fee equal to the amount of the bid.	Not True	58.2

Table 3.A.2: Comprehension questions

Screen 1		Screen 2		Screen 3	
Option A	Option B	Option A	Option B	Option A	Option B
30	$\frac{1}{2}$, 85; $\frac{1}{2}$, 15	30	$\frac{1}{2}$, 75; $\frac{1}{2}$, 25	30	$\frac{1}{2}$, 65; $\frac{1}{2}$, 35
35	$\frac{1}{2}$, 85; $\frac{1}{2}$, 15	35	$\frac{1}{2}$, 75; $\frac{1}{2}$, 25	35	$\frac{1}{2}$, 65; $\frac{1}{2}$, 35
40	$\frac{1}{2}$, 85; $\frac{1}{2}$, 15	40	$\frac{1}{2}$, 75; $\frac{1}{2}$, 25	40	$\frac{1}{2}$, 65; $\frac{1}{2}$, 35
45	$\frac{1}{2}$, 85; $\frac{1}{2}$, 15	45	$\frac{1}{2}$, 75; $\frac{1}{2}$, 25	45	$\frac{1}{2}$, 65; $\frac{1}{2}$, 35
50	$\frac{1}{2}$, 85; $\frac{1}{2}$, 15	50	$\frac{1}{2}$, 75; $\frac{1}{2}$, 25	50	$\frac{1}{2}$, 65; $\frac{1}{2}$, 35
55	$\frac{1}{2}$, 85; $\frac{1}{2}$, 15	55	$\frac{1}{2}$, 75; $\frac{1}{2}$, 25	55	$\frac{1}{2}$, 65; $\frac{1}{2}$, 35
60	$\frac{1}{2}$, 85; $\frac{1}{2}$, 15	60	$\frac{1}{2}$, 75; $\frac{1}{2}$, 25	60	$\frac{1}{2}$, 65; $\frac{1}{2}$, 35
65	$\frac{1}{2}$, 85; $\frac{1}{2}$, 15	65	$\frac{1}{2}$, 75; $\frac{1}{2}$, 25	65	$\frac{1}{2}$, 65; $\frac{1}{2}$, 35
70	$\frac{1}{2}$, 85; $\frac{1}{2}$, 15	70	$\frac{1}{2}$, 75; $\frac{1}{2}$, 25	70	$\frac{1}{2}$, 65; $\frac{1}{2}$, 35
75	$\frac{1}{2}$, 85; $\frac{1}{2}$, 15	75	$\frac{1}{2}$, 75; $\frac{1}{2}$, 25	75	$\frac{1}{2}$, 65; $\frac{1}{2}$, 35
80	$\frac{1}{2}$, 85; $\frac{1}{2}$, 15	80	$\frac{1}{2}$, 75; $\frac{1}{2}$, 25	80	$\frac{1}{2}$, 65; $\frac{1}{2}$, 35

Table 3.A.3: Paired lottery-choice questions. $\frac{1}{2},\pi^{B,high};\frac{1}{2},\pi^{B,low}$ denotes the lottery yielding a high prize $\pi^{B,high}$ with probability 0.5 and a low prize $\pi^{B,low}$ with probability 0.5.

Round	Treatment						
	1	2	3	all	1 vs 2	2 vs 3	1 vs 3
1	50	52	69	60			-2.492 (0.0127)
2	48	40	45	44		-2.357 (0.0184)	
3	28	30	30	30			-1.709 (0.0874)
4	45	40	60	50		-3.073 (0.0021)	-2.884 (0.0039)
5	40	40	45	40			-1.831 (0.0671)
6	30	30	30	30			-1.781 (0.0749)
7	50	40	70	50		-2.968 (0.0030)	-3.000 (0.0027)
8	30	36	45	35			-2.198 (0.0280)
9	20	30	30	30	-2.489 (0.0128)		-2.893 (0.003)
10	66	68	80	70		-1.945 (0.0518)	
11	40	40	45	40			-2.043 (0.0411)
12	35	36	45	40		-1.703 (0.0886)	-1.977 (0.0481)
13	35	40	50	40		-2.296 (0.0217)	-2.430 (0.0151)
14	33	35	43	40		-1.719 (0.0856)	-1.909 (0.0562)
15	30	30	45	40		-1.706 (0.0880)	-1.941 (0.0523)
16	50	40	65	50		-2.586 (0.0097)	-2.916 (0.0035)
17	25	30	30	30			-2.411 (0.0159)
18	40	47	50	48		-1.860 (0.0628)	-2.614 (0.0089)
19	30	31	35	33			
20	80	70	70	72			
all	40	40	50	40			

Table 3.A.4: Median bids. Results of a Mann-Whitney-Wilcoxon rank-sum test (p-values in parenthesis) are reported only for statistically significant cases.

3.B Derivations of Valuation Functions

In this appendix we present the derivation of the measures of positive freedom Φ_1^{pf}, negative freedom Φ_1^{nf} and power Φ_1^{p} under each specification of function $g(o,u)$ ($g = 1$ and $g = |\pi^{high} - \pi^{low}|$) and for each treatment (1, 2 and 3).

The positive freedom measure Φ_1^{pf} under Treatment 1 for a general function g is:

$$
\begin{aligned}
&\Phi_1^{pf}(subg(\eth_1, y), \theta_{1|y}) = \\
&\sum_{r \leq y} \sum_{u \in U_1} \theta_{1|y}(u) \sum_{c \in \{A,B\}} g(o(r,1,c),u)\theta_{1|y}(o(r,1,c)|u) \log_2 \frac{\theta_{1|y}(o(r,1,c)|u)}{\theta_{1|y}(o(r,1,c))} \\
&+ \sum_{r > y} \sum_{u \in U_1} \theta_{1|y}(u) \sum_{c \in \{A,B\}} g(o(r,2,c),u)\theta_{1|y}(o(r,2,c)|u) \log_2 \frac{\theta_{1|y}(o(r,2,c)|u)}{\theta_{1|y}(o(r,2,c))}
\end{aligned}
\tag{3.23}
$$

where we use the fact that $\sum_{o \in O_i} f(o) = \sum_{r=1}^{100} \sum_{i \in \{1,2\}} \sum_{c \in \{A,B\}} f(o(r,i,c))$ for any function $f(o)$ and that $y \geq r$ implies $\theta_{1|y}(o(r,2,c)) = 0$. Moreover, $\theta_{1|y}(o(r,2,c)|u) = \theta_{1|y}(o(r,2,c))$ since if Player 2 has the decision right, the outcome is independent of Player 1's preferences. Since $\log_2 1 = 0$, the measure simplifies to:

$$
\begin{aligned}
&\Phi_1^{pf}(subg(\eth_1, y), \theta_{1|y}) = \\
&\sum_{r \leq y} \sum_{u \in U_1} \theta_{1|y}(u) \sum_{c \in \{A,B\}} g(o(r,1,c),u)\theta_{1|y}(o(r,1,c)|u) \log_2 \frac{\theta_{1|y}(o(r,1,c)|u)}{\theta_{1|y}(o(r,1,c))}
\end{aligned}
\tag{3.24}
$$

The remaining probabilities are as follows:

$$
\begin{aligned}
\forall u \in U_1: \quad & \theta_{1|y}(u) = 1/2 \\
\forall u \in U_1 : \forall r \leq y: \quad & \theta_{1|y}(o(r,1,A)|u) = \begin{cases} \frac{1}{100}, & u = u_1^A \\ 0, & \text{else} \end{cases} \\
\forall u \in U_1 : \forall r \leq y: \quad & \theta_{1|y}(o(r,1,B)|u) = \begin{cases} \frac{1}{100}, & u = u_1^B \\ 0, & \text{else} \end{cases} \\
\forall r \leq y: \quad & \theta_{1|y}(o(r,1,A)) = 1/200 \\
\forall r \leq y: \quad & \theta_{1|y}(o(r,1,B)) = 1/200
\end{aligned} \tag{3.25}
$$

The positive freedom measure therefore simplifies to:

$$
\Phi_1^{pf}(subg(\eth_1, y), \theta_{1|y}) = \frac{1}{200} \sum_{r \leq y} (g(o(r,1,A), u_1^A) + g(o(r,1,B), u_1^B)) \tag{3.26}
$$

Since Treatment 2 differs from Treatment 1 only in that Player 2's endowment w_2 equals 0 instead of 100, it follows that

$$
\Phi_1^{pf}(subg(\eth_1, y), \theta_{1|y}) = \Phi_1^{pf}(subg(\eth_2, y), \theta_{1|y}).
$$

For Treatment 3, instead:

$$
\begin{aligned}
&\Phi_1^{pf}(subg(\eth_3, y), \theta_{1|y}) = \\
&\sum_{r \leq y} \sum_{u \in U_1} \theta_{1|y}(u) \sum_{c \in \{C\}} g(o(r,1,c), u) \theta_{1|y}(o(r,1,c)|u) \log_2 \frac{\theta_{1|y}(o(r,1,c)|u)}{\theta_{1|y}(o(r,1,c))} \\
&+ \sum_{r > y} \sum_{u \in U_1} \theta_{1|y}(u) \sum_{c \in \{A,B\}} g(o(r,2,c), u) \theta_{1|y}(o(r,2,c)|u) \log_2 \frac{\theta_{1|y}(o(r,2,c)|u)}{\theta_{1|y}(o(r,2,c))}
\end{aligned} \tag{3.27}
$$

As in (3.23), $\theta_{1|y}(o(r,2,c)|u) = \theta_{1|y}(o(r,2,c))$: if Player 2 has the decision right, the outcome is independent of Player 1's preferences. In addition, $\theta_{1|y}(o(r,1,C)|u) = \theta_{1|y}(o(r,1,C))$: if Player 1 has the decision right, then only Card C is available, and thus the outcome is independent of Player 1's preferences. Since $\ln_2 1 = 0$, the measure equals $\Phi_1^{pf}(subg(\eth_3,y),\theta_{1|y}) = 0$. This concludes the derivations for positive freedom Φ^{pf}.

The negative freedom measure Φ_1^{nf} for a general function g is:

$$
\begin{aligned}
&\Phi_1^{nf}(subg(\eth,y),\theta_{1|y}) = \\
&-\sum_{r\le y}\sum_{v\in U_2}\theta_{1|y}(v)\sum_{u\in U_1}\theta_{1|y}(u|v) \\
&\cdot\sum_{c\in\{A,B\}} g(o(r,1,c),u)\theta_{1|y}(o(r,1,c)|v)\log_2\frac{\theta_{1|y}(o(r,1,c)|v)}{\theta_{1|y}(o(r,1,c))} \\
&-\sum_{r>y}\sum_{v\in U_2}\theta_{1|y}(v)\sum_{u\in U_1}\theta_{1|y}(u|v) \\
&\cdot\sum_{c\in\{A,B\}} g(o(r,2,c),u)\theta_{1|y}(o(r,2,c)|v)\log_2\frac{\theta_{1|y}(o(r,2,c)|v)}{\theta_{1|y}(o(r,2,c))}
\end{aligned}
\tag{3.28}
$$

In all treatments, $\theta_{1|y}(o(r,1,c)|v) = \theta_{1|y}(o(r,1,c))$: if Player 1 has the decision right, the outcome is independent of Player 2's preferences. Thus, Φ_1^{nf} can be written, for all treatments, as:

$$
\begin{aligned}
&\Phi_1^{nf}(subg(\eth,y),\theta_{1|y}) = \\
&-\sum_{r>y}\sum_{v\in U_2}\theta_{1|y}(v)\sum_{u\in U_1}\theta_{1|y}(u|v) \\
&\cdot\sum_{c\in\{A,B\}} g(o(r,2,c),u)\theta_{1|y}(o(r,2,c)|v)\log_2\frac{\theta_{1|y}(o(r,2,c)|v)}{\theta_{1|y}(o(r,2,c))}
\end{aligned}
\tag{3.29}
$$

Since the negative freedom measure captures 'interferences', it captures what happens if Player 2 has the decision right, and not what happens if Player

1 has the decision right. The remaining probabilities are as follows:

$$
\begin{aligned}
\forall v \in U_2: \quad & \theta_{1|y}(v) = 1/2 \\
\forall v \in U_2 : \forall u \in U_1: \quad & \theta_{1|y}(u|v) = 1/2 \\
\forall v \in U_2 : \forall r \leq y: \quad & \theta_{1|y}(o(r,2,A)|v) = \begin{cases} \frac{1}{100}, & v = u_2^A \\ 0, & \text{else} \end{cases} \\
\forall v \in U_2 : \forall r \leq y: \quad & \theta_{1|y}(o(r,2,B)|v) = \begin{cases} \frac{1}{100}, & v = u_2^B \\ 0, & \text{else} \end{cases} \\
\forall r \leq y: \quad & \theta_{1|y}(o(r,2,A)) = 1/50 \\
\forall r \leq y: \quad & \theta_{1|y}(o(r,2,B)) = 1/50
\end{aligned} \tag{3.30}
$$

The negative freedom measure therefore simplifies to:

$$
\Phi_1^{nf}(subg(\eth,y),\theta_{1|y}) = -\frac{1}{400}\sum_{r>y}\sum_{u\in U_1}\left(g(o(r,2,A),u) + g(o(r,2,B),u)\right) \tag{3.31}
$$

It is then straightforward to insert the values for $g(o,u)$ in the above equations. Summing up, we have for positive freedom:

$$
\begin{aligned}
\Phi^{pf,c}(subg(\eth_1,y),\theta_{1|y}) &= \Phi^{pf,c}(subg(\eth_2,y),\theta_{1|y}) = \frac{y}{100} \\
\Phi^{pf,d}(subg(\eth_1,y),\theta_{1|y}) &= \Phi^{pf,d}(subg(\eth_2,y),\theta_{1|y}) = \frac{y}{100}\left(\pi_1^{high} - \pi_1^{low}\right) \\
\Phi^{pf,c}(subg(\eth_3,y),\theta_{1|y}) &= \Phi^{pf,d}(subg(\eth_3,y),\theta_{1|y}) = 0
\end{aligned} \tag{3.32}
$$

For negative freedom we have for all $\eth \in \{\eth_1, \eth_2, \eth_3\}$:

$$
\begin{aligned}
\Phi^{nf,c}(subg(\eth,y),\theta_{1|y}) &= -\frac{100-y}{100} \\
\Phi^{nf,d}(subg(\eth,y),\theta_{1|y}) &= -\frac{100-y}{100}\left(\pi_1^{high} - \pi_1^{low}\right)
\end{aligned} \tag{3.33}
$$

Power is largely analogous to $\Phi^{pf,d}$ and therefore gives:

$$\Phi^{p,d}(subg(\circlearrowleft_1^p, y), \theta_{1|y}) = \Phi^{p,d}(subg(\circlearrowleft_2^p, y), \theta_{1|y}) = \frac{y}{100}\left(\pi_2^{high} - \pi_2^{low}\right) \tag{3.34}$$

$$\Phi^{p,d}(subg(\circlearrowleft_1^{np}, y), \theta_{1|y}) = \Phi^{p,d}(subg(\circlearrowleft_2^{np}, y), \theta_{1|y}) = 0 \tag{3.35}$$

$$\Phi^{p,d}(subg(\circlearrowleft_3, y), \theta_{1|y}) = 0 \tag{3.36}$$

3.C Locus of Control

We implement the Levenson Multidimensional Locus of Control Test as designed by Levenson (1981) and translated from English to German by Krampen (1981). In personality psychology, locus of control refers to the extent to which individuals believe that they can control events that affect them. A person's 'locus' is either internal (i.e., the person believes that events in his life derive primarily from his own actions) or external (i.e., the person believes that events in his life derive primarily from external factors, such as chance and other people's actions, which he cannot influence). There are three separate scales used to measure one's locus of control: Internal Scale (I scale), Powerful Others External Scale (P scale), and Chance External Scale (C scale). The I-scale measures the degree to which individuals believe that they control their lives. The P-scale measures the degree to which individuals believe that other persons control their lives. Finally, the C-scale measures the degree to which individuals believe that chance plays a role in their lives.

The questionnaire is reported in Table 3.C.2. There are eight items on each of the three scales, which are presented to the subject as one unified attitude scale of 24 items. The specific content areas mentioned in the items are counterbalanced so as to appear equally often for all three dimensions. To score each scale, add up the points of the answers for the items appropriate for that scale (from 1 for strongly disagree to 6 for strongly agree). The possible range on each scale is from 0 to 48. Each subject receives three scores indicative of his or her locus of control on the three dimensions of I, P, and C.

Table 3.C.1 reports summary statistics of the three scales across all participants. Since the empirical distribution does not differ across treatments, Table 3.C.1 pools all treatments together.

scale	No.	mean	std	min	p25%	p50%	p75%	max
I-scale	244	36	4	16	33	36	38	46
P-scale	244	24	5	10	21	24	27	38
C-scale	244	25	5	11	22	25	28	39

Table 3.C.1: Locus of Control: summary statistics of each scale.

There may be several reasons why attitudes towards locus of control may be thought to be intrinsically related to attitudes towards positive freedom and negative freedom. Subjects who believe that other individuals control their lives may have a greater preference for positive freedom and negative freedom. Subjects who believe that chance controls their lives may be more risk averse. Also, after a series of low bids resulting in Player 2 determining the game outcome, Player 1 may be more likely to believe that other individuals control his life.

As shown in Table 3.C.3, we do not find strong evidence that attitudes towards locus of control are correlated with preference for positive freedom or negative freedom. In one model there is some suggestion that the P-scale predicts preference for positive freedom and negative freedom but this does not extend to the other specification. This suggests that general attitudes towards control in their lives are unrelated to their preference for control over outcomes.

		I-scale	C-scale	P-scale
$\alpha V^{f,c} + \beta V^{ni,d}$	α	-0.1632	-0.2132*	-0.2930**
	β	0.1596	0.1799	0.2258*
$\alpha V^{f,d} + \beta V^{ni,c}$	α	-0.0917	0.1124	-0.0096
	β	0.1732	0.0215	0.0363

Table 3.C.3: Correlation of estimated preference parameters with Locus of Control scores. *: $p < 0.05$, **: $p < 0.01$

1. (I) Whether or not I get to be a leader depends mostly on my ability.
2. (C) To a great extent my life is controlled by accidental happenings.
3. (P) I feel like what happens in my life is mostly determined by powerful people.
4. (I) Whether or not I get into a car accident depends mostly on how good a driver I am.
5. (I) When I make plans, I am almost certain to make them work.
6. (C) Of ten there is no chance of protecting my personal interests form bad luck happenings.
7. (C) When I get what I want, it is usually because I'm lucky.
8. (P) Although I might have good ability, I will not be given leadership responsibility without appealing to those positions of power.
9. (I) How many friends I have depends on how nice a person I am.
10. (C) I have often found that what is going to happen will happen.
11. (P) My life is chiefly controlled by powerful others.
12. (C) Whether or not I get into a car accident is mostly a matter of luck.
13. (P) People like myself have very little chance of protecting our personal interests when they conflict with those of strong pressure groups.
14. (C) It's not always wise for me to plan too far ahead because many things turn out to be a matter of good or bad fortune.
15. (P) Getting what I want requires pleasing those people above me.
16. (C) Whether or not I get to be a leader depends on whether I'm lucky enough to be in the right place at the right time.
17. (P) If important people were to decide they didn't like me, I probably wouldnâĂŹt make many friends.
18. (I) I can pretty much determine what will happen in my life.
19. (I) I am usually able to protect my personal interests.
20. (P) Whether or not I get into a car accident depends mostly on the other driver.
21. (I) When I get what I want, it's usually because I worked hard for it.
22. (P) In order to have my plans work, I make sure that they fit in with the desires of people who have power over me.
23. (I) My life is determined by my own actions.
24. (C) It's chiefly a matter of fate whether or not I have a few friends or many friends.

Table 3.C.2: Locus of Control questionnaire

3.D Inequality Aversion

Our experimental design also allows for the estimation of fairness preferences. We implemented the Fehr and Schmidt (1999) model, which gives us the following optimal bid condition:

$$b - (\pi_1^{hi} - \pi_1^{lo})/2 = \lambda V^{dis} + \mu V^{adv} \tag{3.37}$$

$$V^{dis} = \max\left(0, \frac{\pi_2^{high} + \pi_2^{low}}{2} + w_2 - \pi^{high} - w_1 + b\right) - \max\left(0, \pi_2^{high} + w_2 - \frac{\pi_1^{high} + \pi_1^{low}}{2} - w_1\right)$$

$$V^{adv} = \max\left(0, \pi_1^{high} + w_1 - b - \frac{\pi_2^{high} + \pi_2^{low}}{2} - w_2\right) - \max\left(0, \frac{\pi_1^{high} + \pi_1^{low}}{2} + w_1 - \pi_2^{high} - w_2\right)$$

where V^{dis} stands for the difference in disadvantageous inequality between Box L and Box R, and V^{adv} stands for the difference in advantageous inequality between Box L and Box R. An individual behaving according to the above model compares not only the utility values resulting from having or not having the decision right, but also the expected payoff inequalities resulting from having or not having the decision right.

Note that whether an observation is in the area of advantageous or disadvantageous inequality for Player 1 depends not only on the payoffs but also on the bid of Player 1.

For better readability, we define:

$$\eta_1 = \pi_1^{high,L} - \frac{\pi_1^{high,R} + \pi_1^{low,R}}{2} \tag{3.38}$$

$$\eta_2 = \frac{\pi_1^{high,L} + \pi_1^{low,L}}{2} - \pi^{high,R} \tag{3.39}$$

$$\eta_L = \pi_1^{high,L} - \frac{\pi_2^{high,L} + \pi_2^{low,L}}{2} \tag{3.40}$$

$$\eta_R = \frac{\pi_1^{high,R} + \pi_1^{low,R}}{2} - \pi_2^{high,R} \tag{3.41}$$

$$\eta_w = w_1 - w_2 \tag{3.42}$$

b^* is then implicitly defined via:

$$b^*(\lambda,\mu) = \begin{cases} \eta_1 - \frac{\lambda}{1+\lambda}\eta_2 \ , & (\eta_L + \eta_w < b^*) \wedge (\eta_R + \eta_w < 0) \\ \frac{\eta_1+\lambda(\eta_w+\eta_L)+\mu(\eta_w+\eta_R)}{1+\lambda} \ , & (\eta_L + \eta_w < b^*) \wedge (\eta_R + \eta_w > 0) \\ \eta_1 + \frac{\mu}{1-\mu}\eta_2 \ , & (\eta_L + \eta_w > b^*) \wedge (\eta_R + \eta_w > 0) \\ \frac{\eta_1-\mu(\eta_w+\eta_L)+\lambda(\eta_w+\eta_R)}{1-\mu} \ , & (\eta_L + \eta_w > b^*) \wedge (\eta_R + \eta_w < 0) \end{cases} \tag{3.43}$$

Which case is the relevant one depends on the round and preference parameters λ and μ and the optimal bid is nonlinear in λ and μ. We therefore estimated the preference parameters via nonlinear least squares on the bids. We additionally included a constant to account for preference for negative freedom.[28] The estimated model is:

$$b_{i,t} = b^*_{i,t}(\lambda,\mu) + \gamma + \epsilon_{i,t}\,. \tag{3.44}$$

We find only slight evidence for preference for advantageous inequality. In-

[28]Technically, this is not quite the same definition of negative freedom as in the main body of the paper, where the model additionally accounts for risk attitudes. Since in both the risk-neutral and the risk-averse case there is strong evidence for preference for negative freedom, we interpret this as an additional robustness result.

λ	.013038
	(.0514239)
μ	-.0886128*
	(.044491)
γ	12.44203***
	(1.783147)
obs	2440
subjects	122
R^2	0.7604

Table 3.D.1: Estimation results of model (3.44). We used a grid of 10^3 starting points for the three parameters and obtained standard errors via bootstrapping with clusters at the individual level and 100 repetitions. Standard errors are shown in parenthesis: * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$.

dividuals therefore seemed to engage more in competitive bidding when they were in an advantageous situation. The main explanation for overbidding relative to the Nash equilibrium predictions is still negative freedom.

There are several explanations why inequality aversion seems to play a small role in explaining the data. First, in more complex decision tasks individuals may focus more strongly on their own payoffs than on inequality. Second, unlike decision problems such as the dictator game, the decision problem in our experiment is not clearly framed as one where individuals are morally obliged to share. Finally, experiment participants may not have been aware of the effect that their bids had on the payoffs of the other player.

4 Contests with Group-Specific Public Goods and Complementarities in Efforts[1]

Martin Kolmar
Hendrik Rommeswinkel

This paper starts from the observation that in public-goods group contests, group impact can in general not be additively decomposed into some sum (of functions) of individual efforts. We use a CES-impact function to identify the main channels of influence of the elasticity of substitution on the behavior in and the outcome of such a contest. We characterize the Nash equilibria of this game and carry out comparative-static exercises with respect to the elasticity of substitution among group members' efforts. If groups are homogeneous (i.e. all group members have the same valuation and efficiency within the group), the elasticity of substitution has no effect on the equilibrium. For heterogeneous groups, the higher the complementarity of efforts of that group, the lower the divergence of efforts among group members and the lower the winning probability of that group. This contradicts the common intuition that groups can improve their performance by solving the free-rider problem via higher degrees of complementarity of efforts.

[1]This paper has been published as Kolmar and Rommeswinkel (2013).

4.1 Introduction

In many economic situations like R&D races, military conflicts, lobbying, or sports, groups compete for economic rents that are group-specific public goods. Usually, in all these examples, efforts of different group members are to some extent complementary. In R&D races, where teams of researchers develop new technologies, the whole project is often divided into different, more or less complementary sub-projects that are carried out by different researchers. In military conflicts the armed forces are highly specialized and often divided into complementary units. The same is true for the standard lobbying case if representatives of different firms or organizations lobbying for the same policy differ in qualifications and specialize accordingly. In sports contests, team members are usually specialized with respect to qualifications that complement each other in a non-additive way. Another example for a group conflict is competition for a prize between different business partnerships. Management consultants, lawyers, physicians, and architects often organize their companies as partnerships where individual incomes of the partners are determined according to their shares in the partnership (Garicano & Santos, 2004). Consultancies and architect offices competing for projects and physicians competing for patients are all in situations that closely resemble a contest.[2] Furthermore, the substitutability of the partners' efforts depends on the industry as well as on the qualifications of the different partners (and thereby the organizational structure and the business strategy). A medical center that combines physicians with different qualifications has a relatively high degree of complementarity between the different physicians' qualifications. A consulting firm that specializes in only one field of business, on the other hand, is likely to have a higher degree of substitutability between the partners' efforts.

This list of examples could be more or less arbitrarily extended because the mere idea of specialization implies that there is a certain degree of complementarity in team or group production. Individuals differ in talents, qual-

[2]Competition for customers has more the character of an oligopolistic market. However, if market demand is isoelastic, the Tullock contest is isomorphic to a Cournot oligopoly.

ifications, and affections in a way that they will specialize to increase overall productivity. We can therefore expect a certain degree of complementarity between the efforts of the group members. Alchian and Demsetz (1972) see the non-additivity as constitutive for group or team production (pp. 777): *"Resource owners increase productivity through cooperative specialization. [...] With team production it is difficult, solely by observing total output, to either define or determine each individual's contribution to this output of the cooperating inputs. The output is yielded by a team, by definition, and it is not a sum of separable outputs of each of its members. [...] Usual explanations of the gains from cooperative behavior rely on exchange and production in accord with the comparative advantage specialization principle with separable additive production. However [...] there is a source of gain from cooperative activity involving working as a team, wherein individual cooperating inputs do not yield identifiable, separate products which can be summed to measure the total output."*

Despite the growing interest in the influence of heterogeneity within and between groups, with only a few exceptions the literature on group contests (surveyed in Corchón, 2007; Garfinkel & Skaperdas, 2007; Konrad, 2009) has focused attention on situations where the effort levels of group members are perfect substitutes. This case is an important starting point for the analysis of group contests. However, if complementarities are the rule rather than the exception, it is important to understand how the degree of complementarity between individual efforts influences behavior in and the outcome of the contest.

We use a CES production (impact) function in an n-group contest. To be more specific, assume that individual efforts x_i^k are mapped onto group impact (that enters a lottery contest as aggregate contribution by a group)[3] by means of a CES-impact function, $g_i \cdot (\sum a_i^k \cdot (x_i^k)^{\gamma_i})^{1/\gamma_i}$, with variable elasticity of substitution $1/(1-\gamma_i)$, ranging from perfect complements ($\gamma_i \to -\infty$) to perfect substitutes ($\gamma_i \to 1$), and aggregate as well as individual efficiency parameters g_i, a_i^k respectively. The contest is of the Tullock type, and the rent is a

[3]The term 'impact function' is defined and discussed in Wärneryd (2001); Münster (2009).

group-specific public good (i.e. nonrival and nonexcludable in consumption).

If groups instead of individuals compete in a contest, the well-known free-rider problem among group members exists. Every individual bears the full costs of its investments, whereas the benefits partly spill over to the rest of the group (Katz, Nitzan, & Rosenberg, 1990; Esteban & Ray, 2001; Epstein & Mealem, 2009; Nitzan & Ueda, 2009; Ryvkin, 2011). Depending on the sharing rule applied, this problem may also exist for a private good (Nitzan, 1991a, 1991b; Esteban & Ray, 2001; Nitzan & Ueda, 2009). In the recent literature, Baik (2008), Epstein and Mealem (2009), and Lee (2012) have presented contest models with group-specific public goods. A major result in Baik (2008) is that in a model with linear effort costs and additively linear impact functions only those group members with the highest valuation of the rent make positive investments in the contest.[4] In his model, efforts of group members are perfect substitutes and therefore the optimality conditions given by the first-order conditions cannot hold for different valuations. With several group members having the maximal valuation among the group, there exist multiple equilibria, since the first order condition only defines the total effort spent by the group. Epstein and Mealem (2009) stick to the assumption of additive separability of individual effort in the group-production functions but introduce decreasing returns to investment. Using a technology that fulfills standard "Inada" conditions they show that every individual makes positive investments. Their model is isomorphic to a model with linear impact functions and in which individuals face strictly convex costs. In this sense, effort levels are no longer perfect substitutes, but the impact function is still additively separable. Lee (2012) focuses attention on weakest-link or perfectly complementary impact functions. The perfect complementarity of efforts creates a coordination problem between group members which gives rise to multiple equilibria, and the equilibrium with highest efforts is determined by the valuation of the player with minimum valuation within each group. Hence, the models of Baik (2008)

[4]This result has, of course, a counterpart in the literature on the private provision of public goods where it follows as a special case of the seminal contribution by Bergstrom, Blume, and Varian (1986).

and Lee (2012) represent the "polar" cases with respect to the elasticity of substitution between group members for those cases where the iso-impact curves remain convex. Chowdhury, Lee, and Sheremeta (2011) nicely complements our paper. They analyze the case of a best-shot impact function as the most extreme case of non-convex iso-impact curves.

Our model generalizes the "convex" models by allowing for degrees of complementarity among group efforts. It turns out that the equilibrium behavior of each group is unique for all values of $\gamma_i \in (0,1)$. For $\gamma_i \in (-\infty,0)$, the complementarity of efforts is high enough, such that the effort contributions of each member become indispensable. Groups may therefore end up in a high effort equilibrium, in which all members contribute, or in a low effort equilibrium, where none contribute. However, in both cases we can give analytical expressions for equilibrium strategies. In our comparative statics analysis we therefore track equilibria with the same set of groups which fail to coordinate on a high effort equilibrium.

A first corollary is that if there is no within-group heterogeneity with respect to valuations of the prize v_i^k and efficiency a_i^k of each group member and all groups have the same size, the equilibrium is independent of the elasticity of substitution except for the mentioned multiple equilibria issue. This result is a useful starting point because it shows that the elasticity of substitution *per se* has no impact on behavior in the contest, contrary to the cursory idea that increasing the degree of complementarity between group-members' efforts may help to internalize the existing free-rider problem.[5] This point, which has been derived for public-goods games with effort complementarity (Cornes, 1993; Cornes & Hartley, 2007), carries over to the contest environment.[6] As a con-

[5]E.g. Hirshleifer (1983) argues for the special case of perfect complements ("weakest-link" technology) that the complementarity between group members' efforts helps solving the free-rider problem.

[6]Cornes and Hartley (2007) have analyzed a voluntary-contributions to a public-good game with CES production (social-composition) functions where a single group jointly produces a public good. The additional dimension of generality from the contest structure comes at the cost of a more restrictive class of utility functions. Whereas Cornes and Hartley (2007) need binormal utility functions, we assume that utility functions are additively separable between the group-specific public good and some numéraire good that finances individual contributions.

venient side effect, this independence shows that the standard results on group contests are robust with respect to variations in the elasticity of substitution under within-group homogeneity.

The comparative-static analysis of the paper reveals that this effect is even more pronounced in the general case: A larger degree of complementarity within a single group reduces its winning probability. The intuition for this result is as follows. It is true that a larger degree of complementarity brings the effort levels of the group members closer together. Free-riding that is especially pronounced in the boundary case $\gamma_i = 1$ is therefore mitigated. However, the level of effort is increasingly determined by the group member with the lowest valuation, and it is this latter effect that turns out to be dominant. Even though the winning probability is decreasing, the effect on the overall welfare of the group is ambiguous: Highly efficient group members with a low valuation may start to provide effort under higher degrees of complementarity and due to their efficiency raise the overall welfare of the group. The results highlight the importance of accounting for within-group heterogeneity and complementarity for a proper analysis of the provision of group-specific public goods in a contest environment.

While these results are derived for the public good "winning probability in a contest", it may be interesting to see whether they hold in general for the private provision of public goods. So far it has not been possible to analyze this point in public goods models such as Cornes and Hartley (2007), since there exist no analytical solutions for the equilibrium contributions aside from some special cases. The fact that in our model we can explicitly solve for equilibrium strategies enables us to perform this comparative-static analysis with respect to the degree of complementarity in efforts.

The paper is organized as follows. We introduce the model in Section 2 and start with introductory examples in Section 3. We characterize the simultaneous Nash equilibria of the general model in Section 4. In section 5 the comparative-static results are summarized. Section 6 concludes. Large proofs are given in the appendix, and in a special Appendix C we will state convergence results for $\vec{\gamma}$ approaching 1, 0, and $-\infty$.

4.2 The model

Assume that $n \geq 2$ groups compete for a given rent. The set of groups is given by $N = \{1, 2, \ldots, n\}$ while m_i is the number of individuals in group i and k is the index of a generic member of this group. The rent is a group-specific public good that has a value $v_i^k > 0$ to individual k of group i. p_i represents the probability of group $i = 1, \ldots, n$ to win the contest. It is a function of some vector of aggregate group output $q_1, \ldots, q_n$. We focus on Tullock-form contest-success functions where the winning probability of a group i is defined as:

Assumption 4.1.

$$p_i(Q_1, \ldots, Q_n) = \begin{cases} \frac{Q_i}{\sum_{j=1}^{n} Q_j}, & \exists Q_j > 0 \\ \frac{1}{n}, & Q_j = 0 \; \forall \; j \end{cases}, \tag{4.1}$$

with $i = 1, \ldots, n$

The aggregate group output Q_i of each group $i = 1, \ldots, n$ depends on individual effort x_i^k, $Q_i = q_i(x_i^1, \ldots, x_i^{m_i})$. Following the literature we will call $q_i(.)$ *impact functions* in the following and make the assumption that they are of the constant elasticity of substitution (CES) type.

Assumption 4.2.

$$q_i(x_i^1, \ldots, x_i^{m_i}) = g_i \cdot \left(\sum_{l=1}^{m_i} a_i^l \cdot (x_i^l)^{\gamma_i} \right)^{1/\gamma_i}, \tag{4.2}$$

with $\gamma_i \in \{(-\infty, 0), (0, 1)\}$, $i = 1, \ldots n$.

The function has the usual parameters a_i^k for the efficiency of an individual's effort and g_i for the relative strength of the group. Note that we obtain a closed-form solution only if for all i it holds that $\gamma_i \neq 0$. Also, if $\gamma_i < 0$ and

$\exists k : x_i^k = 0$, the function is not well defined. We will therefore take the limit of $q_i(\dots)$ as $x_i^k \to 0$, which means $q_i(\dots) = 0$ in that case. Note that for $\gamma_i > 0$ this is not the case.

Assumption 4.3. Individuals are risk neutral, face linear costs, and maximize their net rent.

It follows from Assumptions 1, 2, and 3 that the individual expected utility functions are as follows:

$$\pi_i^k(x_1^1, \dots, x_n^{m_n}) := \pi_i^k(x_i^k, \vec{x}_{/x_i^k}) = v_i^k \cdot \frac{g_i \cdot \left(\sum_l a_i^l \cdot (x_i^l)^{\gamma_i}\right)^{1/\gamma_i}}{\sum_j g_j \cdot \left(\sum_l a_j^l \cdot \left(x_j^l\right)^{\gamma_j}\right)^{1/\gamma_j}} - x_i^k, \tag{4.3}$$

where $\vec{x}_{/x_i^k}$ refers to the vector $x_1^1, \dots, x_n^{m_n}$ without x_i^k. We are looking for a Nash equilibrium of this game where individuals choose their effort x_i^k simultaneously to maximize their expected utility,

$$x_i^{k*} \in \arg\max_{x_i^k} \pi_i^k\left(x_i^k, \vec{x}^*_{/x_i^k}\right) \quad \forall i, k, \tag{4.4}$$

where "$*$" refers to equilibrium values.

4.3 Introductory examples

In this section we analyze two simple special cases that provide intuition for the relevance of the degree of complementarity in contests. As we will see, the degree of complementarity is only relevant if the valuations between members of the same group differ. The examples restrict attention to a contest between two groups, 1 and 2, with m_1 and m_2 members whose valuations can take two values. The valuation of the group members are either high $\overline{v_i}$ or low $\underline{v_i}$, thus $\overline{v_i} \geq \underline{v_i}$, $i = 1, 2$. The examples are chosen to highlight the central mechanisms of this model, we therefore relegate all technical details about the existence of interior solutions, active and inactive groups and group members, etc. to the next section.

Example 1: Let us restrict attention to groups of equal size $m_1 = m_2 = m$ with only a single valuation of the members of a given group, $\underline{v_i} = \overline{v_i} = v_i, i = 1,2$ and identical technologies with $a_i^k = 1$, $g_i = 1$ and $\gamma_i = \gamma$. In this case

$$x_1^*(v_1,v_2,m) = \frac{(v_1)^2 \cdot v_2}{m \cdot (v_1 + v_2)^2}, \quad x_2^*(v_1,v_2,m) = \frac{v_1 \cdot (v_2)^2}{m \cdot (v_1 + v_2)^2}$$

constitutes an interior equilibrium. Investments in the contest are independent of γ_i. This example shows that the elasticity of substitution does not play a role if there is no within-group heterogeneity and groups are of equal size and have the same impact function. The reason for this result is the combination of a constant-return to scale impact function with a contest success function that is homogeneous of degree zero. Conversely, it must be either within-group heterogeneity and/or differences in group size and technology that may cause behavioral changes due to changes in γ_i. The next example shows that this may in fact be the case.

Example 2: Let us assume again $m_1 = m_2 = m$ and for all i and k that $a_i^k = 1$ and $\gamma_i = \gamma$. However, we allow for heterogeneous valuations within groups: $\underline{v_1} = \underline{v_2} = \underline{v} \leq \overline{v} = \overline{v_1} = \overline{v_2}$. The population of each group is divided into $\overline{m} = \underline{m} = m/2$ of individuals with the high and the low valuation, respectively. One gets the following symmetric equilibrium:

$$\begin{aligned} \overline{x}^*(\underline{v},\overline{v},m,\gamma) &= \frac{\overline{v}}{2 \cdot m \cdot \left((\underline{v}/\overline{v})^{\frac{\gamma}{1-\gamma}} + 1 \right)}, \\ \underline{x}^*(\underline{v},\overline{v},m,\gamma) &= \frac{\underline{v}}{2 \cdot m \cdot \left((\overline{v}/\underline{v})^{\frac{\gamma}{1-\gamma}} + 1 \right)}. \end{aligned} \tag{4.5}$$

where $\overline{x}^*$ and $\underline{x}^*$ are the respective equilibrium efforts of the individuals with the high and low valuation. As expected, γ may influence the outcome of the game if differences among the valuations of the rent among the group members exist.

4.4 The general case

We now turn to the analysis of the general case. In order to have a lean notation, let $X_i = \sum_k x_i^k$, $y_i^k = (x_i^k)^{\gamma_i}$, and $Y_i = (\sum_l a_i^l \cdot y_i^l)$. Further, $Q = \sum_j Q_j = \sum_j g_j \cdot Y_j^{\frac{1}{\gamma_j}} = g_i \cdot Y_i^{\frac{1}{\gamma_i}} + \sum_{j \neq i} g_j Y_j^{\frac{1}{\gamma_j}} = Q_i + Q_{/i}$ in the following. Also, let $\vec{\gamma}$ denote the vector of all γ_i. While deriving the equilibrium strategies, we will omit the parameters of these functions for better readability (e.g y_i^k instead of $y_i^k(\gamma_i, x_i^k)$).

Hillman and Riley (1987) and Stein (2002) have shown that individuals may prefer to stay inactive in a single player contest. Baik (2008) has shown for $\gamma_i = 1$ that only group members with maximum valuation participate in a contest. Hence, it is possible that some individuals and/or groups will stay inactive in our setup. We therefore start with an analysis of active individuals and groups.

Definition 1: An individual k of group i is said to participate if $x_i^k > 0$. A group i is said to participate if there exists some k such that $x_i^k > 0$. A group is said to fully participate if $\forall k : x_i^k > 0$.

Lemma 4.1. *a) In a Nash equilibrium of a contest fulfilling Assumptions 1, 2, and 3 if a group participates, it fully participates.*
b) If $\gamma_i < 0$, $m_i > 1$, and one group member of group i does not participate, it is always a best response for all group members to not participate.

The proof of this as well as the next Lemma can be found in the appendix. Lemma 1 a) implies that in order to determine whether an individual participates, it is sufficient to determine whether its group participates. Lemma 1 b) shows that irrespective of the behavior of the other groups, it may occur that a group does not participate. The reason is that for $\gamma_i < 0$, a positive contribution from each member is indispensable: As soon as some group member k chooses $x_i^k = 0$, we have $q_i(\dots) = 0$. This of course gives rise to multiple equilibria as a group i may either coordinate on *not participating* or *fully participating* if $\gamma_i < 0$. In the following, we will therefore use the notation that there are $\underline{n} \leq n$

groups with $\gamma_i < 0$ and $m_i > 1$ and we will denote their set as $\underline{N}$, which may be empty. For each of the equilibria determined we must specify a subset of $\underline{N}$ of groups that coordinate on not participating, which will be denoted $\underline{N_0}$.

Let $V_i(\gamma_i) \equiv g_i \cdot \left(\sum_l a_i^l \cdot (a_i^l \cdot v_i^l)^{\frac{\gamma_i}{1-\gamma_i}}\right)^{\frac{1-\gamma_i}{\gamma_i}}$. Without loss of generality, suppose that all groups are ordered with descending V_i such that $V_i(\gamma_i) \geq V_{i+1}(\gamma_{i+1})$. $Q_i^*(\vec{\gamma}, \underline{N_0})$ and $Q^*(\vec{\gamma}, \underline{N_0})$ shall denote Q_i and Q in an equilibrium where $\underline{N_0}$ do not participate. We use $|\dots|$ to denote the cardinality of a set of groups. The following Lemma determines the groups that participate in the equilibrium in which $\underline{N_0}$ do not participate.

Lemma 4.2. *a) The best response conditions of the members of a group $i \in N/\underline{N_0}$ can be fulfilled, if and only if the following group best response function is fulfilled:*

$$\hat{Q}_i(\vec{\gamma}, Q_{/i}) = \max\left(0, \sqrt{Q_{/i} \cdot V_i(\gamma_i)} - Q_{/i}\right). \tag{4.6}$$

where $Q_{/i} > 0$ must hold.
b) Groups $N^*(\vec{\gamma}, \underline{N_0}) = \{i \in N/\underline{N_0} : i \leq n^*(\vec{\gamma}, \underline{N_0})\}$ participate, where
$n^*(\vec{\gamma}, \underline{N_0}) \equiv \arg\max_{i \in N/\underline{N_0}} i$ *such that $V_i(\gamma_i) > Q^*(\vec{\gamma}, \underline{N_0})$.*
c) Holding $\underline{N_0}$ fixed, if the resulting Nash equilibrium is unique, $Q_i^*(\vec{\gamma}, \underline{N_0})$ and $Q^*(\vec{\gamma}, \underline{N_0})$ are continuous functions in $\vec{\gamma}$.

Lemma 2 a) gives the necessary and sufficient condition for existence of best response strategies of those groups that do not belong to $\underline{N_0}$, for which it is automatically fulfilled. If (4.6) is not fulfilled, there will be at least one individual who does not play best responses if the group reaches impact $Q_i \neq \hat{Q}_i(\dots)$.

Lemma 2 b) characterizes the participating groups given that the groups in $\underline{N_0}$ do not participate. There are therefore two reasons why a group might not participate: Either because the average valuation of the group members are too low or because it belongs to $\underline{N_0}$. However, once $\underline{N_0}$ is fixed, one can uniquely identify the remaining groups which do not participate.

Lemma 2 c) is useful for the comparative-static analysis if one focuses on a specific equilibrium with given $\underline{N_0}$. Given that the number and identity of active groups then still depends on $\vec{\gamma}$, it is *a priori* not clear that aggregate effort and indirect utilities are continuous in $\vec{\gamma}$. The Lemma reveals that continuity is in fact guaranteed if the identity of groups in $\underline{N_0}$ remains the same. The economic intuition is as follows: Excluding groups among $\underline{N_0}$, assume that $\hat{\gamma}_j$ is a point where a formerly active group becomes inactive or a formerly inactive group becomes active. The aggregate group effort of the active group is continuously reduced to zero as γ_j approaches $\hat{\gamma}_j$, and the formerly inactive group continuously increases its effort from 0 as γ_j increases from $\hat{\gamma}_j$. Hence, there is a "smooth" fade out or fade in of groups at those points.

The following proposition characterizes the set of Nash equilibria of the game. For readability, the strategies x_i^k are defined as functions of $Q^*(\vec{\gamma})$ and $V_i(\gamma_i)$.

Proposition 4.1. *The set of Nash equilibria of the game characterized by Assumptions 1,2, and 3 is given as follows. For each set of groups in $\underline{N_0}$ such that $|N/\underline{N_0}| \geq 2$ there exists a Nash equilibrium given by strategies ${x_i^k}^*(\vec{\gamma},\underline{N_0})$ that fulfill*

$$
{x_i^k}^*(\vec{\gamma},\underline{N_0}) = \begin{cases} \cdot\left(Q^*(\vec{\gamma},\underline{N_0}) - \frac{Q^*(\vec{\gamma},\underline{N_0})^2}{V_i(\gamma_i)}\right) \cdot \frac{(g_i)^{\frac{\gamma_i}{1-\gamma_i}}(a_i^k \cdot v_i^k)^{\frac{1}{1-\gamma_i}}}{V_i(\gamma_i)^{\frac{1}{1-\gamma_i}}}, & i \in N^*(\vec{\gamma},\underline{N_0}) \\ 0, & else \end{cases} \tag{4.7}
$$

where $Q^*(\vec{\gamma},\underline{N_0}) = \frac{|N^*(\vec{\gamma},\underline{N_0})|-1}{\sum_{i\in N^*(\vec{\gamma},\underline{N_0})} V_i(\gamma_i)^{-1}}$ and $N^*(\vec{\gamma},\underline{N_0})$ is defined in Lemma 2 a).

Proof. Suppose $\underline{N_0}$ do not participate. From Lemma 1 b) we then know that the members of these groups play best responses. Lemma 2 b) determines the participating groups. To obtain $Q^*(\vec{\gamma},\underline{N_0})$ we sum (4.6) over all $i \in N^*(\vec{\gamma},\underline{N_0})$:

$$
Q^*(\vec{\gamma},\underline{N_0}) = \frac{|N^*(\vec{\gamma},\underline{N_0})| - 1}{\sum_{i\in N^*(\vec{\gamma},\underline{N_0})} V_i(\gamma_i)^{-1}}. \tag{4.8}
$$

With an explicit solution for $Q^*(\vec{\gamma}, \underline{N_0})$, we can now determine individual expenditures $x_i^{k*}(\vec{\gamma}, \underline{N_0})$ by solving equation (4.6) using (4.8). The participation condition of a group is given by Lemma 2, while Lemma 1 a) ensures that there does not exist an incentive for any group member to deviate to $x_i^k = 0$ in the participating groups. It was further shown that the first-order conditions return local maxima. Since the system of equations given by the first-order conditions of the participating groups has a unique solution this is indeed the unique Nash equilibrium given $\underline{N_0}$. Notice if $N/\underline{N_0}$ has cardinality 1, we have $Q_{/i} = 0$ in (4.6) and thus best responses are no longer well defined for the participating group. There may be therefore some $\underline{N_0}$ for which no equilibrium exists. However, there is always at least one Nash equilibrium for $\underline{N_0} = \varnothing$. □

Several things are noteworthy: Given the set $\underline{N_0}$, the equilibrium is unique if it exists. Therefore, the maximum number of equilibria is the number of possible combinations of $\underline{N_0}$ such that in total either no groups or at least two groups participate. However, it is possible that some of these equilibria are identical since removing group i from $\underline{N_0}$ does not necessarily mean that it enters N^*.

Further, it may be a Nash equilibrium that no group participates if for all i we have $\gamma_i < 0$ and $m_i > 1$. It also may occur that for some $\underline{N_0}$ no Nash equilibrium exists, since for (4.7) to be well defined it is required that at least two groups participate. An $\underline{N_0}$ that leaves only one potentially participating group will therefore not yield a Nash equilibrium.

A focal special case has no intra-group heterogeneity $v_i^k = v_i \forall k \forall i$ and $a_i^k = a_i \forall k \forall i$. The following corollary of Proposition 1 can then be established.

Corollary 4.1. *Let $\underline{N_0} = \varnothing$. Suppose for all groups i and all individuals k, it holds that $a_i^k = a_i$ and $v_i^k = v_i$ and further for all other groups j it holds that $a_i \cdot m_i = a_j \cdot m_j$. Then the equilibrium efforts are independent of $\vec{\gamma}$.*

Proof. Inserting the above values for every individual l $a_i^l = a_i$ and $v_i^l = v_i$ and setting for all other groups j $a_j \cdot m_j = a_i \cdot m_i$ into (4.7) directly yields the result. □

The corollary shows that $\vec{\gamma}$ is only relevant if there is either heterogeneity with respect to valuations within groups and/or heterogeneity with respect to group size. In all other cases equilibrium behavior does not depend on $\vec{\gamma}$ with the exception that groups may fall into the set $\underline{N_0}$ if their γ_i drops below 0. The corollary shows that corresponding results from public-goods games with complementarities in efforts (Cornes, 1993; Cornes & Hartley, 2007) continue to hold in a contest environment. This finding implies that an increase in complementarity between group members' effort *per se* has no effect on the within-group free-rider problem, as could have been conjectured from Hirshleifer (1983). A further implication of the corollary is that the results on group contests that have been derived in the literature for the case of perfect substitutes or perfect complements carry over to arbitrary elasticities of substitution if homogeneous groups differ only in their valuations of the rent and their group efficiency parameter g_i.

4.5 Comparative statics

Before we move on to the comparative-static analysis, let us first note that the winning probability of group i takes the form:

$$p_i(Q_1^*(\vec{\gamma}),...,Q_n^*(\vec{\gamma})) = \frac{Q_i{}^*(\vec{\gamma})}{Q^*(\vec{\gamma})} = \left(1 - \frac{Q^*(\vec{\gamma})}{V_i(\gamma_i)}\right), \tag{4.9}$$

which can be derived from (4.6). An analysis of convergence results for γ_i which can be found in Appendix C suggests that it makes sense to generally impose $\sum_k a_i^k = 1$ to model relative differences in efficiency between group members and use the parameter g_i for the resulting absolute differences in efficiency between groups. Only then the comparative statics with respect to $\vec{\gamma}$ will capture solely the effect of different degrees of substitution and no productivity effects.

We now turn to the comparative-static analysis of the influence of the elasticity of substitution on the behavior in the contest using the approach developed by Cornes and Hartley (2005). Since we have multiple equilibria for $\gamma_i < 0$, it is necessary to exclude jumps from one equilibrium to another. We will therefore focus in the following on the equilibrium given by some $\underline{N_0}$ in which at least two groups participate.

Most interestingly, individual valuations in relation to the valuations of the other group members define the individuals' share of the amount of effort spent by the group, ${x_i^k}^*/X_i^*$. The valuation of other groups have no effect on these shares. As was to be expected, a larger elasticity of substitution γ_i increases *ceteris paribus* the dispersion of these shares, since in equilibrium the exponent discriminates more strongly between differences in (efficiency-weighted) valuations. The next proposition states the effect of γ_i on the individual shares.

Proposition 4.2. *Suppose group i participates. The share of an individual of its group's effort, $\frac{x_i^k}{X_i}$, increases (decreases) strictly in the elasticity of substitution among efforts if the valuation times the efficiency $a_i^k \cdot v_i^k$ of the individual is strictly larger (smaller) than the share-weighted geometric mean of the group members' valuation times efficiency, $\prod_l (a_i^l \cdot v_i^l)^{\left(\frac{x_i^l}{X_i}\right)}$.*

Proof. It is straightforward to derive the following equation from (4.7):

$$\frac{{x_i^k}^*(\vec{\gamma})}{X_i^*(\vec{\gamma})} = \frac{(a_i^k \cdot v_i^k)^{\frac{1}{1-\gamma_i}}}{\sum_l (a_i^l \cdot v_i^l)^{\frac{1}{1-\gamma_i}}} \tag{4.10}$$

Taking the derivative of (4.10) with respect to γ_i yields

$$\frac{\partial \frac{x_i^k}{X_i}}{\partial \gamma_i} = \frac{(a_i^k \cdot v_i^k)^{\frac{1}{1-\gamma_i}}}{\sum_l (a_i^l \cdot v_i^l)^{\frac{1}{1-\gamma_i}}} \frac{1}{(1-\gamma_i)^2} \left(\ln(a_i^k \cdot v_i^k) - \frac{\sum_l (a_i^l \cdot v_i^l)^{\frac{1}{1-\gamma_i}} \ln(a_i^l \cdot v_i^l)}{\sum_l (a_i^l \cdot v_i^l)^{\frac{1}{1-\gamma_i}}} \right). \tag{4.11}$$

The RHS of the above equation is positive whenever the term in brackets is positive. Setting $\ln(a_i^k \cdot v_i^k) \geq \sum_l (a_i^l \cdot v_i^l)^{\frac{1}{1-\gamma_i}} \ln(a_i^l \cdot v_i^l) / \sum_l (a_i^l \cdot v_i^l)^{\frac{1}{1-\gamma_i}}$ and rearranging yields the condition:

$$\frac{\partial \frac{x_i^k}{X_i}}{\partial \gamma_i} \gtreqless 0 \quad \Leftrightarrow \quad a_i^k \cdot v_i^k \gtreqless \prod_l \left(a_i^l \cdot v_i^l\right)^{\left(\frac{(a_i^l \cdot v_i^l)^{\frac{1}{1-\gamma_i}}}{\Sigma_s\, (a_i^s \cdot v_i^s)^{\frac{1}{1-\gamma_i}}}\right)}. \tag{4.12}$$

□

The proposition implies that for all group members with a valuation above the weighted geometric mean, the share of total group effort increases with γ_i. The result shows that the dispersion of valuations plays a crucial role for the comparative-static effects of γ_i. In the easiest case of a two-member group i with individuals j and k, the proposition boils down conveniently: Individual j's share increases in γ_i if and only if $a_j v_j > a_k v_k$: The individual with the higher efficiency-weighted valuation increases its relative contributions if γ_i goes up. In the context of the partnership example from the introduction, the finding implies that the relative burden for group success is increasingly carried by the individuals with either the highest stakes and/or the highest productivity if it becomes easier to substitute between the partners' efforts. The reason is as follows. A higher elasticity of substitution has two effects. From the point of view of the high-stake / high productivity player, effort becomes less dependent on the other players' efforts, which *ceteris paribus* gives an additional stimulus to invest relatively more. And from the point of view of his fellow team mates, the negative effects of slacking off become less detrimental, which *ceteris paribus* implies that it pays to invest relatively less.

A second interesting question may be whether the winning probability of groups can be increased by a higher degree of complementarity of efforts. The intuition behind this may be twofold: First, with higher complementarity, the free-rider problem is solved better, such that also individuals with low valuations participate. Second, there often exist gains from specialization. While the latter intuition is induced by the technology itself, which is exogenous in our model, the first intuition can be examined through comparative statics of the model.

Proposition 4.3. *Suppose* $\sum_k a_i^k = 1$*. Then the winning probability of a participating group* i *is weakly increasing in* γ_i *and strictly increasing whenever there exist two group members* k *and* l *such that* $a_i^k \cdot v_i^k \neq a_i^l \cdot v_i^l$ *and the change in* γ_i *does not turn any group from a participative into a non-participative status.*

Proof. Using (4.8), the winning probability of group i, (4.9), can be written as

$$\frac{Q_i{}^*(\vec{\gamma},\underline{N_0})}{Q^*(\vec{\gamma},\underline{N_0})} = \left(1 - \frac{n^*(\vec{\gamma}) - 1}{1 + \sum_{j \neq i} \frac{V_i(\gamma_i)}{V_j(\gamma_j)}}\right), \tag{4.13}$$

where the sum refers to all active groups $1,...,n^*(\vec{\gamma})$ except i. Two cases have to be distinguished: (a) A change in γ_i turns group i from a participative to a non-participative status or leaves its non-participative status intact. In this case, the change in γ_i has no influence on group i's winning probability because of the smooth fade out of the group's investments. (b) A change in γ_i has no influence on the participative status of i. In this case, note that (4.13) is strictly increasing in $V_i(\gamma_i)$. $V_i(\gamma_i)$ has under the assumption of $\sum_k a_i^k = 1$ the structure of an a_i^k weighted power mean of the $a_i^k \cdot v_i^k$ values of the group members. By the weighted power mean inequality (Bullen, 2003) we know that $V_i(\gamma_i)$ is strictly increasing in $\vec{\gamma}$ whenever there exist two individuals with $a_i^k \cdot v_i^k \neq a_i^l \cdot v_i^l$. Whenever all individuals have the same $a_i^k \cdot v_i^k$, $V_i(\gamma_i) = g_i \cdot a_i^k \cdot v_i^k$ and is therefore independent of γ_i. □

This result contradicts the common intuition that higher complementarity leads to a better solution of the free-rider problem and thus a better performance of the group. The result shows exactly the opposite: All things equal, heterogeneous groups with higher complementarity perform worse than similar groups with low complementarity. The intuition behind this is that a lower γ_i puts more emphasis on the lower values of x_i^k, so the lower γ_i, the more the equilibrium will reflect the optimal q_i of lower valuation group members. This has an important implication for the provision of public goods by groups in general: Highly complementary technologies will only be used if there are sufficient gains of specialization coming with them. While higher complementarity solves the free-rider problem, it solves it in the worst possible way: By reducing the incentives of high valuation individuals more than increasing the incentives of low valuation individuals.

For the partnership example from the introduction, Proposition 3 implies that if the production of impact of a partnership becomes more complementary, the equilibrium share or winning probability for this partnership goes down. If as a thought experiment one defines the sum of prizes of the partners in a partnership as total profit, the distribution of these profits depends on the shares of the partners in the firm, and can therefore be considered a design element. If in addition one considers the degree of complementarity also as a design element (because it depends at least to a certain extend on the organizational structure and the business model of the partnership), Proposition 3 reveals a rather odd implication for the share- or winning-probability maximizing design: The partnership would try to minimize complementarities. If it is possible to reach perfect substitutability, it would allocate all the profit shares to the single, most productive and / or highest-stake individual (Olson, 1965 and Ray, Baland, & Dagnelie, 2007). This conclusions runs counter to the intuition that complementarity in efforts encourages division of labor. Our finding isolates the *pure* effect of complementarity and shows that this pure effect alone is not only insufficient but counterproductive to explain gains from the division of labor. It is true that the division of labor comes with specialization, which makes individual efforts complementary. But the gains from specialization must result from an increase in group productivity, and this increase must be sufficiently strong to overcompensate the negative effect resulting from an increase in complementarity. If groups cannot use incentive mechanisms to internalize the within-group externalities, a free-rider problem exists for all degrees of complementarity and the effects are the more severe the higher the complementarity.

Our result is also novel in the literature on public-goods games in which no general comparative-static results have been provided for the effect of complementarity in efforts on the provision of public goods for heterogeneous contributors. The fact that we can solve for equilibrium strategies analytically allows us to perform this analysis here. This may also motivate a reexamination of the public-goods games in Cornes (1993) and Cornes and Hartley (2007) outside a contest setting to verify whether this result carries over to other public-goods games. Since it is not generally possible to solve for equilibrium strategies analytically in these models, one can expect this to be a nontrivial task, however.

Given that the winning probability of group i is monotonically increasing in γ_i, we may be interested in whether the same is true for the expected payoff. It turns out that the effect on the expected payoff of the group members is ambiguous both for the aggregate of players as well as the individual players. Inserting (4.7) into (4.4) we obtain:

$$\pi_i^k = p_i \left(v_i^k - Q^*(\vec{\gamma}, \underline{N_0})(g_i)^{\frac{\gamma_i}{1-\gamma_i}} \frac{(a_i^k \cdot v_i^k)^{\frac{1}{1-\gamma_i}}}{V_i(\gamma_i)^{\frac{1}{1-\gamma_i}}} \right). \tag{4.14}$$

As we know from Proposition 4.3, p_i is increasing in γ_i. However, also $Q^*(\vec{\gamma}, \underline{N_0})$ is increasing in γ_i and for sufficiently high $a_i^k v_i^k$ the term $(g_i)^{\frac{\gamma_i}{1-\gamma_i}} \cdot \frac{(a_i^k \cdot v_i^k)^{\frac{1}{1-\gamma_i}}}{V_i(\gamma_i)^{\frac{1}{1-\gamma_i}}}$ may be increasing as well. Therefore, for the group members with the highest $a_i^k \cdot v_i^k$, expected utility may be decreasing in γ_i. It is also clear that the group members of the lowest $a_i^k \cdot v_i^k$ will always improve their expected payoff by lower complementarity, since they will strictly reduce their effort and the group has a higher winning probability. The optimal γ_i a utilitarian planner who maximizes the sum of the group members' expected payoffs would impose is ambiguous: A lower γ_i may induce individuals with a lower valuation v_i^k but higher efficiency a_i^k to exert higher effort. If the highest type has a high valuation but a low efficiency, this may lead to overall efficiency gains for the group. From a group-production perspective one can understand the underlying mechanism in the following way: By changing the incentives of the group members, different degrees of complementarity also change the shares of effort provided by them. In turn, under heterogeneous technologies, this also changes the shares of total effort used by the different technologies. Different γ_i will therefore not only influence the effort X_i provided by the group, but also the average efficiency of the group in converting this effort into impact. To get a better intuition for this result we turn to an example.

Example 4: Since we are only interested in the effects of higher complementarity for one group, let the aggregate of the valuations of the first group be $V_1(\gamma_1) = 10$. Since this is the only way in which parameters from the first group enter the decision problem of the second, no more information about group one would be necessary. One could for example think of a group of a single individual with $v_1^1 = 10$, $a_1^k = 1$, and $g_1 = 1$. For the second group, assume two individuals with efficiency parameters $a_2^1 = 0.2$ and $a_2^2 = 0.8$. Thus, $\sum_l a_2^l = 1$ and comparative statics over γ_2 contain no effects from changes in productivity. Further, let valuations be heterogeneous such that $v_2^1 = 30$ and $v_2^2 = 5$. Finally, the efficiency parameter of the group is $g_2 = 1$.

From the fact that $v_2^1 \cdot a_2^1 = 6 > 4 = v_2^2 \cdot a_2^2$, we know that for $\gamma_2 = 1$ only the first individual will participate and for $\gamma_2 \to -\infty$, both individuals will participate. Proposition 4.3 tells us that the winning probability will decrease with lower values of γ_2.

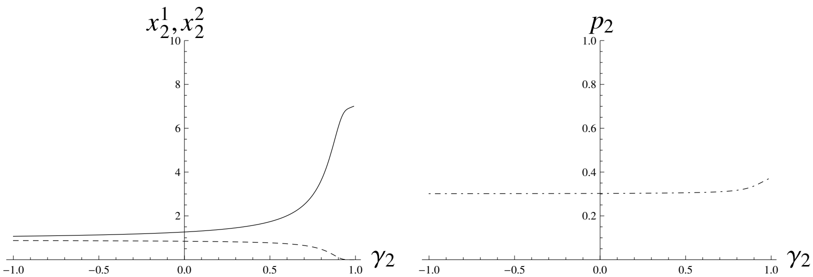

Figure 4.1: Effort levels and winning probability for different values of γ_2.

From Figure 4.1 we can see how this translates into our example. The effort level of individual 2 (with high efficiency and low valuation, dashed line) slowly increases as we reduce γ_2, while the effort of individual 1 (solid line)

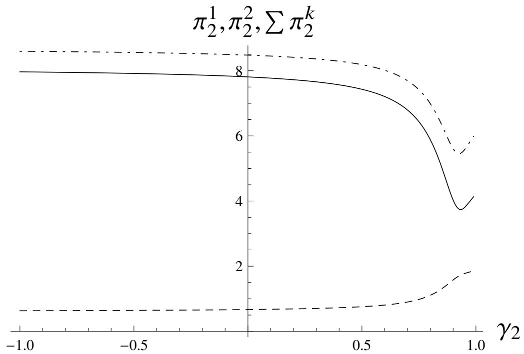

Figure 4.2: Expected utility for different values of γ_2.

falls. Both converge as $\gamma_2 \to -\infty$. We also see that the winning probability is falling with lower values of γ_2, as expected. The free-rider problem is thus solved with lower γ_2, but in a way such that the overall winning probability of the group is decreased. The more interesting result is, however, how this translates into the expected utility of the individuals.

In Figure 4.2 we see the expected utility of individuals 1 and 2 (again, represented by solid and dashed lines) and the aggregate expected utility (dotdashed line). The expected utility of individual 2 is of course rising in γ_2 (falling with higher complementarity), since in the case of perfect substitutes, i.e. $\gamma_2 = 1$, individual 2 can fully free ride. The change of expected utility of individual 1 is ambiguous with respect to changes in γ_2. For very high values of γ_2, it is also increasing with γ_2, while for low values it is decreasing in γ_2. Aggregate expected utility is mainly influenced by individual 1 and thus total expected utility of the group members behaves similarly: It is also maximal for very high degrees of complementarity and has a minimum below $\gamma_2 = 1$. The result is driven by the fact that the efficiency of individual 2 is much higher than that of individual 1 and at the same time the valuation of individual 1 is much higher than that of individual 2. In the perfect-substitutes case $\gamma_2 = 1$, only the less efficient individual 1 contributes effort and individual 2 takes a free ride. As we move away from this case, individual 2's incentives to provide effort increase only slowly. Due to the complementarity, individual 1 incurs very high losses in these cases. Reducing γ_2 even further provides much stronger incentives for individual 2. Individual 1 can thus reduce its effort further and in turn gain utility from the higher complementarity.

4.6 Concluding Remarks

This paper has started from the observation that group effort can in general not be additively decomposed into some sum (of functions) of individual efforts. The use of a CES-impact function has allowed to identify the main channels of influence of the elasticity of substitution on the behavior in and the outcome of contests. If groups have are homogeneous (i.e. all group members have the same valuation and efficiency within the group), the elasticity of substitution does not matter. For heterogeneous groups, the higher the complementarity of efforts of that group, the lower the divergence of efforts among group members and the lower the winning probability of that group. This contradicts the common intuition that groups can improve their performance by solving the free-rider problem via higher degrees of complementarity of efforts. Only if very high valuation members are also very inefficient at effort production the total expected utility may be higher for higher degrees of complementarity: At high levels of complementarity, highly efficient individuals with low valuations may replace some of the effort that is provided by less efficient group members at low levels of complementarity. The beneficial or detrimental role of complementarity for a group is therefore undetermined without further information on the composition of a group.

Appendix

4.A Proof of Lemma 4.1

Proof. For the proof of Lemma a) we first check that the interior solution is a local maximum if all group members participate. The first-order condition of the maximization problem (4.4) can be written as

$$\frac{Q_{/i}}{Q^2} Y_i^{\frac{1}{\gamma_i}-1} = \frac{(y_i^k)^{\frac{1}{\gamma_i}-1}}{v_i^k}. \tag{4.15}$$

The second-order condition is satisfied if

$$\frac{v_i^k \cdot Q_{/i} \cdot Y_i^{\frac{1}{\gamma_i}-2}}{\gamma_i \cdot Q^2} \left(\frac{1 - 2 \cdot \frac{Q_i}{Q}}{\gamma_i} - 1 \right) - \frac{\frac{1}{\gamma_i} - 1}{\gamma_i} \cdot (y_i^k)^{\frac{1}{\gamma_i}-2} < 0. \tag{4.16}$$

Solving the first-order condition for v_i^k and inserting the expression into the second-order condition we obtain, upon rearranging:

$$\frac{1 - \frac{1}{\gamma_i}}{\gamma_i} \left(1 - \frac{y_i^k}{Y_i} \right) - 2 \cdot \frac{1}{\gamma_i^2} \cdot \frac{Q_i \cdot y_i^k}{Q \cdot Y_i} < 0, \tag{4.17}$$

which holds for all $\gamma_i \in \{(-\infty,0),(0,1)\}$. Therefore, all solutions of the first-order condition are local maxima taking the other players' strategies as given. The best responses are either given by the solution to the first-order condition, or by a corner solution. From equation (4.3) it is clear that the only possible corner solutions are non-participation with $x_i^k = 0$. We thus need to verify that whenever the best response of one member of the group is given by the solution to the first-order condition, it is not possible for any member of the group to have the best response $x_i^k = 0$.

First, we will show that whenever there exists a solution of the first-order condition for one individual of a group, it exists for all individuals: From the first-order conditions of two representative group members l, k we obtain the within-group equilibrium condition:

$$\forall l,k: \quad \frac{(y_i^k)^{\frac{1}{\gamma_i}-1}}{v_i^k} = \frac{(y_i^l)^{\frac{1}{\gamma_i}-1}}{v_i^l} \tag{4.18}$$

for all members k,l of group i. Both, the left-hand side (LHS) and right-hand side (RHS) of (4.18) are strictly increasing in y_i^k, y_i^l if $\vec{\gamma} \in (0,1)$. For $\vec{\gamma} \in (-\infty,0)$ both LHS and RHS of (4.18) are strictly decreasing in y_i^k, y_i^l. Thus, for each y_i^k there exists a y_i^l such that the within-group equilibrium condition holds. Since for all group members the LHS of (4.15) is equal, there exists a positive solution to the first-order condition (FOC) for either all group members or none.

Second, we need to show that $x_i^k = 0$ is not a best response if it is a best response for another individual l in the group to play $x_i^l > 0$. We do so by contradiction: Obviously, for a corner solution with $x_i^k = 0$ and $x_i^l > 0$ the following condition needs to hold:

$$\frac{\partial \pi_i^k}{\partial x_i^k} = \frac{Q_{/i}}{Q^2} \cdot Y_i^{\frac{1}{\gamma_i}-1} \cdot (x_i^k)^{\gamma_i - 1} \cdot v_i^k - 1\Big|_{x_i^k = 0, x_i^l > 0} \leq 0. \tag{4.19}$$

From the fact that there is an individual l in the group, which participates with strictly positive effort, we know that

$$\frac{\partial \pi_i^l}{\partial x_i^l} = \frac{Q_{/i}}{Q^2} \cdot Y_i^{\frac{1}{\gamma_i}-1} \cdot (x_i^l)^{\gamma_i - 1} \cdot v_i^l - 1\Big|_{x_i^k = 0, x_i^l > 0} = 0. \tag{4.20}$$

Inserting (4.20) into (4.19) yields:

$$\frac{(x_i^l)^{1-\gamma_i}}{v_i^l} - \frac{(x_i^k)^{1-\gamma_i}}{v_i^k}\Big|_{x_i^k = 0, x_i^l > 0} \leq 0 \tag{4.21}$$

from which we obtain by inserting $x_i^k = 0$:

$$(x_i^l)^{1-\gamma_i}\Big|_{x_i^l > 0} \leq 0 \tag{4.22}$$

which is a contradiction for all $\gamma_i < 1$. Thus there does not exist an equilibrium in which for one player in the group a corner solution at zero effort investments is obtained while for another an interior solution holds.

Part b) can be shown as follows: Suppose $x_i^k = 0$ for some k, $m_i \geq 2$ and $\gamma_i < 0$. Then $q_i(x_i^1, \ldots, x_i^{m_i}) = 0$. The expected payoff $\pi_i^l(x_i^1, \ldots, x_n^{m_i})$ of any other group member is then strictly decreasing in its own effort x_i^l independent of $Q_{/i}$. Therefore, $x_i^k = 0$, $x_i^l = 0$ are mutually best responses for all group members, independent of the behavior of other groups reflected in $Q_{/i}$.

□

4.B Proof of Lemma 4.2

Proof. Suppose $i \notin \underline{N_0}$. If there exists a solution to the FOC, it is characterized by the following equation, obtained by solving (4.18) for y_i^l and summing over all l,

$$Y_i = y_i^k \cdot \sum_l \left(\frac{v_i^l}{v_i^k}\right)^{\frac{\gamma_i}{1-\gamma_i}}. \tag{4.23}$$

We can now solve equation (4.15) for Y_i explicitly:

$$Y_i = \left(\sqrt{Q_{/i} \cdot V_i(\gamma_i)} - Q_{/i}\right)^{\gamma_i}. \tag{4.24}$$

Thus, the condition for a strictly interior solution is $\left(\sum_l v_i^{l\,\frac{\gamma_i}{1-\gamma_i}}\right)^{\frac{1-\gamma_i}{\gamma_i}} > Q_{/i}$. Note that this condition is the same for all members of a group. In all other cases, we get $y_i^k = 0$ for $\gamma_i \in (0,1)$ and $y_i^k = \infty$ for $\gamma_i \in (-\infty,0)$ as was to be expected and which corresponds to $x_i^k = 0$. In these cases we have $\forall l : y_i^k = y_i^l$ by equation (4.18) and by the definition of Q_i, we have: $Q_i = Y_i^{\frac{1}{\gamma_i}} = 0$. We can write a group best-response function as

$$\hat{Q}_i(\gamma_i, Q_{/i}) = \max\left(0, \sqrt{Q_{/i} \cdot V_i(\gamma_i)} - Q_{/i}\right). \tag{4.25}$$

establishing part a), since by Lemma 1 either for all group members we obtain an interior solution or for none. Since the best-response function is continuous in $\gamma_i \neq 0$ and in the strategies of the other groups $Q_{/i}$, if a unique Nash equilibrium exists, the equilibrium strategies must also be continuous in all γ_i. This establishes part c) of Lemma 2. What remains to be shown is which groups participate in equilibrium given that $\underline{N_0}$ do not participate. Suppose a group ζ participates in equilibrium with strictly positive effort, while a group $\zeta+1$ does not participate. Let $Q_i^*(\vec{\gamma}, \underline{N_0})$ be Q_i in equilibrium (the notation ignores here that these are best responses and should thus be functions of $Q_{/i}^*$) and let the other variables introduced above be defined correspondingly in equilibrium. Then by the above condition in equilibrium we have for any given $\vec{\gamma}$:

$$\begin{aligned} V_\zeta(\gamma_i) &> Q_{/\zeta}^*(\vec{\gamma}, \underline{N_0}) \\ V_{\zeta+1}(\gamma_i) &\leq Q_{/\zeta+1}^*(\vec{\gamma}, \underline{N_0}) \end{aligned} \tag{4.26}$$

Since by assumption $Q_{\zeta+1}^*(\vec{\gamma}, \underline{N_0}) = 0$, we have $Q_{/\zeta+1}^*(\vec{\gamma}, \underline{N_0}) = Q^*(\vec{\gamma}, \underline{N_0})$. Solving (4.6) for $Q_{/i}$ tells us that in an equilibrium where group ζ participates, the following needs to be true:

$$Q_{/\zeta}^*(\vec{\gamma}, \underline{N_0}) = \frac{Q^*(\vec{\gamma}, \underline{N_0})^2}{V_\zeta(\gamma_i)}. \tag{4.27}$$

We now insert (4.27) into the first equation of (4.26) and the condition $\hat{Q}_{/\zeta+1} = \hat{Q}$ into the second equation. Thus, the condition (4.26) becomes

$$\begin{aligned} V_\zeta(\gamma_\zeta) &> Q^*(\vec{\gamma}, \underline{N_0}) \\ V_{\zeta+1}(\gamma_{\zeta+1}) &\leq Q^*(\vec{\gamma}, \underline{N_0}) \end{aligned} \tag{4.28}$$

in equilibrium. It follows that $V_\zeta(\gamma_\zeta) > V_{\zeta+1}(\gamma_{\zeta+1})$. We can thus order the groups such that $V_i(\gamma_i) \geq V_{i+1}(\gamma_{i+1})$ and define $n^*(\vec{\gamma}, \underline{N_0})$ as the group with the highest index number in $N/\underline{N_0}$ that still participates with strictly positive effort. By (4.28), all groups with $i \in N/\underline{N_0}$ and $i \leq n^*(\vec{\gamma})$ participate. This establishes part b) of Lemma 2. □

4.C Convergence Results

We will now state convergence results where for all groups j, γ_j approaches 1, 0, and $-\infty$. X_i^* denotes X_i in equilibrium. Throughout we will assume $\underline{N_0} = \emptyset$.

Proposition 4.4. *For $\gamma_i \to 1^-$, we get $\frac{x_i^{k\,*}}{X_i^*} = 0$ if $\exists a_i^l v_i^l > a_i^k v_i^k$ and $\frac{1}{\sharp\{l : a_i^l \cdot v_i^l = a_i^k \cdot v_i^k\}}$ otherwise.*

Proof. It is straightforward to derive the following equation from (4.7):

$$\frac{x_i^{k\,*}(\vec{\gamma})}{X_i^*(\vec{\gamma})} = \frac{(a_i^k \cdot v_i^k)^{\frac{1}{1-\gamma_i}}}{\sum_l (a_i^l \cdot v_i^l)^{\frac{1}{1-\gamma_i}}} \tag{4.29}$$

For the limit it then holds:

$$\begin{aligned} \lim_{\gamma_i \to 1} \frac{(a_i^k \cdot v_i^k)^{\frac{1}{1-\gamma_i}}}{\sum_l (a_i^l \cdot v_i^l)^{\frac{1}{1-\gamma_i}}} &= \lim_{\gamma_i \to 1} \left(\sum_l \left(\frac{a_i^l \cdot v_i^l}{a_i^k \cdot v_i^k} \right)^{\frac{1}{1-\gamma_i}} \right)^{-1} \\ &= \begin{cases} 0, & \exists a_i^l \cdot v_i^l > a_i^k \cdot v_i^k \\ \frac{1}{\sharp\{a_i^l \cdot v_i^l : a_i^l \cdot v_i^l = a_i^k \cdot v_i^k\}}, & else \end{cases} \end{aligned} \tag{4.30}$$

□

Proposition 4.4 shows that for γ_i increasing towards one, the group members with lower valuations will decrease their efforts towards zero, and only the group members with the highest valuations contribute. If there is more than one individual with the highest valuation, we converge to an equilibrium where those individuals contribute equally. In this case we get multiple equilibria if $\gamma_i = 1$ with the property that the sum of contributions is always identical (Baik, 2008). In this sense, our convergence result can be interpreted as an equilibrium-selection mechanism which selects the equal-contributions equilibrium from the multiple equilibria in Baik (2008).

Next we will analyze the other boundary case when all γ_j approach $-\infty$. In order to have a lean notation we denote $\gamma_j = \gamma$ and $\lim_{\gamma\to-\infty} f(\gamma)$ by $f(-\infty)$ for all functions $f(.)$:

Proposition 4.5. *For $\gamma \to -\infty$, we obtain:*
a) $\lim_{\gamma\to-\infty} V_i(\gamma) = \frac{g_i}{m_i} HM(v_i^1, \dots, v_i^{m_i})$
b) $\lim_{\gamma\to-\infty} \frac{x_i^{k*}(\gamma)}{X_i^*(\gamma)} = \frac{1}{m_i}$
c) $\lim_{\gamma\to-\infty} Q^*(\gamma) = \frac{n^*(-\infty)}{\sum_j \sum_l 1/(v_j^l \cdot g_i)}$
d) x_i^{k*} *is independent of* a_j^l $\forall j, l$
where $HM(v_i^1, \dots, v_i^{m_i}) = \frac{m_i}{\sum_l \frac{1}{v_i^l}}$ *is the harmonic mean of the valuations within the group.*

The results follow directly from the determination of the limit of (4.7).

Since relative strength of groups is determined by V_i, the limit behavior of V_i is of course of great interest. From Proposition 4.5 b) we see that the distribution and level of relative strengths a_i^k of each group member have no effect on V_i. The irrelevance of a_i^k is further shown by part d) of the proposition, where we see that even equilibrium efforts x_i^{k*} are unaffected by a_i^k. This was to be expected, since under perfect complements in fact all inputs are crucial for the level of q_i. Proposition 4.5 b) shows that (as expected given the results by Lee (2012)) all group members participate with equal amounts. In this sense, for γ near $-\infty$, we obtain similar results as for a min(...) impact function. However, this function creates multiple equilibria with an associated equilibrium-selection problem. Given the uniqueness of equilibria for all finite $\vec{\gamma}$, our limit result can be interpreted as an equilibrium-selection mechanism where individual contributions depend on the harmonic mean of the valuations.

Next we look at the limit behavior for $\gamma \to 0$. It turns out that we have to consider $\gamma \to 0^+$ and $\gamma \to 0^-$ separately because the problem may not be continuous at this point.

Proposition 4.6. *At $\gamma_i = 0$, $V_i(\gamma_i)$ is discontinuous if $\sum_l a_i^l \neq 1$.*

$$\lim_{\gamma_i \to 0^+} V_i = \begin{cases} \infty, & \sum a_i^k > 1 \\ \prod (a_i^k \cdot v_i^k)^{a_i^k}, & \sum a_i^k = 1 \\ 0, & \sum a_i^k < 1 \end{cases}, \tag{4.31}$$

$$\lim_{\gamma_i \to 0^-} V_i = \begin{cases} 0, & \sum a_i^k > 1 \\ \prod (a_i^k \cdot v_i^k)^{a_i^k}, & \sum a_i^k = 1 \\ \infty, & \sum a_i^k < 1 \end{cases}. \tag{4.32}$$

Since the winning probability, the equilibrium efforts, and impacts are all functions of all V_i, it follows that these values will in general also be discontinuous in γ_i. In particular, the winning probability and the participation condition of group i are increasing functions of V_i. For $\gamma \to 0^+$ the group with the strictly highest $\sum a_i^k$ will therefore win with probability one while for $\gamma \to 0^-$ the group with the strictly lowest $\sum a_i^k$ will win with probability one. Only if all groups have $\sum a_i^k = 1$, these effects do not occur and we obtain for V_i the a_i^k-weighted geometric mean of $v_i^k \cdot a_i^k$. To obtain a proper intuition for the behavior near $\gamma = 0$, it is helpful to show an example.

Example 3: Assume that $v_1 = v_2$ but allow for differences in group size with $m_i > 1$. Further, we fix $a_i^k = 1$, $g_i = g_j = 1$, and $\gamma_i = \gamma$. Therefore, we are always in a situation with $\sum a_i^k = m_i > 1$. In this case, (4.7) implies

$$x_1(m_1,m_2,\gamma,v) = \frac{v \cdot m_1^{\frac{1-2\gamma}{\gamma}} \cdot m_2^{\frac{1-\gamma}{\gamma}}}{\left(m_1^{\frac{1-\gamma}{\gamma}} + m_2^{\frac{1-\gamma}{\gamma}}\right)^2},\ x_2(m_1,m_2,\gamma,v) = \frac{v \cdot m_1^{\frac{1-\gamma}{\gamma}} \cdot m_2^{\frac{1-2\gamma}{\gamma}}}{\left(m_1^{\frac{1-\gamma}{\gamma}} + m_2^{\frac{1-\gamma}{\gamma}}\right)^2}, \tag{4.33}$$

in a within-group symmetric equilibrium. In this case, individual efforts depend on the size of the groups. Coming back to Example 2, (4.33) can be used to determine that the values of the impact functions are

$$q_1(m_1,m_2,\gamma,v) = v \cdot \frac{m_1^{\frac{1-\gamma}{\gamma}} \cdot m_2^{\frac{1-\gamma}{\gamma}}}{\left(m_1^{\frac{1-\gamma}{\gamma}} + m_2^{\frac{1-\gamma}{\gamma}}\right)^2}, \quad q_2(m_1,m_2,\gamma,v) = v \cdot \frac{m_1^{\frac{1-\gamma}{\gamma}} \cdot m_2^{\frac{1-\gamma}{\gamma}}}{\left(m_1^{\frac{1-\gamma}{\gamma}} + m_2^{\frac{1-\gamma}{\gamma}}\right)^2},$$

which in turn can be used to determine the equilibrium winning probabilities:

$$p_1(m_1,m_2,\gamma) = \frac{m_2^{\frac{\gamma-1}{\gamma}}}{m_1^{\frac{\gamma-1}{\gamma}} + m_2^{\frac{\gamma-1}{\gamma}}}, \quad p_2(m_1,m_2,\gamma) = \frac{m_1^{\frac{\gamma-1}{\gamma}}}{m_1^{\frac{\gamma-1}{\gamma}} + m_2^{\frac{\gamma-1}{\gamma}}}. \tag{4.34}$$

The limit behavior of these probabilities is

$$\lim_{\gamma \to 0^-} p_1(m_1,m_2,\gamma) = \begin{cases} 1, & m_1 < m_2 \\ 0, & m_1 > m_2 \end{cases},$$

$$\lim_{\gamma \to 0^+} p_1(m_1,m_2,\gamma) = \begin{cases} 0, & m_1 < m_2 \\ 1, & m_1 > m_2 \end{cases},$$

and analogously for $p_2(m_1,m_2,\gamma)$. Figure 4.C.1 shows $p_1(m_1,m_2,\gamma)$ (dashed line) and $p_2(m_1,m_2,\gamma)$ (solid line) for the case $m_1 > m_2$. We will focus on $p_1(m_1,m_2,\gamma)$ in the following. The graph starts at 0.5 at $\gamma = 1$. This is the well-known case where group size has no impact on the winning probability (Baik, 2008). $p_1(m_1,m_2,\gamma)$ steadily rises to 1 as γ converges to 0. At this point it jumps to 0 and increases to 0.5 again as γ converges to $-\infty$. In this

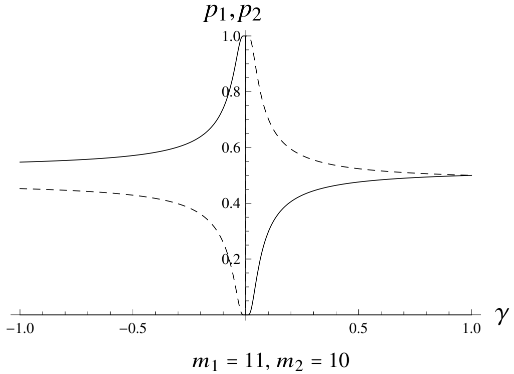

Figure 4.C.1: Equilibrium probabilities for different values of γ .

case, group-size again does not matter because only the minimum contribution counts (Lee, 2012). As evident from the left panel of Figure 4.C.2, for the

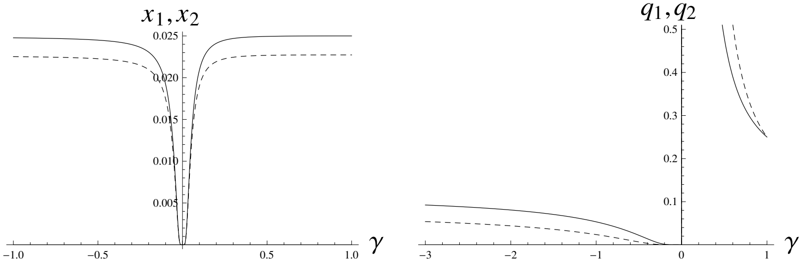

Figure 4.C.2: Effort levels (left) and impacts (right) for different values of γ.

smaller group the efforts are larger over the whole range of γ. Therefore, the changes in the winning probability at $\gamma = 0$ are due to a changing productivity of the larger and the smaller group with γ. This is evident from the right panel of Figure 4.C.2, where the impact of group 2 is consistently higher than the one of group 1 for $\gamma < 0$ and vice versa for $\gamma > 0$. The driving force behind these results is thus the CES function which for $\sum a_i^k \neq 1$ changes not only the degree of complementarity with γ but also the efficiency as becomes apparent when inserting $x_i^k = x_i$ and $\gamma_i = \gamma$ into the impact function:

$$q_i(x_i, ..., x_i) = g_i \cdot x_i \cdot \left(\sum_{k=1}^{m_i} a_i^k \right)^{1/\gamma} . \tag{4.35}$$

Whenever $\sum a_i^k > 1$, the function becomes infinitely large for $\gamma \to 0^+$ and infinitely small for $\gamma \to 0^-$. The rate of convergence depends on the sum of all a_i^k, which was smaller for group 2 in the above case. Therefore it had a disadvantage for positive γ and an advantage for negative γ.

5 Determinants of the Group-Size Paradox[1]

Martin Kolmar
Hendrik Rommeswinkel

This paper analyzes the occurrence of the group-size paradox in situations in which groups compete for rents, allowing for degrees of rivalry of the rent among group members. We provide two intuitive criteria for the group-impact function which for groups with homogeneous valuations of the rent determine whether there are advantages or disadvantages for larger groups: social-interactions effects and returns to scale. For groups with heterogeneous valuations, the complementarity of group members' efforts and the composition of valuations are shown to play a role as further factors.

[1]We thank Stefan Bühler, Philipp Denter, Jörg Franke, Reto Föllmi, Magnus Hoffmann, Nick Netzer, Marco Runkel, Dana Sisak, and Felix Várdy for very helpful comments.

5.1 Introduction

The group-size paradox is perceived as being a result of an unresolved free-rider problem between group members that becomes the more accentuated the larger the group is. Olson (1965) already discussed the alleged advantage of small interest groups over larger ones. His arguments gave rise to a debate about the so-called group-size paradox, which Esteban and Ray (2001) define as: *"larger groups may be less successful than smaller groups in furthering their interests"* (p.663).

The starting point of our paper is to ask which properties of a conflict environment between groups explain the relative advantage or disadvantage of larger compared to smaller groups. We focus on three properties of the group impact function:[2] social-interactions effects, returns to scale, and complementarities between group-members' efforts. As will become clear throughout the paper, all three technological factors are independent. Since we also allow group members to differ in valuations within the group, a fourth crucial property will be the heterogeneity of the valuations of a group. In order to analyze the impact of group size on group performance we use a comparative-static approach where we ask for the effect of adding an additional set of group members to a given group.[3] The main contribution of the paper is the complete characterization of the influence of the above factors on the group-size paradox.

Whereas returns to scale, complementarities in efforts and heterogeneity of valuations are standard concepts, the use of the term social-interactions effects has to be clarified.[4] We say that (positive or negative) social-interactions effects

[2]Impact functions are defined as the functions with which individuals transform effort into relative chance of success in a contest (Wärneryd, 2001). Group impact functions correspondingly play the role of production functions with which group members jointly "produce" a higher relative chance of their group winning the contest.

[3]In Appendix K we show that this approach yields the same results as a comparison between groups and argue that it is slightly more general.

[4]The term "social-interactions effect" has a number of different meanings in the literature. Definitions reach from the very narrow concept of direct interdependencies between preferences (Scheinkman, 2008; Bernheim, 1994; Akerlof, 1997) to the very wide concept of aggregative games (e.g. Manski, 2000).

exist if group impact changes in group size while holding the total group effort constant. There are diverse causes for social-interactions effects in contests such as returns to the division of labor, network effects among group members, or learning between group members.

If group members have equal valuations of winning the contest (which may still differ *between* groups), returns to scale and social-interactions effects completely determine whether the group-size paradox occurs or not.[5] Social-interactions effects work in the predictable way: positive social-interactions effects *ceteris paribus* make it less likely that the group-size paradox occurs. Returns to scale play the role of the discriminatory power of the contest and may thus favor either smaller or larger groups, depending on whether the valuation of winning the contest decreases or increases with group size.

Empirical research emphasizes that within-group heterogeneity is an important mediator for the impact of group size on group success (Hardin, 1982; Ostrom, 1997). Despite the fact that there is a growing interest in the influence of heterogeneity within and between groups[6], with only a few exceptions the literature on group contests[7] has either focused on homogenous individuals or on situations where the effort levels of group members are perfect substitutes, i.e. are aggregated by summation.

For the analysis of the case of heterogeneous individuals, this paper employs CES-impact functions with varying degrees of complementarity. In order to analyze this case, we were able to characterize two useful technical properties that help to simplify future research on group conflicts and comparative statics for CES production functions in general. First, the generalized-mean structure of the impact functions maps onto a generalized mean structure of the valuations of group members that explains equilibrium behavior. This is a quite useful technical property because it allows to analyze the impact of the

[5]This claim may appear to be at odds with Esteban and Ray (2001) who focus on convexities in the cost-of effort functions. However, their model is isomorphic to a model with linear costs and nonlinear impact functions that is a special case of our model.

[6]See Esteban and Ray (2008, 2010).

[7]The literature on contests between groups has recently been surveyed by Corchón (2007, Section 4.2), Garfinkel and Skaperdas (2007, Section 7), and Konrad (2009, Chapters 5.5 and 7).

composition of valuations within a group on group performance using properties of generalized means. Second, we derive a theorem for comparative statics of the elasticity of substitution for a ratio of two generalized means over vectors that differ in heterogeneity. This theorem helps us perform comparative statics in the present model, but is applicable in any other setting where such ratios occur, for example New Keynesian models of inflation, where the inflation rate is a ratio of two CES aggregates.

The effect of adding additional individuals to a group ("new" group members as opposed to "old" ones) depends on the relationship between social-interactions effects and returns to scale on the one hand, and the ratio of power means of the valuations of the group with smaller and larger group size on the other hand. The latter effect is new, and we further explore how adding new group members influences this power mean. The above findings give a precise theoretical underpinning for the results by Hardin (1982) and Ostrom (1997): heterogeneity plays an important role for group success if effort levels are imperfect substitutes. As a general conjecture that follows from the above results one would expect that the group-size paradox becomes more likely for higher levels of complementarity between group-members' efforts if adding new group members to an additional group makes the extended group weakly more heterogeneous.

Our paper is most closely related to Esteban and Ray (2001) who argue that in a contest between groups of different sizes, larger groups may profit from cost advantages if the costs of effort are sufficiently convex. In this case, *ceteris paribus*, members of larger groups face sufficiently lower marginal costs that reverse the group-size paradox. This is a very important insight that helps to explain the prevalence of groups in conflicts.[8] Our model differs from the model by Esteban and Ray (2001) in several ways. First, we take a comparative-statics view on the group-size paradox instead of a comparison between groups. As we show in Appendix K, this approach is slightly more general. Moreover, it allows us to transfer our results and methods to other collective action prob-

[8]See also Marwell and Oliver (1993); Pecorino and Temini (2008); Nitzan and Ueda (2009, 2010).

lems as we show in Appendix L. Second, we allow for heterogeneous valuations within a group. Third, given that the model by Esteban and Ray (2001) is isomorphic to a specific contest model with linear costs and impact functions which are sums of concave functions of efforts (Siegel, 2009), their model is a special case of the model analyzed in this paper. In addition, our results are directly relevant for models of Cournot-competition in oligopolistic markets with hyperbolic demand if firms consist of teams (Raab & Schipper, 2009) and team output is some (in general nonadditive) function of team-members' efforts.

The paper is organized as follows. We introduce the model in Section 5.2 and cover the case of homogeneous group members in Section 5.3. In Section 5.4 we allow for heterogeneity of agents and use a CES type impact function to aggregate group members' efforts. We characterize the simultaneous Nash equilibrium and we show the effect of complementarity on the group-size paradox for heterogeneous agents. Section 5.5 concludes.

5.2 The model

Assume that n groups compete for a given rent. Let $\overline{m} \in \mathbb{N}$ be the maximum possible number of group members and let $m_i \in 2,\ldots,\overline{m}$ be the number of individuals in group i where k is the index of a generic member of this group. We refer to the set of group members by $M_i = \{1,\ldots,m_i\}$. The rent can be completely rival or completely non-rival between group members, and every intermediate case where additional group members dilute the value of the rent for the remaining group are also taken into consideration. To cover these cases it suffices to assume that the valuation of the rent for each individual k of group i is a function of the size of the group, $v_i^k(m_i) > 0$. If $v_i^k(\hat{m}_i) < v_i^k(m_i)$, whenever $\hat{m}_i > \tilde{m}_i$, then the rent is partly rival among group members as some degree of crowding is involved as group size is increased. If $v_i^k(\hat{m}_i) = v_i^k(m_i)$ for all $\hat{m}_i, m_i$ the rent is a group-specific public good[9]. In the following it will

[9]See Cornes and Sandler (1996) for a precise and ample discussion of different types of public goods with crowding. Note that the linear case (as for example in Esteban & Ray, 2001) $v_i^k = \alpha \frac{w}{m_i} + (1-\alpha)w'$, where w is the utility from the rival dimensions (with an equal-sharing

be assumed that $v_i^k(\hat{m}_i) \leq v_i^k(m_i)$ whenever $\hat{m}_i > m_i$. [10]

Sometimes it will be necessary to refer to vectors of valuations of (subsets of) the group members: $\vec{v}_{i,M}(m_i) \equiv (v_i^{k_1}(m_i), \ldots, v_i^{k_{\sharp M}}(m_i))$ where $M \subset M_i$ and $v_i^{k_1}(m_i)$ refers to the valuation of the first element of M.[11] This somewhat elaborate notation is necessary since later on we will analyze comparative statics if sets of individuals are added to a group. It is the easiest to think of $\vec{v}_{i,M}(m_i)$ as the vector of valuations of a subset of group members M of group i if the total group size is m_i.

p_i represents the probability of group $i = 1, ..., n$ to win the contest. Individuals can influence the winning probability by contributing effort x_i^k. The group members' efforts are then aggregated by a function $q_i(x_i^1, ..., x_i^{m_i}) = q_i(\vec{x}_i) \geq 0$. Following the literature, it will be called *impact function*. Since we are most of the time interested in comparative statics with respect to the size of a single group, we define a class of impact functions for this group to specify the impact functions which are used at different sizes of the group $\{q_{i,m_i}(x_i^1, ..., x_i^{m_i})\}_{m_i=2}^{\overline{m}}$. The winning probability p_i is a function of these impacts. $p_i(.)$ is called a *contest-success function*. We focus on Tullock-form contest-success functions where the winning probability of a group i is defined as:[12]

Assumption 5.1.

$$p_i(q_1, ..., q_n) = \begin{cases} \dfrac{q_i}{\sum_{j=1}^n q_j}, i = 1, ...n, & \exists j : q_j > 0 \\ \dfrac{1}{n}, & \forall j : q_j = 0 \end{cases}.$$

---

rule being applied) of the rent and w' the utility from the non-rival part, is a special case of our formulation.

[10]It is also possible to consider other cases, but for reasons of space these will only shortly be discussed.

[11]To illustrate this notation assume that group i has three members, $m_i = 3$ and $M_i = (1, 2, 3)$ with valuations $v_i^1(3) = 5$, $v_i^2(3) = 10$, $v_i^1(3) = 15$. Let $M' = (1, 2)$ and $M'' = (2, 3)$ be two subsets of group members. Then, $\vec{v}_{i,M'}(3) = (5, 10)$ and $\vec{v}_{i,M''}(3) = (10, 15)$.

[12]An axiomatic foundation for the Tullock function for group contests can be found in Münster (2009). An interpretation of the Tullock contest as a perfectly discriminatory noisy ranking contest can be found in Fu and Lu (2008).

Further, we impose the following assumptions on the individuals:

Assumption 5.2. Individuals are risk neutral, face linear costs, and maximize their net rent.

Assumptions 5.1 and 5.2 imply that we can write expected utility as:

$$\pi_i^k(x_i^k, \vec{x}_{-x_i^k}) = \frac{q_i(x_i^1, \ldots, x_i^{m_i})}{\sum_{j=1}^n q_j(x_j^1, \ldots, x_j^{m_j})} v_i^k(m_i) - x_i^k. \tag{5.1}$$

We are looking for Nash equilibria of this game where individuals choose their effort x_i^k simultaneously to maximize their expected utility,

$$x_i^{k*} \in \arg\max_{x_i^k} \pi_i^k(x_i^k, \vec{x}^*_{-x_i^k}) \quad \forall i, k. \tag{5.2}$$

where "$*$" refers to equilibrium values and $\vec{x}^*_{-x_i^k}$ to the vector of efforts by all individuals except k in group i. In order to facilitate the analysis, we will focus on situations where a unique Nash equilibrium with respect to the total effort produced by each group exists. Formally,

Assumption 5.3. $q_i(.)$ is at least twice continuously differentiable,
$\forall k, \vec{x}_i$: $\frac{\partial q(\vec{x}_i)}{\partial x_i^k} > 0$,
$\forall k, \vec{x}_i$: $\frac{\partial^2 q(\vec{x}_i)}{\partial (x_i^k)^2} \leq 0$, and
$\forall \lambda \geq 1, k, \vec{x}_i$: $q_i(\lambda \vec{x}_i) \leq \lambda q_i(\vec{x}_i)$.[13]

Assumption 5.4. $q_i(.)$ has symmetric partial derivatives at $\{x, \ldots, x\}$, i.e.

$$\partial q_i(x, \ldots, x)/\partial x_i^k = \partial q_i(x, \ldots, x)/\partial x_i^l \ \forall x \ \forall \ k, l \ \forall \ i$$

Assumption 5.5. If $\vec{x}_i$ is such that $x_i^k > x_i^l$, then $\frac{\partial q_i(\vec{x}_i)}{\partial x_i^k} < \frac{\partial q_i(\vec{x}_i)}{\partial x_i^l}$.

[13] These assumptions rule out impact functions with for example hyperbolic (Cobb-Douglas) or L-shaped (perfect complements) indifference curves. Impact functions with $\partial q_i(0, \ldots, 0)/\partial x_i^k = 0$ usually lead to multiple equilibria because $\{0, \ldots, 0\}$ at the group as well as as the aggregate level is always a Nash equilibrium. See Skaperdas (1992) for an extensive discussion in a somewhat different context. This would cause additional and merely technical problems that would divert attention from the main focus of the paper.

In some of the below results we also need the assumption that the impact functions are homogeneous.

Assumption 5.6. $q_i(.)$ is homogeneous of degree r_i, i.e. $\forall \lambda \geq 0, \vec{x}_i$: $q_i(\lambda \vec{x}_i) = \lambda^{r_i} \cdot q_i(\vec{x}_i)$.

5.3 Homogeneous valuations within groups

Before we turn to the analysis of the effects of group size on winning probabilities, we establish that a unique equilibrium exists. A proof of existence of a Nash equilibrium cannot rely on standard fixed-point arguments because with a Tullock lottery contest the best-response function for an individual k of group i are not well defined if all other groups exert zero effort, and the alternative approach to make use of the aggregative nature of contests does not work because group contests lack such an aggregative structure.[14] The proof of this and all of the following results can be found in the appendix.

Theorem 5.1. *Suppose a contest fulfills Assumptions 5.1, 5.2, 5.3, and 5.4 for all groups i. Then, a Nash equilibrium exists where the equilibrium efforts are symmetric such that* $\forall i, k : x_i^{k*} = x_i^*$. *There exists only one symmetric equilibrium given* $\forall i, k : x_i^{k*} = x_i^*$.

Therefore, under the given assumptions there may exist other, nonsymmetric equilibria. Using a stronger assumption instead of Assumption 5.4 we obtain a unique equilibrium:

Theorem 5.2. *Suppose a contest fulfills Assumptions 5.1, 5.2, 5.3, and 5.5 for all groups i. Then, there exists a unique Nash equilibrium.*

Since Assumption 5.4 is weaker than 5.5, some cases are of course not covered by the latter theorem. Most prominent is the case of additive impact functions where infinitely many equilibria exist in which only the level of total effort of each group is fixed.

[14]See Acemoglu and Jensen (2009) for a definition of aggregative games.

In some cases, a group may decide to exhibit zero effort, which implies that it makes sense to distinguish between active and inactive groups:

Definition 5.1. (**Participation**) An individual k of group i is said to *participate* if $x_i^{k*} > 0$. A group i is said to *participate* if there exists some k such that $x_i^{k*} > 0$. A group is said to *fully participate* if $\forall k : x_i^{k*} > 0$.

The group-size paradox was first discussed by Olson (1965), who stated that *"the larger the group, the farther it will fall short of providing an optimal amount of a collective good"* (p. 35). One particular interpretation of the statement has been given by Esteban and Ray (2001): In a contest environment in which different groups compete for a rent, larger groups should win with lower probability if the group-size paradox was true. We take a comparative-static perspective on the group-size paradox:

Definition 5.2. (**Group-size paradox**) Suppose there are n groups competing for a prize and each group $j \neq i$ consists of a set $M_j = \{1, \ldots, m_j\}$ of individuals with equal valuations v_j. Let group i have either members $M_i = \{1, \ldots, m_i\}$ or $\hat{M}_i = \{1, \ldots, \hat{m}_i\}$ with $m_i < \hat{m}_i$ with valuations $v_i(m_i)$ and $v_i(\hat{m}_i)$, respectively. Let the corresponding equilibrium winning probabilities be p_i^* and $\hat{p}_i^*$. Then the group-size paradox holds strictly (weakly) for group i at sizes m_i and $\hat{m}_i$ if and only if $p_i^* > (\geq) \hat{p}_i^*$.

In order to have a simple language we will refer to M_i as the "old" group members and to $\Xi_i = \hat{M}_i / M_i$ as the "new" group members in the following. The definition of the group size paradox is therefore local with respect to the original group size m_i and the size of the group after the increase in group members, $\hat{m}_i$. We will give precise conditions under which the group-size paradox occurs but it may well be the case that an impact function is such that the group size paradox only occurs for small group sizes but not for large ones or vice versa. One can naturally also take the perspective of a comparison across groups of different size in the same contest. In Appendix K, we show for all propositions in this paper that the comparative static perspective on the group size paradox yields the same results as a comparison of winning probabilities across groups. However, for some cases that can be analyzed via the comparative-static perspective, no corresponding contest exists which can be analyzed by comparing groups of different size in the same contest. Moreover, using our approach one can also analyze the group-size paradox in other collective action problems, as we show in Appendix L.

We will also consider welfare effects and their relation to the group size paradox.

Definition 5.3. (**Group Welfare**) The total group welfare is defined as the sum of expected utilities $\pi_i^T = \sum_{k \in M_i} \pi_i^k(x_i^{k*}, \vec{x}^*_{-x_i^k})$ and the average group welfare is defined as $\pi_i^A = \frac{1}{m_i}\pi_i^T$.

Next we formulate two intuitive criteria that will turn out to be able to explain the occurrence of the group-size paradox if individuals of a group have identical valuations of the rent. The first one defines the concept of social-interactions effects for within-group symmetric effort contributions.

Definition 5.4. (**Symmetric Social-interactions effects (SSIE)**) A class of impact functions $\{q_{m_i}(x_i^1, \ldots, x_i^{m_i})\}_{m_i=2}^{\overline{m}}$ with m_i being the number of group members is said to have *absent (positive, negative) symmetric social-interactions effects* at effort level x_i for an increase in group size from m_i to $\hat{m}_i$ if it holds that $q_{\hat{m}_i}(\frac{x_i m_i}{\hat{m}_i} \ldots \frac{x_i m_i}{\hat{m}_i}) = (>,<)\ q_{m_i}(x_i, \ldots, x_i)$.

Definition 5.4 can be used to define a measure of SSIE:

Definition 5.5. For a class of impact functions, $\{q_{m_i}(x_i^1, \ldots, x_i^m)\}_{m_i=2}^{\overline{m}}$, SSIE are measured by $s_i(x_i, m_i, \hat{m}_i) = q_{\hat{m}_i}(x_i m_i/\hat{m}_i, \ldots, x_i m_i/\hat{m}_i)/q_{m_i}(x_i, \ldots, x_i)$. SSIE are absent (positive, negative), if and only if $s_i(x_i, m_i, \hat{m}_i) = (>,<)\ 1$.

Note that SSIE are defined as a local measure and may change for different values of x_i, m_i, and $\hat{m}_i$. If impact functions are homogeneous, then $s_i(\ldots)$ does not depend on x_i and can be written as $s_i(m_i, \hat{m}_i)$.

To gain intuition it is instructive to look at an impact function that is the sum of efforts of all group members, $\sum_{k=0}^{m_i} x_i^k$. This function has absent SSIE:

If all group members k exert the same effort $x_i^k = x_i$, then $\sum_{k=1}^{m_i} x_i = m_i \cdot x_i = \sum_{k=1}^{\hat{m}_i} x_i m_i / \hat{m}_i$. In this case, adding additional group members has no influence on the productivity of the group, social-interactions effects are absent.

Another property of an impact function is its returns to scale:

Definition 5.6. (**Returns to scale (RTS)**) A class of impact functions $\{q_{m_i}(\vec{x}_i)\}_{m_i=2}^{\overline{m}}$ is said to have *constant (increasing, decreasing) returns to scale* if $\forall m_i : q_{m_i}(\lambda \vec{x}_i) = (>,<)\ \lambda \cdot q_{m_i}(\vec{x}_i)$ where $\lambda > 0$.

Based on this definition, it is plausible to measure returns to scale in the following way:

Definition 5.7. For a class of homogeneous impact functions, $\{q_{m_i}(\vec{x}_i)\}_{m_i=2}^{\overline{m}}$, returns to scale are measured by the degree of homogeneity r_i, such that for all m_i: $q_{m_i}(\lambda \vec{x}_i) = \lambda^{r_i} \cdot q_{m_i}(\vec{x}_i)$

Note that Definition 5.7 immediately implies that if we speak of a class of impact functions having certain returns to scale, each of the impact functions of this class has the same returns to scale. Further, since we focus on concave impact functions (see Assumption 5.3), the results will only be stated for decreasing or constant returns to scale. However, our results also hold in those cases where even with increasing returns to scale there still exists a unique interior equilibrium. One example would be the case of two groups with symmetric valuations $v_i = v_j$ and $r_i = r_j < 2$.

Both properties, SSIE and RTS are independent: Assume that the impact functions have the generalized CES-form

$$q_{m_i}(\vec{x}_i) = m_i^{s_i + r_i} \cdot \left(\frac{1}{m_i} \sum \left(x_i^k \right)^{\gamma_i} \right)^{r_i/\gamma_i}. \tag{5.3}$$

In this case, we get $q_{\hat{m}_i}(\frac{x_i m_i}{\hat{m}_i}, \ldots, \frac{x_i m_i}{\hat{m}_i}) = (\hat{m}_i / m_i)^{s_i} \cdot q_{m_i}(x_i, \ldots, x_i)$ for the SSIE and $q_{m_i}(\lambda x_i, \ldots, \lambda x_i) = \lambda^{r_i} q_{m_i}(x_i, \ldots, x_i)$ for the RTS, which shows that RTS and SSIE can be chosen independently.

Before presenting the main results of this section it makes sense to discuss them informally. In the graphs in Figure 1, RTS are measured along the ordinate and SSIE, which are independent of efforts under Assumption 5.6, are measured along the abscissa. At the point $\{1,1\}$, SSIE are absent and RTS are constant. Moving right from this point creates positive and moving left creates negative SSIE, moving downwards reduces RTS.

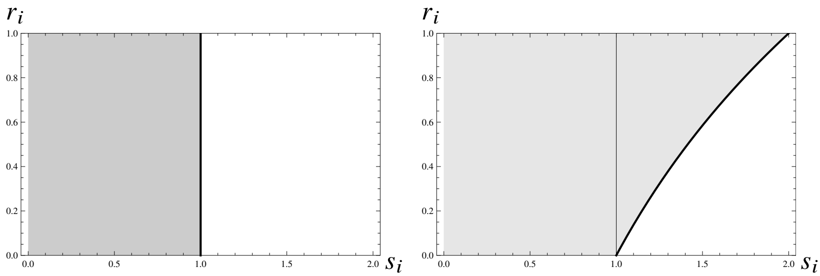

Figure 5.1: Group-size paradox for the case of a non-rival rent (left) and for the case of crowding (right).

The left panel of Figure 1 focuses on the special case that the rent is a pure public good among group members. In the case of a perfectly non-rival rent, only SSIE turn out to be relevant for the occurrence of the group-size paradox: the winning probability decreases in group-size in the left quadrant (shaded gray) whereas it increases in group size in the right quadrant (shaded white).

Allowing for crowding makes the group-size paradox more likely, due to the dilution of per-capita rents that follows from increases in group-size. The separating line moves to the right compared to the case of a pure public good and is given by the upward-sloping line in the right panel of Figure 1. The group-size paradox again holds in the gray shaded areas of the figure, which means the RTS now also play a role. The adverse effect of crowding must be compensated by an increase in SSIE, and the increase has to be the stronger, the larger the RTS. This is due to the fact that the RTS of the impact function control the discriminatory power of the contest with respect to the average valuation of that group. If due to crowding there is an inherent disadvantage from larger group size, then this disadvantage is amplified by higher RTS.

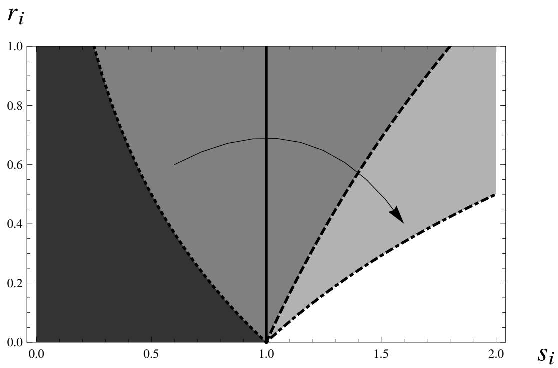

Figure 5.2: Group-size paradox for increasing rivalry of the rent.

Though unlikely, we can also imagine cases where an increase in group size leads to an increase in the valuation. In this case, the RTS will favor the larger group, as evident from the dotted line in Figure 2.

This is again due to the role of the RTS as the discriminatory power, which amplifies the effect of differences in valuation on the winning probabilities. Since larger groups have higher valuations than smaller ones, they are favored by large RTS and therefore the larger the RTS, the lower the SSIE must be in order for the group-size paradox to occur. Figure 2 also shows the effect of an increase in rivalry on the occurrence of the group-size paradox. The dividing line pivots clockwise around the point of zero RTS and absent SSIE. The more rival the rent becomes, the higher the level of SSIE that is necessary to compensate for the increase in the dilution of the rent.

We now turn to the formal presentation of the results.

Proposition 5.1. *Consider two contests fulfilling Assumptions 5.1, 5.2, 5.3, and 5.4 for all groups, which differ only in the group size of group i,* m_i *and* $\hat{m}_i > m_i$*. For all* j,k: $v_j^k = v_j$ *and let the equilibrium winning probabilities in the symmetric equilibrium be* p_i^* *and* $\hat{p}_i^*$*, respectively. Group i participates at group size* m_i *with effort level* x_i^**. The class of impact functions of group i has* $s_i(x_i^*, m_i, \hat{m}_i) = 1$ *and constant or decreasing RTS. If* $v_i(m_i) = v_i(\hat{m}_i)$, *then* $p_i^* = \hat{p}_i^*$, $\pi_i^A < \hat{\pi}_i^A$, *and* $\pi_i^T < \hat{\pi}_i^T$. *If* $v_i(m_i) > v_i(\hat{m}_i)$, *then* $p_i^* > \hat{p}_i^*$ *and* $\exists \underline{v_i(\hat{m}_i)} : \pi_i^T \geq \hat{\pi}_i^T$.

The case of a pure public good establishes a link between our model and the special case of additively linear impact functions which have been standard in the literature so far (e.g. Baik, 2008; Konrad, 2009, Chapters 5.5 and 7). For the case of non-rival rents, the equilibrium group impact and the winning probability are independent of group size as long as the valuation remains unchanged. This leads to a welfare advantage for larger groups. If rents are rival, the increasing dilution of rents (and therefore lower marginal returns) for larger group sizes, bring larger groups into a worse position. If the rent is sufficiently rival, both total and average welfare will decrease after an increase in group size.

The results on the group-size paradox can be strengthened if we assume that the impact functions are homogeneous and allow for SIE. However, welfare effects will be less clear in this case:

Proposition 5.2. *Consider two contests fulfilling Assumptions 5.1, 5.2, 5.3, and 5.4 for all groups, which differ only in the group size of group i,* m_i *and* $\hat{m}_i > m_i$. *For all j,k:* $v_j^k = v_j$ *and let the equilibrium winning probabilities in the symmetric equilibrium be* p_i^* *and* $\hat{p}_i^*$, *respectively. The class of impact functions* $\{q_{m_i}(.)\}_{m_i=2}^{\overline{m}}$ *fulfills Assumption 5.6 with* $s_i(m_i, \hat{m}_i)$ *as the measure of SSIE. Suppose group i participates at group size* m_i. *Then:*

$$p_i^* \gtreqless \hat{p}_i^* \quad \Leftrightarrow \quad \frac{v_i(m_i)}{v_i(\hat{m}_i)} \gtreqless s_i(m_i, \hat{m}_i)^{1/r_i} \tag{5.4}$$

$$\pi_i^A \gtreqless \hat{\pi}_i^A \quad \Leftrightarrow \quad \frac{p_i^* v_i(m_i)}{\hat{p}_i^* v_i(\hat{m}_i)} \gtreqless \frac{\left(1 - (1 - \hat{p}_i^*)\frac{r_i}{\hat{m}_i}\right)}{\left(1 - (1 - p_i^*)\frac{r_i}{m_i}\right)} \tag{5.5}$$

$$\pi_i^T \gtreqless \hat{\pi}_i^T \quad \Leftrightarrow \quad \frac{p_i^* m_i v_i(m_i)}{\hat{p}_i^* \hat{m}_i v_i(\hat{m}_i)} \gtreqless \frac{\left(1 - (1 - \hat{p}_i^*)\frac{r_i}{\hat{m}_i}\right)}{\left(1 - (1 - p_i^*)\frac{r_i}{m_i}\right)}. \tag{5.6}$$

In other words, if for some class of impact functions the group-size paradox holds, then increasing the RTS or decreasing the SSIE further will imply that the group-size paradox still holds if the rent is partly rival. The reverse holds for classes of impact functions for which the group-size paradox does not hold: With crowding, decreasing the RTS or increasing the SSIE will imply that for the new class of impact functions, the group-size paradox also does not hold. It also follows from the proposition as a corollary that RTS play no role in the case of non-rival rents, since in that case the LHS of (5.4) equals one. In the case of $v_i(m_i) < v_i(\hat{m}_i)$, the effect of the SSIE remains the same, but the effect of the RTS is opposite: If for some class of impact functions the group-size paradox holds, then it will *ceteris paribus* continue to hold under lower RTS, but not necessarily under higher RTS. An example where $v_i(m_i) < v_i(\hat{m}_i)$ is meaningful is the case when groups can rely on mechanisms to internalize within-group externalities.[15]

Solving (5.4) for r_i further reveals that (a) the locus of RTS–SSIE pairs that constitute the dividing line between group-size paradox and no group-size paradox has a positive slope in Figures 5.1 and 5.2 and (b) an increase in the privateness of the rent shifts this dividing line in the direction of either more increasing SSIE and/or lower returns to scale as seen in Figure 5.2.

The above analysis shows that SSIE and RTS fully explain the occurrence of the group-size paradox if individuals of the same group have the same valuation of the rent. They enable us to understand how the technological and cultural determinants of group impact influence the relative success of larger or smaller groups.[16] For the non-rival case, the case of absent SSIE is the watershed for the existence of the group-size paradox so that this very simple rule is easy to check empirically. In case that dilution is important, empirical tests

[15]Suppose a group has access to a mechanism solving its collective action problem. In this case, agents fully internalize their effect on the payoff of others and thus optimize as if their valuation of the rent were $v_i(m_i) = \sum_{k=1}^{m_i} v_i^k(m_i)$. Therefore, equilibrium efforts (and thus winning probabilities) will be equal to those obtained in a contest with a homogeneous group and valuations $v_i(m_i)$. If the rent is not too rival and the new group members' valuations are not too low, we will further have $v_i(m_i) < v_i(\hat{m}_i)$ and thus the described case. For details see Kolmar and Wagener (2011).

[16]In Appendix L we show that this also holds for voluntary contriution games as Bergstrom et al. (1986).

are more difficult because the quantitative extent of SSIE becomes important, but it nevertheless gives a clear guideline.

The welfare effects do not follow such a clear pattern, since we cannot solve for them explicitly. While it is obvious that an increase in winning probability ceteris paribus increases average and total group welfare, it becomes clear from setting $p_i^* = \hat{p}_i^*$ in (5.6), that even for equal winning probabilities it is not clear whether $\pi_i^T > \hat{\pi}_i^T$ or the opposite holds. Also, the public good case where in Proposition 5.1 larger groups still held an advantage may have $\pi_i^T > \hat{\pi}_i^T$ if SIE are sufficiently low.

5.4 Heterogeneous valuations within groups

While the literature on the group-size paradox has focused on the case of homogeneous groups, we will now proceed to examine the heterogeneous case. Naturally then, the above mentioned connection between an analysis relying on cost functions and one allowing for more general impact functions with SSIE and RTS as the main properties no longer holds. Since individuals may have different valuations, they may end up with different marginal returns on impact to effort. The following analysis however establishes that SSIE and RTS continue to play the same role, thus generalizing the results from the previous section. We also introduce a further parameter that will gain importance, the complementarity between members' efforts, whose effect depends on the heterogeneity of the new and old group members.

To simplify the analysis, we concentrate on a CES impact function with SSIE given by $s_i(m_i, \hat{m}_i) = (\frac{\hat{m}_i}{m_i})^{s_i}$ and RTS r_i. These properties are fulfilled by the following CES-type impact function:[17]

[17] It could be argued that a class of impact functions should fulfill a condition such as $\forall m_i, \hat{m}_i$: $q_{m_i}(x_i^1, \ldots, x_i^{m_i}) = q_{\hat{m}_i}(x_i^1, \ldots, x_i^{m_i}, 0, \ldots, 0)$. We do not require (and indeed violate) this for the following reason: Consider 100 farmers participating in a demonstration. It may matter for their impact on policies a lot whether they belong to a group of 100 or 1000 farmers. The notion of complementarity captures this: High complementarity means that an interest group of 1000 farmers will only have an impact if all farmers demonstrate and not only a subset. The above condition however violates this intuition.

Assumption 5.7. $q_i(x_i^1,...,x_i^{m_i}) = m_i^{s_i+r_i} \cdot \left(\sum_{l=1}^{m_i} \frac{1}{m_i}(x_i^l)^{\gamma_i}\right)^{r_i/\gamma_i}$, $\gamma_i \in (0,1)$, $r_i \in (0,1]$, $s_i \in \mathbb{R}$ $i = 1,...n$.

γ_i accounts for different elasticities of substitution of the group members' efforts.[18] Since the CES-type impact function is essentially a power mean of the contributions, and power means will play an important role in the following, it is useful to introduce them formally:

Definition 5.8. (**Power Mean**) If $\vec{a}$ is a vector with s elements $a_1, a_2, \ldots, a_s$, then the θ-power mean of $\vec{a}$ is defined as:

$$\mathcal{M}(\vec{a},\theta) \equiv \left(\sum_{i=1}^{s} \frac{a_i^\theta}{s}\right)^{1/\theta}. \tag{5.7}$$

Therefore, we can express the CES-type impact function as: $q_i(\vec{x}_i) = m_i^{s_i+r_i} \cdot \mathcal{M}(\vec{x}_i,\gamma_i)^{r_i}$. To analyze the interplay of γ_i and the heterogeneity of a group, one needs a tractable definition of heterogeneity. The most common idea associated with higher heterogeneity is that of a mean-preserving spread:

Definition 5.9. A vector $\vec{v} = (v^{(1)}, \ldots, v^{(m)})$ is a θ-power mean preserving spread of a vector $\vec{v}' = (v'^{(1)}, \ldots, v'^{(m)})$ if there exist i,j such that $v^{(i)} > v'^{(i)} \geq v'^{(j)} > v^{(j)}$ with $\mathcal{M}(\vec{v}) = \mathcal{M}(\vec{v}')$ and $v^{(k)} = v'^{(k)}$ for all $k \neq i,j$.

The definition of a power mean preserving spread differs from mean preserving spreads by Rothschild and Stiglitz (1970) in two important ways: First, it is generalized to power means since – as discussed before – an arithmetic mean preserving spread of valuations is not always neutral to the winning probability of a group. Second, it is restricted by the assumption of equal weights of each element, since in the CES impact function employed here, all individual efforts have equal weights. We may want to compare groups with different average

[18]We restrict attention to $\gamma_i \in (0, 1)$ to guarantee uniqueness of the equilibrium. If $\gamma_i \leq 0$ multiple equilibria can occur because of a within-group coordination failure: If at least one group member exerts zero effort, group impact is zero and it is rational for the other group members to also exert zero effort. However, Propositions 5.3 and 5.4 continue to hold for $\gamma_i \leq 0$ in all but the extreme equilibrium where all members of all groups exert zero effort.

effort levels and thus employ a slightly more general definition of heterogeneity than power mean preserving spreads:

Definition 5.10. $\vec{v}'$ is more heterogeneous than $\vec{v}$ at mean parameter θ if $\vec{v}'$ is a permutation of $\vec{v''} \cdot \omega$ where $\omega \in \mathbb{R}^+$ and $\vec{v''}$ can be obtained from a sequence of θ-power mean preserving spreads of $\vec{v}$.

According to this definition, a vector is more heterogeneous if it can be obtained from another vector via the application of power mean preserving spreads and multiplying it with a positive constant. From the definition of heterogeneity, the following theorem follows.

Theorem 5.3. *Suppose $\vec{v}'$ is more heterogeneous than $\vec{v}$ at power mean parameter θ, then:*

$$\theta \gtreqless \phi \quad \Leftrightarrow \quad \frac{\mathcal{M}(\vec{v}',\theta)}{\mathcal{M}(\vec{v},\theta)} \gtreqless \frac{\mathcal{M}(\vec{v}',\phi)}{\mathcal{M}(\vec{v},\phi)}$$

While this theorem will be applied in the context of contests in this paper, it is applicable in many other settings with heterogeneity and CES aggregates. For example, it also applies to ratios of price indices P_{t+1}/P_t in models with monopolistic competition and heterogeneous producers (e.g. New Keynesian models such as Yun, 1996). Theorem 5.3 can be used in these contexts to analyze the effects of changes in the elasticity of substitution on the inflation measure if heterogeneity differs across periods. Similarly, growth rates of consumption in models with CES production functions can be analyzed.

It follows from Assumptions 5.1, 5.2, and 5.7 that the individual expected utility functions are as follows:

$$\pi_i^k(x_1^1,\dots,x_n^{m_n}) = v_i^k(m_i)\frac{m_i^{s_i+r_i}\cdot\mathcal{M}(\vec{x}_i,\gamma_i)^{r_i}}{\sum_j m_j^{s_j+r_j}\cdot\mathcal{M}(\vec{x}_j,\gamma_j)^{r_j}} - x_i^k. \qquad (5.8)$$

The Nash equilibrium of this model can only be obtained explicitly for the case $r_i = 1$.[19] For $r_i < 1$ it turns out that comparative statics results can still be

[19]These results are given in Chapter 4. For our purposes, explicit results are not necessary.

derived. We proceed as follows: First, existence and uniqueness of the Nash equilibrium will be proven. Second, it will be shown that the winning probability of a group is strictly increasing in an aggregate valuation V_i of the group (to be determined). This reduces the question of whether the group-size paradox holds to the question whether V_i increases or decreases after adding a set of individuals to the group. Third, we will examine how various combinations of heterogeneity and complementarity affect V_i.

Theorem 5.4. *Suppose a contest fulfills Assumptions 5.1, 5.2, 5.7 for all groups. Then, a unique Nash equilibrium exists in which $\forall r_i < 1$ all groups fully participate and $\forall r_i = 1$, $n^* \geq 2$ groups fully participate.*

Having established existence and uniqueness of the Nash equilibrium, we can now turn to the comparative-static analysis. It follows from the proof of Theorem 5.4 that if Q^* is the equilibrium total impact, the following equilibrium relation must hold for all participating groups i with members M_i:

$$V_i \cdot (1 - p_i^*) = (Q^*)^{1/r_i} \cdot (p_i^*)^{1/r_i - 1}, \tag{5.9}$$

where $V_i \equiv r_i m_i^{s_i/r_i} \cdot \mathcal{M}(\vec{v}_{i,M_i}(m_i), \frac{\gamma_i}{1-\gamma_i})$. Notice that (as discussed in Section 5.2) $\vec{v}_{i,M_i}(m_i)$ is a vector valued function of total group size m_i. For example, if there are three members in M_i, then $\vec{v}_{i,M_i}(5)$ would give the vector of valuations which these three members would have if the actual group size was five. $\mathcal{M}(\vec{v}_{i,M_i}(m_i), \frac{\gamma_i}{1-\gamma_i})$ is then a power mean of these valuations.

In the following the term "average valuation" will refer to this power mean (which does not have to coincide with the arithmetic mean). There is a tight relation between changes in p_i^* and V_i which we can use for comparative statics of our model:

Theorem 5.5. *Consider two contests fulfilling Assumptions 5.1, 5.2, 5.7 for all groups, which differ only by the set of group members of group i, M_i and $\hat{M}_i$ and their valuations $\vec{v}_{M_i}$, $\vec{v}_{\hat{M}_i}$,. Let the equilibrium winning probabilities in each equilibrium be p_i^* and $\hat{p}_i^*$, respectively. Then, $p_i^* \geq \hat{p}_i^*$ if and only if $V_i \geq \hat{V}_i$.*

The above theorem holds for changes in valuations and group size from M_i to $\hat{M}_i$. Hence, we can obtain comparative-static results on p_i^* by only examining the effect of a change in group size on V_i. The question whether a change in group size increases or decreases the winning probability of that group reduces to whether the change in group size increases or decreases V_i. This is a noteworthy result because it implies that the strategic interaction *between* groups has no *qualitative* influence on the comparative-static properties of the model. We show this in Appendix L by extending our results to voluntary contribution games with linear costs of effort.

The following proposition summarizes the effect of adding a set of individuals Ξ_i to group i on its winning probability:

Proposition 5.3. *Consider two contests fulfilling Assumptions 5.1, 5.2, 5.7 for all groups, which differ only by the set of group members M_i and $\hat{M}_i = M_i \cup \Xi_i$. Let the equilibrium winning probabilities in each equilibrium be p_i^* and $\hat{p}_i^*$, respectively. Suppose group i participates at group size m_i. Then $p_i^* \gtrless \hat{p}_i^*$ iff:*

$$\frac{\mathcal{M}(\vec{v}_{i,M_i}(m_i), \frac{\gamma_i}{1-\gamma_i})}{\left(\frac{m_i}{\hat{m}_i} \cdot \mathcal{M}(\vec{v}_{i,M_i}(\hat{m}_i), \frac{\gamma_i}{1-\gamma_i})^{\frac{\gamma_i}{1-\gamma_i}} + (1-\frac{m_i}{\hat{m}_i}) \cdot \mathcal{M}(\vec{v}_{i,\Xi_i}(\hat{m}_i), \frac{\gamma_i}{1-\gamma_i})^{\frac{\gamma_i}{1-\gamma_i}}\right)^{\frac{1-\gamma_i}{\gamma_i}}} \gtrless s_i(m_i, \hat{m}_i)^{1/r_i}. \tag{5.10}$$

The proposition shows that the results for heterogeneous groups are very similar to the ones derived for the case of homogeneous groups given in Proposition 2. However, because of the heterogeneity of the group, there is no longer an obvious choice for the valuation of the new group members. The LHS of the above expression may therefore be smaller than one even with crowding if high-valuation individuals join the group and the crowding effect is not too strong. In this case, the effect of the RTS is opposite to the effect we have observed for homogeneous groups: The higher the RTS, the lower the minimal SSIE such that the group-size paradox does not occur. The addition of new group members with high valuations may therefore compensate for average valuation losses due to crowding. With this exception, all other results from the homogeneous valuation analysis carry over to the heterogeneous case.

The relaxation of the assumption of homogeneous valuations introduces another important property which has an influence on the performance of large and small groups and that is somewhat hidden in Proposition 5.3. The complementarity of efforts influences the degree to which the new group members influence group effort.

To better understand the interplay of group heterogeneity and complementarity, we impose a further assumption to simplify the LHS of (5.10):

Assumption 5.8. The valuation of an individual k in group i is given by $v_i^k(m_i) = w_i^k \alpha_i(m_i)$, where $\alpha_i : \mathbb{N}_+ \to \mathbb{R}$ is a weakly decreasing function.

This assumption encompasses the often used functional form $v_i^k = \alpha w_i^k + (1 - \alpha) w_i^k / m_i$ by setting $\alpha_i(m_i) = \alpha + (1 - \alpha)/m_i$, where α denotes the fraction of the rent that is a public good. Other functional forms are also possible, for example $\alpha_i(m_i) = 1/m_i^{1-\alpha}$ corresponds to Cobb-Douglas preferences of the form $v_i^k = (w_i^k)^\alpha (w_i^k/m_i)^{1-\alpha}$. It is however restrictive in the sense that it does not allow for heterogeneity in the way group members' valuations respond to additional group members which join the group: All valuations in the group are reduced by a common factor.

Assumption 5.8 yields a natural measure for the degree of rivalry in the rent:

Definition 5.11. The degree of rivalry of the rent is measured by:

$$R_i(m_i, \hat{m}_i) = \frac{\alpha_i(m_i)}{\alpha_i(\hat{m}_i)}. \tag{5.11}$$

Notice that $R_i(m_i, \hat{m}_i) \geq 1$ if we focus the analysis on public goods and rents that are partly rival. Also, under homogeneous valuations, this ratio is equivalent to the LHS ratio in (5.4), which neatly extends the homogeneous case.

Assumption 5.8 allows us to simplify (5.10) and obtain comparative statics on γ_i for cases where new and old group members can clearly be ranked in their heterogeneity:

Proposition 5.4. *Consider two contests fulfilling Assumptions 5.1, 5.2, 5.7 for all groups, which differ only by the set of group members M_i and $\hat{M}_i = M_i \cup \Xi_i$. The valuations of M_i fulfill Assumption 5.8. Let the equilibrium winning probabilities in each equilibrium be p_i^* and $\hat{p}_i^*$, respectively. Suppose group i participates for the set of group members M_i.*
a) Then:

$$p_i^* \gtreqless \hat{p}_i^* \quad \Leftrightarrow \quad \Gamma(\gamma_i, m_i, \hat{m}_i, R_i(m_i, \hat{m}_i), s_i, r_i) \gtreqless \frac{\mathcal{M}(\vec{w}_{i,\Xi_i}, \frac{\gamma_i}{1-\gamma_i})}{\mathcal{M}(\vec{w}_{i,M_i}, \frac{\gamma_i}{1-\gamma_i})}. \tag{5.12}$$

where $\Gamma(\dots) \equiv \left(\frac{\hat{m}_i}{\hat{m}_i - m_i} \left(\frac{R_i(m_i,\hat{m}_i)}{s_i(\hat{m}_i/m_i)^{1/r_i}} \right)^{\frac{\gamma_i}{1-\gamma_i}} - \frac{m_i}{\hat{m}_i - m_i} \right)^{\frac{1-\gamma_i}{\gamma_i}}$.
b) Suppose $\vec{w}_{\Xi_i}$ is more heterogeneous than $\vec{w}_{M_i}$ at mean parameter $\frac{\gamma_i}{1-\gamma_i}$. Then:

$$\gamma_i \gtreqless \gamma_i' \quad \Leftrightarrow \quad \frac{\mathcal{M}(\vec{w}_{i,\Xi_i}, \frac{\gamma_i}{1-\gamma_i})}{\mathcal{M}(\vec{w}_{i,M_i}, \frac{\gamma_i}{1-\gamma_i})} \gtreqless \frac{\mathcal{M}(\vec{w}_{i,\Xi_i}, \frac{\gamma_i'}{1-\gamma_i'})}{\mathcal{M}(\vec{w}_{i,M_i}, \frac{\gamma_i'}{1-\gamma_i'})}$$

c) Then $\Gamma(\gamma_i, m_i, \hat{m}_i, R_i, s_i, r_i)$ is weakly decreasing in γ_i, increasing in R_i, and decreasing in s_i. It is strictly decreasing in γ_i if $R_i \neq s_i^{1/r_i}$.

The occurence of the group-size paradox is therefore dependent on the behavior of Γ_i and the ratio of power means on the RHS of (5.12). $\Gamma(.)$ captures the complex interplay between the rivalry of the rent R_i, social interaction effects s_i, returns to scale r_i, and the complementarity in efforts γ_i. Obviously, the SSIE and the rivalry of the rent have opposite effects, with high SSIE making the group-size paradox less likely and higher rivalry making it more likely. This is in line with our results from the first part. $\Gamma(.)$ is decreasing in γ_i, which makes the group-size paradox more likely under lower γ_i when not considering the effect on the RHS.

The effects on the RHS can easily be derived from Theorem 5.3. If Ξ_i is more heterogeneous than M_i, the RHS of (5.12) will increase for a discrete increase in γ_i. We therefore know that in case the new group members are less heterogeneous than the old group members, the effects of γ_i the RHS of (5.12) and Γ_i will work in opposite directions and the total effect remains undetermined. However, if the new group members are equally or more heterogeneous than the former group members, a higher level of complementarity will make the group-size paradox more likely to occur.

5.5 Concluding remarks

Some empirical findings support the existence of a group-size paradox, but as noted by Marwell and Oliver (1993), it also stands in contrast to a significant body of empirical findings pointing to a positive relationship between group size and group performance in conflicts. Oliver (1993) has complained that in most theoretical studies, the results on the group-size paradox depend on some implicit assumptions that drive the result and that the theoretical understanding of the problem is too weak to permit confident conclusions, especially in light of the fact that empirical results reveal complex interactions that prevent simple generalizations. According to our model one can expect that four crucial factors determine the effect of group size on the outcome of a group contest: social-interactions effects, returns to scale, complementarity between group members' efforts, and the composition of their valuations in case of heterogeneous valuations within groups. We are confident that our analysis helps to clarify the different dimensions that contribute to the logic of collective action.

Appendix

5.A Proof of Theorem 5.1

For ease of notation, define $Q_{-i} = \sum_{j \neq i} q_j(\vec{x}_j)$ and $Q = \sum_j q_j(\vec{x}_j)$. The first order conditions (FOCs) for all i, k are:

$$\frac{Q_{-i}}{Q^2} \cdot \frac{\partial q_i}{\partial x_i^k} \cdot v_i(m_i) - 1 \leq 0 \quad \wedge \quad x_i^k \geq 0 \quad \wedge \quad \left(\frac{Q_{-i}}{Q^2} \cdot \frac{\partial q_i}{\partial x_i^k} \cdot v_i(m_i) - 1 \right) \cdot x_i^k = 0. \tag{5.13}$$

We start by showing that the first order conditions are sufficient conditions for an equilibrium. The second order conditions for a local maximum are:

$$\frac{Q_{-i} \cdot v_i(m_i)}{Q^2} \left(\frac{\partial^2 q_i(\vec{x}_i)}{\partial (x_i^k)^2} - \frac{\partial q_i(\vec{x}_i)}{\partial x_i^k} \frac{2}{Q} \right) < 0$$

which holds for $Q_{-i} \geq 0$ and $\frac{\partial^2 q_i(\vec{x}_i)}{\partial (x_i^k)^2} \leq 0$, which holds by Assumption 5.3. Since the above concavity condition holds for all $x_i^k \in [0, \infty)$ we only need to verify that $\pi(\infty, \vec{x}_{/x_i^k}) \leq \pi(x_i^{k*}, \vec{x}_{/x_i^k})$ and $\pi(0, \vec{x}_{/x_i^k}) \leq \pi(x_i^{k*}, \vec{x}_{/x_i^k})$. Since the FOC is strictly decreasing in x_i^k, we must have for all $x_i^k \in [0, x_i^*)$:

$$\frac{\partial \pi_i^k(x_i^k, \vec{x}_{/x_i^k})}{\partial x_i^k} > \frac{\partial \pi_i^k(x_i^*, \vec{x}_{/x_i^k})}{\partial x_i^k}.$$

This means profits are strictly increasing in x_i^k over the interval $[0, x_i^*)$ and thus $\pi(0, \vec{x}_{/x_i^k}) < \pi(x_i^{k*}, \vec{x}_{/x_i^k})$. Further, since $\pi(\infty, \vec{x}_{/x_i^k}) = -\infty < 0 \leq \pi(0, \vec{x}_{/x_i^k})$ the solution to the FOCs indeed yields a global maximum of the expected pay-off for each player.

What is left to show is that there exists a unique solution to the system of FOCs given that $\forall i, k : x_i^{k*} = x_i^*$.[20] By Assumption 5.4 we have for all k, l: $\frac{\partial q_i(x_i,\ldots,x_i)}{\partial x_i^k} = \frac{\partial q_i(x_i,\ldots,x_i)}{\partial x_i^l}$. Therefore, if $\forall i, k : x_i^{k*} = x_i^*$, then the system of FOCs can be reduced to for all i:

$$\begin{aligned} \frac{Q_{-i}}{Q^2} \cdot \frac{\partial q_i(x_i^*, \ldots, x_i^*)}{\partial x_i^1} \cdot v_i(m_i) - 1 \leq 0 \\ \wedge \\ x_i^* \geq 0 \\ \wedge \\ \left(\frac{Q_{-i}}{Q^2} \cdot \frac{\partial q_i(x_i^*, \ldots, x_i^*)}{\partial x_i^1} \cdot v_i(m_i) - 1 \right) \cdot x_i^* = 0\,. \end{aligned} \tag{5.14}$$

We furthermore have the following relation between Q, p_i and x_i:

$$p_i = \frac{q_i(x_i, \ldots, x_i)}{Q} \quad \wedge \quad \underline{p}_i(Q) = \frac{q_i(0, \ldots, 0)}{Q}$$

where $\underline{p}_i(Q)$ is the lower bound on the winning probability given a specific Q. Since $q_i(x_i, \ldots, x_i)$ is strictly increasing in x_i, we can solve this for $x_i(Q, p_i)$ as long as $Q > 0$ and $p_i \geq \underline{p}_i(Q)$. Finally, we can rewrite

$$\frac{\partial q_i(x_i(Q, p_i), \ldots, x_i(Q, p_i))}{\partial x_i^k} = \rho(Q, p_i), \quad \forall p_i \geq \underline{p}_i(Q)$$

which by Assumption 5.3 is weakly decreasing in Q and p_i. The left equation

[20]In a contest with $q_i(0, \ldots, 0) = 0$ at least two groups participate. Since we do no make this assumption, it may be that all groups contribute zero effort because the starting advantage of one group is too large. However, then at least one group will have $q_i(0, \ldots, 0) > 0$.

of the Kuhn-Tucker conditions then becomes:

$$\frac{1-p_i}{Q} \cdot v_i(m_i) \cdot \rho(Q,p_i) - 1 \leq 0, \quad \forall i. \tag{5.15}$$

Since ρ is weakly decreasing in Q and p_i, the LHS of (5.15) is strictly decreasing in p_i and Q. Further, for $p_i \to \infty$, the LHS is negative while for $p_i = q_i(0,\ldots,0)/Q$ it can be negative or positive. Therefore, for each strictly positive Q there exists a unique $p_i \in [0,\infty)$ which solves the Kuhn-Tucker conditions where $p_i = q_i(0,\ldots,0)/Q$ if $v_i(m_i)/Q \cdot \rho(Q,0) \leq 1$ and $p_i = 1$ if $Q = 0$. We can therefore form the function $p_i(Q)$ as the solution to the FOC of each group i.

What remains to be shown is that a unique strictly positive Q^* exists such that the winning probabilities $p_i(Q^*)$ sum to one. Notice that $p_i(Q)$ has the following properties: It is continuous, $\lim_{Q\to 0} p_i(Q) = 1$ and $\lim_{Q\to\infty} p_i(Q) = 0$ and it is strictly decreasing. Therefore, $\sum_i p_i(Q)$ is also strictly decreasing, continuous and has $\lim_{Q\to 0} \sum_i p_i(Q) > 1$ as well as $\lim_{Q\to\infty} \sum_i p_i(Q) = 0$. It follows by the intermediate value theorem that a $Q^* \in (0,\infty)$ exists such that $\sum_i p_i(Q^*) = 1$. Since $\sum_i p_i(Q)$ is strictly decreasing, this Q^* is unique. Given a unique Q^*, we can obtain unique solutions for $p_i(Q^*)$ and thus x_i^* and via $\forall i,k : x_i^{k*} = x_i^*$ also for all x_i^{k*}.

5.B Proof of Theorem 5.2

First notice that Assumption 5.5 implies Assumption 5.4, i.e. we are only considering a subset of the impact functions, therefore the results from the proof of Theorem 5.1 carry over. Since we thus know that the equilibrium is unique given $\forall i,k : x_i^{k*}$, we only need to show that under the more strict Assumption 5.5, any equilibrium must fulfill $\forall i,k : x_i^{k*}$.

From Assumption 5.5 we have that

$$x_i^k > x_i^l \quad \Leftrightarrow \quad \frac{\partial q_i(\vec{x}_i)}{\partial x_i^k} < \frac{\partial q_i(\vec{x}_i)}{\partial x_i^l}.$$

Therefore, in equilibrium it can never be that case that $x_i^{l*} = 0$ if $x_i^{k*} > 0$ since then the above FOC (5.13) does not hold for at least one group member. Thus, either $\forall k : x_i^{k*} = 0$ or $\forall k : x_i^{k*} > 0$. If $\overset{k}{x_i}{}^{*} > 0$ then inserting the FOC for player k into the FOC for player l yields $x_i^{k*} = x_i^{l*}$ and thus the desired condition.

5.C Proof of a useful Lemma

Lemma 5.1. *Suppose a contest fulfills Assumptions 5.1, 5.2, 5.3, and 5.4 for all groups. Further,* $\forall j,k : v_j^k(m_j) = v_j(m_j)$*. Consider two within-group symmetric equilibria, which only differ by the group sizes* $m_i \neq \hat{m}_i$ *and/or the impact functions,* $q_{m_i}(\dots) \neq q_{\hat{m}_i}(\dots)$*. Suppose group* i *participates under group size* m_i *and impact function* $q_{m_i}(\dots)$ *with winning probability* p_i^**. Let the winning probability under group size* $\hat{m}_i$ *and impact function* $q_{\hat{m}_i}(\dots)$ *be* $\hat{p}_i^*$*. Then the following equivalence holds:*

$$p_i^* \gtreqless \hat{p}_i^* \quad \Leftrightarrow \quad v_i(m_i)\frac{\partial q_{m_i}(x_i^*,\dots,x_i^*)}{\partial x_i^k} \gtreqless v_i(\hat{m}_i)\frac{\partial q_{\hat{m}_i}(\hat{x}_i,\dots,\hat{x}_i)}{\partial x_i^k} \tag{5.16}$$

where $\hat{x}_i$ *is defined such that*

$$q_{m_i}(x_i^*,\dots,x_i^*) = q_{\hat{m}_i}(\hat{x}_i,\dots,\hat{x}_i).$$

This Lemma has a very intuitive explanation: Iff for a switch from m_i to $\hat{m}_i$ while holding the winning probability constant the LHS of the FOC is too low, the group will respond by increasing the effort from which a higher winning probability results. The only complication in the proof is that one has to address the possibility of a response by other groups which overcompensates this effect.

Proof. The first-order condition for an interior solution, evaluated at the solution, becomes after rearranging terms:

$$\forall i,k : \quad v_i(m_i)\frac{\partial q_{m_i}(x_i^*,\dots,x_i^*)}{\partial x_i^k} = \frac{Q^*}{(1-p_i^*)}. \tag{5.17}$$

We first show that

$$p_i^* \gtreqless \hat{p}_i^* \quad \Rightarrow \quad v_i(m_i)\frac{\partial q_{m_i}(x_i^*,\ldots,x_i^*)}{\partial x_i^k} \gtreqless v_i(\hat{m}_i)\frac{\partial q_{\hat{m}_i}(\hat{x}_i,\ldots,\hat{x}_i)}{\partial x_i^k} \tag{5.18}$$

and since the cases are exhaustive, the reverse implication is then automatically proven.

$p_i^* > \hat{p}_i^*$: This implies that either $q_{m_i}(x_i^*,\ldots,x_i^*) > q_{\hat{m}_i}(\hat{x}_i^*,\ldots,\hat{x}_i^*)$ or $Q^* < \hat{Q}^*$. We first show that $Q^* < \hat{Q}^*$ yields a contradiction: If $p_i^* > \hat{p}_i^*$ then there exists a group j: $p_j^* < \hat{p}_j^*$. Together with $Q^* < \hat{Q}^*$, this implies that

$$\frac{Q^*}{1-p_j^*} < \frac{\hat{Q}^*}{1-\hat{p}_j^*}.$$

By (5.17) this is equivalent with:

$$v_j\frac{\partial q_j(x_j^*,\ldots,x_j^*)}{\partial x_j^k} < v_j\frac{\partial q_j(\hat{x}_j^*,\ldots,\hat{x}_j^*)}{\partial x_j^k} \tag{5.19}$$

Since $q_j(\ldots)$ has constant or decreasing RTS, this implies $q_j(x_j^*,\ldots,x_j^*) \geq q_j(\hat{x}_j^*,\ldots,\hat{x}_j^*)$. But since $Q^* < \hat{Q}^*$, we have $p_j^* > \hat{p}_j^*$ and thus a contradiction. From this follows that if $p_i^* > \hat{p}_i^*$, then $Q^* \geq \hat{Q}^*$ and $q_{m_i}(x_i^*,\ldots,x_i^*) > q_{\hat{m}_i}(\hat{x}_i^*,\ldots,\hat{x}_i^*)$. The latter implies $\hat{x}_i > \hat{x}_i^*$ via the definition of $\hat{x}_i$. Since q_i has constant or decreasing RTS, we have:

$$\frac{\partial q_{\hat{m}_i}(\hat{x}_i,\ldots,\hat{x}_i)}{\partial x_i^k} \leq \frac{\partial q_{\hat{m}_i}(\hat{x}_i^*,\ldots,\hat{x}_i^*)}{\partial x_i^k} \tag{5.20}$$

From $p_i^* > \hat{p}_i^*$ and $Q^* \geq \hat{Q}^*$ follows $Q^*/(1-p_i^*) > \hat{Q}^*/(1-\hat{p}_i^*)$. Using the FOCs, we have:

$$v_i(m_i)\frac{\partial q_{m_i}(x_i,\ldots,x_i)}{\partial x_i^k} > v_i(\hat{m}_i)\frac{\partial q_{\hat{m}_i}(\hat{x}_i^*,\ldots,\hat{x}_i^*)}{\partial x_i^k}.$$

Combining this equation with 5.20, immediately yields the $p_i^* > \hat{p}_i^*$ part of

(5.18). The proof for $p_i^* < \hat{p}_i^*$ follows the same steps with reverse inequalities and is therefore omitted.

$p_i^* = \hat{p}_i^*$: This implies that either $q_{m_i}(x_i^*, \ldots, x_i^*) = q_{\hat{m}_i}(\hat{x}_i^*, \ldots, \hat{x}_i^*)$ and $Q^* = \hat{Q}^*$ or $q_{m_i}(x_i^*, \ldots, x_i^*) \lesseqgtr q_{\hat{m}_i}(\hat{x}_i^*, \ldots, \hat{x}_i^*)$ and $Q^* \lesseqgtr \hat{Q}^*$.

Suppose $Q^* = \hat{Q}^*$ and $q_{m_i}(x_i^*, \ldots, x_i^*) = q_{\hat{m}_i}(\hat{x}_i^*, \ldots, \hat{x}_i^*)$ hold. Then it immediately follows from the FOCs that

$$v_i(m_i)\frac{\partial q_{m_i}(x_i^*, \ldots, x_i^*)}{\partial x_i^k} = v_i(m_i)\frac{\partial q_{\hat{m}_i}(\hat{x}_i^*, \ldots, \hat{x}_i^*)}{\partial x_i^k}.$$

By definition of $\hat{x}_i$ we then have the symmetric part of (5.16).

For the case $q_{m_i}(x_i^*, \ldots, x_i^*) \lesseqgtr q_{\hat{m}_i}(\hat{x}_i^*, \ldots, \hat{x}_i^*)$ and $Q^* \lesseqgtr \hat{Q}^*$ we can show that this yields a contradiction. It follows from these assumptions that there exists a group j with $p_j^* \gtreqless \hat{p}_j^*$ such that:

$$q_j(x_j^*, \ldots, x_j^*) \gtrless q_j(\hat{x}_j^*, \ldots, \hat{x}_j^*). \tag{5.21}$$

Furthermore,

$$\frac{Q^*}{1 - p_j^*} \gtrless \frac{\hat{Q}^*}{1 - \hat{p}_j^*}.$$

Applying the FOCs gives us:

$$\frac{\partial q_j(x_j^*, \ldots, x_j^*)}{\partial x_j^k} \gtrless \frac{\partial q_j(\hat{x}_j^*, \ldots, \hat{x}_j^*)}{\partial x_j^k}.$$

But from this follows $x_j^* < \hat{x}_j^*$ and thus

$$q_{m_i}(x_j^*, \ldots, x_j^*) \lesseqgtr q_{\hat{m}_i}(\hat{x}_j^*, \ldots, \hat{x}_j^*) \tag{5.22}$$

which contradicts (5.21).

Since the cases considered are exhaustive, it follows that the implication holds in both directions. $\square$

5.D Proof of Proposition 5.1

Proof. We employ the total differential:

$$\Delta q_{m_i}(\vec{x}_i) = \sum_k \Delta x_i^k \frac{\partial q(\vec{x}_i)}{\partial x_i^k} \tag{5.23}$$

For equal inputs we can write $q_{m_i}(x_i,\ldots,x_i) = g(x_i,m_i)$. The total differential then becomes for symmetric efforts:

$$\Delta g(x_i,m_i) = m_i \cdot \Delta x_i \frac{\partial q(x_i,\ldots,x_i)}{\partial x_i^k}, \tag{5.24}$$

and thus

$$\frac{\partial q(x_i,\ldots,x_i)}{\partial x_i^k} = \frac{g(x_i,m_i) - g(x_i,m_i)}{m_i \cdot (x_i - x_i')} \tag{5.25}$$

for $x_i - x_i' \to 0$. Similarly, we have:

$$\frac{\partial q_{\hat{m}_i}(x_i \frac{m_i}{\hat{m}_i},\ldots,x_i \frac{m_i}{\hat{m}_i})}{\partial x_i^k} = \frac{g(x_i \frac{m_i}{\hat{m}_i},\hat{m}_i) - g(x_i' \frac{m_i}{\hat{m}_i},\hat{m}_i)}{m_i \cdot (x_i - x_i') \cdot \frac{m_i}{\hat{m}_i}} \tag{5.26}$$

for $x_i - x_i' \to 0$. It follows from (5.25) and (5.26) that:

$$\frac{\partial q(x_i,\ldots,x_i)}{\partial x_i^k} = \frac{\partial q_{\hat{m}_i}(x_i \frac{m_i}{\hat{m}_i},\ldots,x_i \frac{m_i}{\hat{m}_i})}{\partial x_i^k} \tag{5.27}$$

since by absent SSIE it holds that $g(x_i,m_i) = g(x_i \frac{m_i}{\hat{m}_i},\hat{m}_i)$ and $g(x_i,m_i) = g(x_i' \frac{m_i}{\hat{m}_i},\hat{m}_i)$.

To apply Lemma 5.1, we need to know what the symmetric effort level $\hat{x}_i$ of the group after the increase in size would need to be in order to obtain $q_{m_i}(x_i^*,\ldots,x_i^*) = q_{\hat{m}_i}(\hat{x}_i,\ldots,\hat{x}_i)$. With absent SSIE we have:

$$\hat{x}_i = \frac{x_i^* m_i}{\hat{m}_i}.$$

Lemma 5.1 then yields:

$$p_i^* \gtreqless \hat{p}_i^* \quad \Leftrightarrow \quad v_i(m_i)\frac{\partial q_{m_i}(x_i^*,\ldots,x_i^*)}{\partial x_i^k} \gtreqless v_i(\hat{m}_i)\frac{\partial q_{\hat{m}_i}(\frac{x_i^* m_i}{\hat{m}_i},\ldots,\frac{x_i^* m_i}{\hat{m}_i})}{\partial x_i^k}$$

which given (5.27) reduces to the desired condition:

$$p_i^* \gtreqless \hat{p}_i^* \quad \Leftrightarrow \quad v_i(m_i) \gtreqless v_i(\hat{m}_i)$$

Therefore, if the rent is a public good, the winning probability is independent of group size for an impact function with absent SSIE at the equilibrium effort x_i^* for group size m_i. If the rent is partly private, it is strictly decreasing in group size.

For the welfare effects, we have:

$v_i(m_i) = v_i(\hat{m}_i)$: It follows from $p_i^* = \hat{p}_i^*$ and $s_i(x_i^*, m_i, \hat{m}_i)$ that $x_i^* > \hat{x}_i^*$. Inserting this into $\pi_i^* - \hat{\pi}_i^*$ we get: $(p_i^* - \hat{p}_i^*)(v_i(m_i)) - (x_i^* - \hat{x}_i^*)$. Since the first term is equal to zero and the second term negative, we get that $\pi_i^{k*} < \hat{\pi}_i^{k*}$ from which the statements for average and total utility follow for $v_i(m_i) = v_i(\hat{m}_i)$.

$v_i(m_i) > v_i(\hat{m}_i)$: Let $v_i(\hat{m}_i) \to 0$. Then $\hat{\pi}_i^{k*} \to 0$. Since $\pi_i^{k*} > 0$, and $\hat{\pi}_i^{k*}$ is continuous in $\hat{v}_i(m_i)$ the existence of $\underline{v_i(\hat{m}_i)}$ follows.

We cannot specify $\underline{v_i(\hat{m}_i)}$ more precisely under the very general assumptions. Especially, we cannot know, whether $\underline{v_i(\hat{m}_i)} \lesseqgtr m_i v_i(m_i)/\hat{m}_i$ which is the private good case. □

5.E Proof of Proposition 5.2

Proof. We assume throughout that we are in a symmetric, interior equilibrium. By homogeneity of degree r_i, we have from Euler's theorem

$$r_i \cdot q_{m_i}(x_i,\ldots,x_i) = m_i \cdot x_i \cdot \frac{\partial q_{m_i}(x_i,\ldots,x_i)}{\partial x_i^k}, \tag{5.28}$$

and further

$$\frac{\partial q_{m_i}(x_i,\ldots,x_i)}{\partial x_i^k} = \frac{r_i \cdot q_{m_i}(x_i,\ldots,x_i)}{m_i \cdot x_i} = \frac{r_i \cdot q_{m_i}(1,\ldots,1)}{m_i \cdot (x_i)^{1-r_i}}. \tag{5.29}$$

Using homogeneity of the impact function and the above expression for the partial derivative, we get for the measure of SSIE:

$$s_i(m_i,\hat{m}_i) = \frac{q_{\hat{m}_i}(1,\ldots,1) \cdot m_i^{r_i}}{q_{m_i}(1,\ldots,1) \cdot \hat{m}_i^{r_i}}. \tag{5.30}$$

Lemma 5.1 now tells us that

$$p_i^* \gtreqless \hat{p}_i^* \quad \Leftrightarrow \quad v_i(m_i)\frac{\partial q_{m_i}(x_i^*,\ldots,x_i^*)}{\partial x_i^k} \gtrless v_i(\hat{m}_i)\frac{\partial q_{\hat{m}_i}(\hat{x}_i,\ldots,\hat{x}_i)}{\partial x_i^k} \tag{5.31}$$

where $\hat{x}_i$ is defined such that

$$q_{m_i}(x_i^*,\ldots,x_i^*) = q_{m_i}(\hat{x}_i,\ldots,\hat{x}_i).$$

We can make use of homogeneity of degree r_i and solve for $\hat{x}_i$:

$$\hat{x}_i = \frac{x_i^* m_i}{\hat{m}_i}\left(\frac{q_{m_i}(1,\ldots,1) \cdot \hat{m}_i^{r_i}}{q_{\hat{m}_i}(1,\ldots,1) \cdot m_i^{r_i}}\right)^{1/r_i} = \frac{x_i^* \cdot m_i}{s_i(m_i,\hat{m}_i)^{1/r_i} \cdot \hat{m}_i} \tag{5.32}$$

where the last step follows from (5.30). Plugging this definition back into (5.31), we get using (5.29):

$$p_i^* \gtreqless \hat{p}_i^* \quad \Leftrightarrow \quad v_i(m_i)\frac{r_i \cdot q_{m_i}(x_i^*,\ldots,x_i^*)}{m_i \cdot x_i^*} \gtrless v_i(\hat{m}_i)\frac{r_i \cdot q_{\hat{m}_i}(\hat{x}_i,\ldots,\hat{x}_i)s_i(m_i,\hat{m}_i)^{1/r_i}}{m_i x_i^*} \tag{5.33}$$

By canceling terms, this simplifies to:

$$p_i^* \gtreqless \hat{p}_i^* \quad \Leftrightarrow \quad \frac{v_i(m_i)}{v_i(\hat{m}_i)} \gtrless s_i(m_i,\hat{m}_i)^{1/r_i} \tag{5.34}$$

For the welfare effects, we have:

$\pi_i^A \gtreqless \hat{\pi}_i^A \Leftrightarrow p_i^* v_i(m_i) - x_i^* \gtreqless \hat{p}_i^* v_i(\hat{m}_i) - \hat{x}_i^*$. We furthermore have:

$$x_i^* = \frac{r_i v_i(m_i)}{m_i} p_i^* (1 - p_i^*)$$

from inserting (5.29) into (5.13). Inserting this into the above equation for x_i^* and $\hat{x}_i^*$ and rearranging terms yields the result. The total welfare effects follow by multiplying with m_i and $\hat{m}_i$ on the LHS and RHS, respectively.

□

5.F Proof of Theorem 5.3

A useful result will be the following:

Lemma 5.2. *If $a_1 \geq a_2 > a_3$ or $a_3 > a_2 \geq a_1$ and f is a convex function, then*

$$f(a_1 + a_2 - a_3) > f(a_1) + f(a_2) - f(a_3) \tag{5.35}$$

Proof. Case $a_3 < a_1$: From convexity of f, we have for all h

$$f'(a_1 - a_3 + h) > f'(h) \quad \Leftrightarrow \quad a_3 < a_1 \tag{5.36}$$

Integrating both sides gives:

$$\int_{a_3}^{a_2} f'(a_1 - a_3 + h)dh > \int_{a_3}^{a_2} f'(h)dh \tag{5.37}$$

which yields the desired condition:

$$f(a_1 + a_2 - a_3) - f(a_1) > f(a_2) - f(a_3). \tag{5.38}$$

Case $a_3 > a_1$: From convexity of f, we have for all h

$$f'(a_1 - a_3 + h) < f'(h) \quad \Leftrightarrow \quad a_3 > a_1 \tag{5.39}$$

Integrating both sides gives:

$$\int_{a_2}^{a_3} f'(a_1 - a_3 + h)dh > \int_{a_2}^{a_3} f'(h)dh \tag{5.40}$$

which yields the desired condition:

$$f(a_1) - f(a_1 + a_2 - a_3) < f(a_3) - f(a_2). \tag{5.41}$$

□

The above Lemma can be used to derive the following result:

Lemma 5.3. *Suppose $\vec{v}''$ is obtained from a sequence of θ-power mean preserving spreads of $\vec{v}$. Then*

$$\theta \gtreqless \phi \quad \Leftrightarrow \quad \mathcal{M}(\vec{v},\phi) \gtreqless \mathcal{M}(\vec{v}'',\phi) \tag{5.42}$$

Proof. Suppose $\vec{v}^{(1)},\ldots \vec{v}^{(n)}$ is a sequence of vectors generated by a sequence of θ-power mean preserving spreads. If for all i it holds that

$$\theta \gtreqless \phi \quad \Leftrightarrow \quad \mathcal{M}(\vec{v}^{(i)},\phi) \gtreqless \mathcal{M}(\vec{v}^{(i+1)},\phi) \tag{5.43}$$

then it clearly also holds that:

$$\theta \gtreqless \phi \quad \Leftrightarrow \quad \mathcal{M}(\vec{v}^{(1)},\phi) \gtreqless \mathcal{M}(\vec{v}^{(n)},\phi) \tag{5.44}$$

We therefore only need to show this property for vectors which differ by a single power mean preserving spread. Notice that for any ϕ:

$$\mathcal{M}(\vec{v}^{(i)},\phi) \gtreqless \mathcal{M}(\vec{v}^{(i+1)},\phi) \quad \Leftrightarrow \quad \mathcal{M}((v_H^{(i)},v_L^{(i)}),\phi) \gtreqless \mathcal{M}((v_H^{(i+1)},v_L^{(i+1)}),\phi) \tag{5.45}$$

where $(v_H^{(i)},v_L^{(i)})$ refers to the vector of the two elements that are changed by the spreading operation and $(v_H^{(i+1)},v_L^{(i+1)})$ to the vector of these two elements after application of the spreading operation. Let w.l.o.g. $v_H^{(i)} \geq v_L^{(i)}$ from which immediately follows $v_H^{(i+1)} > v_H^{(i)} \geq v_L^{(i)} > v_L^{(i+1)}$ by the properties of the power mean preserving spread. That is, $v_H^{(i)}$ refers to the element of $\vec{v}^{(i)}$ which is increased to $v_H^{(i+1)}$ and $v_L^{(i)}$ to the decreased element of $\vec{v}^{(i)}$.

From the power mean preserving spread also follows via evaluating (5.45) at equality:

$$\left(\frac{1}{2}(v_H^{(i+1)})^\theta + \frac{1}{2}(v_L^{(i+1)})^\theta\right)^{1/\theta} = \left(\frac{1}{2}(v_H^{(i)})^\theta + \frac{1}{2}(v_L^{(i)})^\theta\right)^{1/\theta}, \tag{5.46}$$

which – after solving for $v_H^{(i+1)}$ – yields:

$$(v_H^{(i+1)})^\theta = \left((v_H^{(i)})^\theta + (v_L^{(i)})^\theta - (v_L^{(i+1)})^\theta\right)^{1/\theta}. \tag{5.47}$$

Combining this condition with (5.45) and (5.43), what is left to show is:

$$\theta \gtreqless \phi \quad \Leftrightarrow \quad \left((v_H^{(i)})^\phi + (v_L^{(i)})^\phi\right)^{1/\phi} \gtreqless \left(\left((v_H^{(i)})^\theta + (v_L^{(i)})^\theta - (v_L^{(i+1)})^\theta\right)^{\phi/\theta} + (v_L^{(i+1)})^\phi\right)^{1/\phi} \tag{5.48}$$

which is implied by the following, more general condition:

$$\forall\, \psi > \eta: \quad \left((v_H^{(i)})^\eta + (v_L^{(i)})^\eta - (v_L^{(i+1)})^\eta\right)^{1/\eta} > \left((v_H^{(i)})^\psi + (v_L^{(i)})^\psi - (v_L^{(i+1)})^\psi\right)^{1/\psi} \tag{5.49}$$

Notice that standard mean inequalities or the reverse Jensen inequality from the previous appendix do not apply to prove (5.49). This would also be counterintuitive as then the proof would not rely on $v_H^{(i)} \geq v_L^{(i)} > v_L^{(i+1)}$. We have to distinguish the cases $\psi > 0$ and $\psi < 0$.

$\psi > 0$: Define $f(a) = a^{\psi/\phi}$, which is strictly convex and $a_1 = (v_H^{(i)})^\phi$, $a_2 = (v_L^{(i)})^\phi$, and $a_3 = (v_L^{(i+1)})^\phi$. If $\phi > 0$, we have $a_1 \geq a_2 > a_3$, while if $\phi < 0$, we have $a_3 > a_2 \geq a_1$. In both cases Lemma 5.2 applies. Employing these definitions in Lemma 5.2 gives:

$$((v_H^{(i)})^\phi + (v_L^{(i)})^\phi - (v_L^{(i+1)})^\phi)^{\psi/\phi} > ((v_H^{(i)})^\phi)^{\psi/\phi} + ((v_L^{(i)})^\phi)^{\psi/\phi} - ((v_L^{(i+1)})^\phi)^{\psi/\phi}, \tag{5.50}$$

which simplifies to (5.49).

$\psi < 0$: Define $f(a) = a^{\phi/\psi}$, which is strictly convex and $a_1 = (v_H^{(i)})^\psi$, $a_2 = (v_L^{(i)})^\psi$, and $a_3 = (v_L^{(i+1)})^\psi$. Since $\phi < 0$, we have $a_3 > a_2 \geq a_1$. Employing these definitions in Lemma 5.2 gives us:

$$((v_H^{(i)})^\psi + (v_L^{(i)})^\psi - (v_L^{(i+1)})^\psi)^{\phi/\psi} > ((v_H^{(i)})^\psi)^{\phi/\psi} + ((v_L^{(i)})^\psi)^{\phi/\psi} - ((v_L^{(i+1)})^\psi)^{\phi/\psi} \tag{5.51}$$

which is equivalent with (5.49) since ϕ is negative and the inequality sign thus changes direction once we exponentiate both sides with ϕ. □

We now turn to the main proof of the theorem.

Proof. Suppose $\vec{v}'$ is more heterogeneous than $\vec{v}$ at mean parameter θ. Then $\exists \omega : \omega \cdot \vec{v}'' = \vec{v}'$ and $\vec{v}''$ is obtained from a sequence of θ-power mean preserving spreads of $\vec{v}$. Thus by Lemma 5.2,

$$\theta \gtreqless \phi \quad \Leftrightarrow \quad \mathcal{M}(\vec{v},\phi) \gtreqless \mathcal{M}(\vec{v}'',\phi) \tag{5.52}$$

By definition of a θ-power mean preserving spread we have equal θ-power means $\mathcal{M}(\vec{v},\theta) = \mathcal{M}(\vec{v}'',\theta)$. We therefore obtain:

$$\theta \gtreqless \phi \quad \Leftrightarrow \quad \frac{\omega \mathcal{M}(\vec{v}'',\theta)}{\mathcal{M}(\vec{v},\theta)} \gtreqless \frac{\omega \mathcal{M}(\vec{v}'',\phi)}{\mathcal{M}(\vec{v},\phi)} \tag{5.53}$$

Making use of the homogeneity of degree 0 of $\mathcal{M}$ and the definition of $\omega \vec{v}''$ we have:

$$\theta \gtreqless \phi \quad \Leftrightarrow \quad \frac{\mathcal{M}(\vec{v}',\theta)}{\mathcal{M}(\vec{v},\theta)} \gtreqless \frac{\mathcal{M}(\vec{v}',\phi)}{\mathcal{M}(\vec{v},\phi)} \tag{5.54}$$

□

5.G Proof of Theorem 5.4

Proof. The proof proceeds similarly to the one for single player contests Cornes and Hartley (2005), with the main difference that first one has to obtain equilibrium conditions that fix relative efforts within each group. The optimality condition for individual k in group i yields:

$$\frac{\partial q_i}{\partial x_i^k} \frac{Q_{-i} \cdot v_i^k}{Q^2} \leq 1 \tag{5.55}$$

with equality if $x_i^k > 0$. Notice first that in equilibrium it can never be the case that $Q_{-i} = 0$, since then some individual will have an incentive to provide effort $x_i^k = \epsilon$ with $\epsilon \to 0$ to win the rent with probability 1. The expression for the partial derivative of the impact function becomes after rearranging terms:

$$\frac{\partial q_i}{\partial x_i^k} = \frac{r_i \cdot Q_i^{1-\gamma/r_i} \cdot (x_i^k)^{\gamma-1}}{m_i^{1-(s_i+r_i)\gamma/r_i}} \tag{5.56}$$

From this expression can already be derived that if one group member participates in equilibrium, all group members do: Notice that if some group member l of group i participates in equilibrium, $Q_i > 0$ and therefore at $x_i^k = 0$, $\frac{\partial q_i}{\partial x_i^k} = \infty$. But then the first order condition cannot hold for individual k in that equilibrium, since we have $Q > 0$ and $Q_{-i} > 0$ and thus the RHS of the optimality condition is infinite, which is greater than the RHS.

Since either all group members participate or none, we can express the following relationship among efforts within a group:

$$(x_i^k)^{\gamma-1} \cdot v_i^k = (x_i^l)^{\gamma-1} \cdot v_i^l \quad \forall l,k \tag{5.57}$$

Notice that this relation trivially also holds for groups that do not participate. Rearranging and summing over all l yields:

$$\left(\frac{1}{m_i} \sum (x_i^l)^\gamma\right)^{1/\gamma} = \frac{x_i^k}{(v_i^k)^{\frac{1}{1-\gamma}}} \left(\frac{1}{m_i} \sum (v_i^l)^{\frac{\gamma}{1-\gamma}}\right)^{1/\gamma} \tag{5.58}$$

Substituting this relation into the optimality condition yields:

$$Q_i^{1-\frac{1}{r_i}} \cdot Q_{-i} \cdot V_i \leq Q^2 \tag{5.59}$$

where $V_i = r_i m_i^{s_i/r_i} \cdot \left(\frac{1}{m_i}\sum_l (v_i^l)^{\frac{\gamma}{1-\gamma}}\right)^{\frac{1-\gamma}{\gamma}}$. We now differentiate our analysis between the cases $\forall i : r_i < 1$ and $\forall i : r_i = 1$.

Notice that for $\forall i : r_i < 1$ we have that $Q_i = 0$ can never be a best response to any positive Q_{-i}, since then the LHS of the above optimality condition is infinite. Therefore, if there exists a Nash equilibrium, it must be such that all groups fully participate. We rewrite the optimality condition therefore in terms of winning probabilities $p_i = Q_i/Q$:

$$V_i \cdot (1 - p_i) = Q^{1/r_i} \cdot p_i^{1/r_i - 1} \tag{5.60}$$

It is now easy to see that for all Q there exists a $p_i(Q)$ such that the optimality condition is fulfilled: Notice that for $p_i = 1$, the LHS is strictly smaller than the RHS, while for $p_i = 0$, the RHS is strictly smaller than the LHS. Since both are continuous functions of p_i, by the intermediate value theorem there then exists at least one $p_i(Q)$ such that the optimality condition holds with equality. Further, this point is unique, since the LHS is strictly decreasing in p_i, while the RHS is strictly increasing in p_i. This proves that there exists a unique best response $p_i(Q)$ to any level of Q. $p_i(Q)$ corresponds to a share function of a single player contest (Cornes & Hartley, 2005), only with the change of interpretation that it is the share of the whole group on which the within-group equilibrium condition (5.57) has been imposed.

Naturally, the remainder of the proof proceeds similarly. $p_i(Q)$ is decreasing in Q as can be verified from the following argument: Suppose Q increases, then the RHS is larger than the LHS of the optimality condition. Since the RHS is strictly increasing in p_i and the LHS strictly decreasing, p_i must decrease in order to maintain equality. A Nash equilibrium is now given by a Q^* such that $\sum_i p_i(Q^*) = 1$. Notice that for $Q = 0$, the solution to the optimality condition is $p_i(0) = 1$, while for $Q \to \infty$, we have that $p_i(\infty) \to 0$. Therefore, $\sum_i p_i(0) > 1 > \sum_i p_i(\infty)$. Since $\sum_i p_i(Q)$ is strictly decreasing in Q and continuous, there must then exist exactly one Q^* such that the equilibrium condition is fulfilled. Thus, there exists a unique Nash equilibrium, where all groups fully participate.

For the case of $r_i = 1$, we instead have the simplified optimality condition:

$$Q_{-i} \cdot V_i \leq Q^2 \tag{5.61}$$

We can therefore directly solve for the best response winning probability:

$$p_i(Q) = \max[0, 1 - Q/V_i] \tag{5.62}$$

Which has the properties $p_i(0) = 1$ and $p_i(V_i) = 0$. Noticing that the best response $p_i(Q)$ is weakly decreasing in Q, we can repeat a similar argument as above to prove that there exists a unique Nash equilibrium: Without loss of generality reorder the groups such that $V_1 > V_2 > \cdots > V_n$. We have

$$\sum_i p_i(0) = n > 1 > 0 = \sum_i p_i(V_1) \tag{5.63}$$

Since $p_1(Q)$ is strictly decreasing in Q for $Q \in [0, V_1]$ and strictly decreasing if , we have that $\sum_i p_i(Q)$ is also strictly decreasing in Q, since it is the sum of a strictly decreasing function and weakly decreasing functions in Q. From this then readily follows existence and uniqueness of a Q^* such that $\sum_i p_i(Q^*) = 1$. Depending on the level of this Q^*, it may very well be for some low enough V_i, that $0 \geq 1 - Q^*/V_i$, such that group i does not participate. Define n^* as the index of the group with the lowest V_i such that $0 > 1 - Q^*/V_i$, which completes the proof. □

5.H Proof of Theorem 5.5

Proof. The proof goes by contradiction. Suppose we have that $V_i \geq \hat{V}_i$ and $p_i^* < \hat{p}_i^*$. Then it follows that $V_i \cdot (1 - p_i^*) > \hat{V}_i \cdot (1 - \hat{p}_i^*)$. By (5.9) this is equivalent to:

$$(Q^*)^{1/r_i} \cdot (p_i^*)^{1/r_i - 1} > (\hat{Q}^*)^{1/r_i} \cdot (\hat{p}_i^*)^{1/r_i - 1}$$

and thus $Q^* > \hat{Q}^*$.

Since $p_i^* < \hat{p}_i^*$ there must exist at least one group j such that: $p_j^* > \hat{p}_j^*$. Therefore, $V_j \cdot (1 - p_j^*) > V_j \cdot (1 - \hat{p}_j^*)$, since V_j does not differ between both equilibria. Using (5.9) for group j gives us:

$$(Q^*)^{1/r_j} \cdot (p_j^*)^{1/r_j - 1} < (\hat{Q}^*)^{1/r_j} \cdot (\hat{p}_j^*)^{1/r_j - 1}.$$

and thus $Q^* < \hat{Q}^*$ which yields a contradiction. By an analogous proof for $V_i \leq \hat{V}_i$ and $p_i^* > \hat{p}_i^*$ then follows the theorem.

□

5.I Proof of Proposition 5.3

Proof. From Theorem 5.5, we have that:

$$p_i^* \gtreqless \hat{p}_i^* \quad \Leftrightarrow \quad r_i m_i^{s_i/r_i} \cdot \mathcal{M}(\vec{v}_{i,M_i}(m_i), \tfrac{\gamma_i}{1-\gamma_i}) \gtreqless r_i (\hat{m}_i)^{s_i/r_i} \cdot \mathcal{M}(\vec{v}_{i,M_i \cup \Xi_i}(\hat{m}_i), \tfrac{\gamma_i}{1-\gamma_i}). \tag{5.64}$$

Rearranging terms, using the definition of $s_i(m_i, \hat{m}_i)$ and writing out the power mean $\mathcal{M}(\vec{v}_{i,M_i \cup \Xi_i}(\hat{m}_i), \frac{\gamma_i}{1-\gamma_i})$ gives us:

$$p_i^* \gtreqless \hat{p}_i^* \quad \Leftrightarrow \quad \frac{\mathcal{M}(\vec{v}_{i,M_i}(m_i), \frac{\gamma_i}{1-\gamma_i})}{\left(\frac{\sum_{k=1}^{m_i} \left(v_i^k(\hat{m}_i)\right)^{\frac{\gamma_i}{1-\gamma_i}} + \sum_{k=m_i+1}^{\hat{m}_i} \left(v_i^k(\hat{m}_i)\right)^{\frac{\gamma_i}{1-\gamma_i}}}{\hat{m}_i} \right)^{\frac{1-\gamma_i}{\gamma_i}}} \gtreqless s_i(m_i, \hat{m}_i)^{1/r_i}, \tag{5.65}$$

which yields:

$$p_i^* \gtreqless \hat{p}_i^* \Leftrightarrow \frac{\mathcal{M}(\vec{v}_{i,M_i}(m_i), \frac{\gamma_i}{1-\gamma_i})}{\left(\frac{m_i}{\hat{m}_i} \sum_{k \in M_i} \frac{\left(v_i^k(\hat{m}_i)\right)^{\frac{\gamma_i}{1-\gamma_i}}}{m_i} + (1 - \frac{m_i}{\hat{m}_i}) \sum_{k \in \Xi_i} \frac{\left(v_i^k(\hat{m}_i)\right)^{\frac{\gamma_i}{1-\gamma_i}}}{\hat{m}_i - m_i} \right)^{\frac{1-\gamma_i}{\gamma_i}}} \gtreqless s_i(m_i, \hat{m}_i)^{1/r_i} \tag{5.66}$$

from which follows (5.10), since $m_i = |M_i|$ and $\hat{m}_i - m_i = |\Xi_i|$. □

5.J Proof of Proposition 5.4

Proof. a) Inserting $R_i(m_i,\hat{m}_i)$ into (5.10) and simplifying gives the desired result.

b) Follows directly from Theorem 5.3 and the assumption that $\vec{w}_{i,\Xi_i}$ is more heterogeneous than $\vec{w}_{i,M_i}$ at $\gamma_i/(1-\gamma_i)$. Note that we only consider cases with $\gamma_i < 1$ and thus $\gamma_i/(1-\gamma_i) < \infty$.

c) It is immediately obvious that $\Gamma(\gamma_i,m_i,\hat{m}_i,R_i,s_i,r_i)$ is increasing in R_i and decreasing in s_i. The only difficulty is thus the proof of the behavior of $\Gamma(\gamma_i,m_i,\hat{m}_i,R_i,s_i,r_i)$ with changes in γ_i. A useful result on which the proof is based is the reverse Jensen inequality (for a more general version and its proof, see Bullen, 2003, p. 43):

Lemma 5.4. *If f is convex, $\varrho_1 > 0$ and $\varrho_i < 0$ for all $2 \geq i \geq n$ and $\sum_{j=1}^{n} \varrho_j = 1$, then $f(\sum_{j=1}^{n} \varrho_j a_j) \geq \sum_{j=1}^{n} \varrho_j f(a_j)$ for all $\sum_{j=1}^{n} \varrho_j a_j \in I$, and the inequality holds strictly, if f is strictly convex and $\exists i,j : a_i \neq a_j$.*

We now show that the following term is decreasing in γ_i:

$$\Gamma(\gamma_i,m_i,\hat{m}_i,R_i,s_i,r_i) = \left(\frac{\hat{m}_i}{\hat{m}_i - m_i} \cdot \left(\frac{R_i(m_i,\hat{m}_i)}{s_i(\hat{m}_i/m_i)^{1/r_i}} \right)^{\frac{\gamma_i}{1-\gamma_i}} - \frac{m_i}{\hat{m}_i - m_i} \right)^{\frac{1-\gamma_i}{\gamma_i}}. \tag{5.67}$$

Define $\theta_i = \frac{\gamma_i}{1-\gamma_i}$, which is increasing in γ_i. What is to be shown is that the above term is decreasing in θ_i, thus:

$$\left(\frac{\hat{m}_i}{\hat{m}_i - m_i} \Psi^{\theta_i} - \frac{m_i}{\hat{m}_i - m_i} \right)^{1/\theta_i} > \left(\frac{\hat{m}_i}{\hat{m}_i - m_i} \Psi^{\phi_i} - \frac{m_i}{\hat{m}_i - m_i} \right)^{1/\phi_i} \tag{5.68}$$

whenever $\phi_i > \theta_i$ and where $\Psi = \frac{R_i(m_i,\hat{m}_i)}{s_i(\hat{m}_i/m_i)^{1/r_i}}$. Note that $\phi_i,\theta_i \in (-1,\infty)$ and therefore we will distinguish the cases $0 < \phi_i$, and $\phi_i < 0$. We will furthermore assume that θ_i and ϕ_i are not zero (which is equivalent to assuming that $\gamma_i \neq 0$.

$0 < \phi_i$: Since ϕ_i is positive, we can rewrite condition (5.68) to:

$$\left(\frac{\hat{m}_i}{\hat{m}_i - m_i}\Psi^{\theta_i} - \frac{m_i}{\hat{m}_i - m_i}(1^{\theta_i})\right)^{\phi_i/\theta_i} > \left(\frac{\hat{m}_i}{\hat{m}_i - m_i}(\Psi^{\theta_i})^{\phi_i/\theta_i} - \frac{m_i}{\hat{m}_i - m_i}(1^{\theta_i})^{\phi_i/\theta_i}\right). \tag{5.69}$$

Setting $f(a) = a^{\phi_i/\theta_i}$ (which is strictly convex also for negative θ_i) and $\varrho_1 = \frac{\hat{m}_i}{\hat{m}_i - m_i}$, $\varrho_2 = -\frac{m_i}{\hat{m}_i - 1}$, and $a_1 = (\Psi)^{\theta_i}$, $a_2 = 1^{\theta_i}$ in the above reverse Jensen inequality directly yields equation (5.69).

$\phi_i < 0$: Since θ_i is negative, we can rewrite condition (5.68) to:

$$\left(\frac{\hat{m}_i}{\hat{m}_i - m_i}\Psi^{\phi_i} - \frac{m_i}{\hat{m}_i - m_i}(1^{\phi_i})\right)^{\theta_i/\phi_i} > \left(\frac{\hat{m}_i}{\hat{m}_i - m_i}(\Psi^{\phi_i})^{\theta_i/\phi_i} - \frac{m_i}{\hat{m}_i - m_i}(1^{\phi_i})^{\theta_i/\phi_i}\right). \tag{5.70}$$

Setting $f(a) = a^{\theta_i/\phi_i}$ (which is strictly convex since the absolute value of ϕ_i is smaller than that of θ_i) and $\varrho_1 = \frac{\hat{m}_i}{\hat{m}_i - m_i}$, $\varrho_2 = -\frac{m_i}{\hat{m}_i - m_i}$, and $a_1 = (\Psi)^{\phi_i}$, $a_2 = 1^{\phi_i}$ in the above reverse Jensen inequality directly yields equation (5.70). Therefore, condition (5.68) follows, which concludes the proof of part c) of the proposition. □

5.K Relation between comparative statics analysis and inter-group comparisons

Theorem 5.6. *Consider two contests fulfilling Assumptions 5.1, 5.2, 5.3, and 5.4 for all groups, which differ only in the group size of group i, m_i and $\hat{m}_i$. Moreover, let $m_j = \hat{m}_i$, $q_j = q_{i,\hat{m}_i}$, and $v_i(\hat{m}_i) = v_j(m_j)$. For all h,k: $v_h^k = v_h$ and let the equilibrium winning probabilities in the symmetric equilibria be p_i^*, p_j^* and $\hat{p}_i^*$, $\hat{p}_j^*$ respectively. Group i participates at group size m_i with effort level x_i^*.*

Then:

$$p_i^* \gtreqless p_j^* \quad \Leftrightarrow \quad p_i^* \gtreqless \hat{p}_i^*$$

The theorem shows that the comparative static interpretation of the group-size paradox and the interpretation of inter-group comparisons yield the same results for equal valuations within groups.

Proof. Define first $\hat{x}$ as the solution to $q_{m_i}(x_i^*,\ldots,x_i^*) = q_{\hat{m}_i}(\hat{x},\ldots,\hat{x})$. By the first order conditions (5.13) evaluated at the equilibrium effort of group i, we have for arbitrary group members k and l of groups i and j, respectively: $p_i^* > p_j^*$ iff

$$v_i(m_i)\frac{\partial q_{i,m_i}(x_i^*,\ldots,x_i^*)}{\partial x_i^k} < v_j(m_j)\frac{\partial q_j(\hat{x},\ldots,\hat{x})}{\partial x_j^l} \tag{5.71}$$

since $\partial q_j(x_j,\ldots,x_j)/\partial x_j^l$ is weakly decreasing in x_j. (5.71) is equivalent with:

$$v_i(m_i)\frac{\partial q_{i,m_i}(x_i^*,\ldots,x_i^*)}{\partial x_i^k} < v_i(\hat{m}_i)\frac{\partial q_{i,\hat{m}_i}(\hat{x},\ldots,\hat{x})}{\partial x_i^k}. \tag{5.72}$$

This is by Lemma 5.1 equivalent with $p_i^* > \hat{p}_i^*$. The proof for $p_i^* = \hat{p}_i^*$ and $p_i^* < \hat{p}_i^*$ is analogous. □

For the heterogeneous case, a similar statement can be made:

Theorem 5.7. *Consider two contests fulfilling Assumptions 5.1, 5.2, 5.7 for all groups, which differ only by the set of group members M_i and $\hat{M}_i = M_i \cup \Xi_i$. Moreover, let $M_j = \hat{M}_i$, $r_i = r_j$, $\gamma_i = \gamma_j$. Let the equilibrium winning probabilities in each equilibrium be p_i^*, p_j^* and $\hat{p}_i^*$, $\hat{p}_j^*$ respectively. Suppose group i participates with group members M_i. Then,*

$$p_i^* \gtreqless p_j^* \quad \Leftrightarrow \quad p_i^* \gtreqless \hat{p}_i^*.$$

To show this, note the following Lemma:

Lemma 5.5. *Consider a contest fulfilling Assumptions 5.1, 5.2, 5.7 for all groups. Suppose $r_i = r_j$. Let the equilibrium winning probabilities in equilibrium be p_i^*, p_j^*. Then,*

$$p_i^* \gtreqless p_j^* \quad \Leftrightarrow \quad V_i \gtreqless V_j.$$

Proof. Suppose $V_i > V_j$, then from (5.9) of groups i and j and $r_i = r_j$ we have:

$$\frac{(p_i^*)^{1/r_i-1}}{(1-p_i^*)} > \frac{(p_j^*)^{1/r_i-1}}{(1-p_j^*)} \tag{5.73}$$

Since the RHS is increasing in p_i (note that $1/r_i - 1 \geq 0$ by assumption) and the LHS in p_j, it follows that $p_i > p_j$. By the symmetry of the problem, for $V_i < V_j$ it follows that $p_i > p_j$. Next suppose $V_i = V_j$, then by (5.9) of groups i and j and $r_i = r_j$ we directly have:

$$\frac{(p_i^*)^{1/r_i-1}}{(1-p_i^*)} = \frac{(p_j^*)^{1/r_i-1}}{(1-p_j^*)} \tag{5.74}$$

which only holds for $p_i^* = p_j^*$. Since the considered cases are exhaustive, it directly follows that: $p_i^* \gtreqless p_j^*$ if and only if $V_i \gtreqless V_j$. □

From here the proof of Theorem 5.7 directly follows from Theorem 5.5 and Lemma 5.5 and the fact that $V_j = \hat{V}_i$.

5.L Extensions of Propositions 5.2 and 5.3 to voluntary contributions games

It turns out that the key properties which have been examined for the group-size paradox in a contest setting are also at work in collective action problems without the contest environment. In this appendix we show that for two collective action problems without the contest environment, our methods and to some extent even the results can be transfered.

We use the model by Bergstrom et al. (1986) with the simplification of identical preferences across players and the generalization of allowing $v_i(m_i)$ to depend on group size.

Assumption 5.9. Individuals k maximize:

$$\pi_i^k(x_i^k, \vec{x}_{-x_i^k})) = u(w - x_i^k, v_i(m_i) q_{m_i}(\vec{x}_i)) \tag{5.75}$$

where u is a binormal utility function increasing in both arguments.

(We could drop the group index i here since there is only one group, but leave it for cross-referencing to our results on group contests.) As discussed in Bergstrom et al. (1986) and Cornes and Hartley (2007), binormality implies that the marginal rate of substitution $MRS_i^k(x_i^k, v_i(m_i)q_{m_i}(\vec{x}_i)) = \frac{\frac{\partial u(\ldots)}{\partial v_i(m_i)q_{m_i}(\vec{x}_i)}}{\frac{\partial u(\ldots)}{\partial w_i - x_i^k}}$ is decreasing in $v_i(m_i)q_{m_i}(\vec{x}_i)$ and non-increasing in x_i^k. Equilibrium existence has been proven by Cornes and Hartley (2007). Symmetry of the equilibrium follows from Assumption 5.4.

We can now obtain similar results to Proposition 5.2:

Proposition 5.5. *Consider two voluntary contribution games fulfilling Assumptions 5.9, 5.3, and 5.4, which differ only in the group size of group i, m_i and $\hat{m}_i > m_i$. For all k: $v_i^k = v_i$ and the class of impact functions $\{q_{m_i}(.)\}_{m_i=2}^{\overline{m}}$ fulfills Assumption 5.6 with $s_i(m_i,\hat{m}_i)$ as the measure of SSIE. Suppose group i contributes $0 < x_i^* < w_i$ at group size m_i and $0 < \hat{x}_i^* < w_i$ at group size $\hat{m}_i$. Then:*

$$(d_i)^{r_i} v_i(m_i) q_i(x_i^*,\ldots,x_i^*) \gtreqless v_i(\hat{m}_i) q_i(\hat{x}_i^*,\ldots,\hat{x}_i^*) \quad \Leftrightarrow \quad 1 \gtreqless s_i(m_i,\hat{m}_i)^{1/r_i} \cdot d_i^{r_i} \tag{5.76}$$

where

$$d_i \gtreqless 1 \quad \Leftrightarrow \quad 1 \gtreqless \frac{v_i(m_i)}{v_i(\hat{m}_i)} \frac{1}{s_i(m_i,\hat{m}_i)} \left(\frac{m_i}{\hat{m}_i}\right)^{r_i} \tag{5.77}$$

In this proposition we used the value-adjusted consumption of the public good as the criterion for the group-size paradox. Results on group welfare are naturally even more difficult to obtain than in the contest case, since they strongly depend on the shape of $u(\ldots)$. Since we assumed a very general form of preferences, we also do not obtain a closed form solution for d_i. However, both SSIE and the rivalness in the rent still work in the predictable manner of making the group-size paradox less and more likely, respectively.[21] We also

[21] An increase in d_i is unanimously good for a larger group: It helps fulfilling the $1 \gtreqless s_i(m_i, \hat{m}_i)^{1/r_i} \cdot d_i^{r_i}$ condition and at the same time increases the critical level $(d_i)^{r_i} v_i(m_i) q_i(x_i^*)$ which will be surpassed if the former condition is met.

see that the term $m_i/\hat{m}_i$ (in the expression determining the orientation of d_i) provides a starting advantage for larger groups. The larger the returns to scale, the more pronounced this starting advantage is.

Proof. We will prove the result only for the case of $MRS(\dots)$ being strictly decreasing in the first argument. The extension to the case where the utility function can be locally linear in the first argument is trivial, but would require many case distinctions.[22] The method of the proof is similar to the contest case. We first find an effort level $\breve{x}_i$ of the group with size $\hat{m}_i$ such that the RHS of the FOC is identical to the RHS of the FOC under the equilibrium efforts x_i^* and then compare the LHS of the FOC to determine whether $\breve{x}_i \gtreqless \hat{x}_i^*$.

The first order condition yields for all k:

$$\frac{\partial q(\vec{x}_i)}{\partial x_i^k} v_i(m_i) = \left(MRS(x_i^k, v_i(m_i) q(\vec{x}_i))\right)^{-1} \tag{5.78}$$

Evaluated in a symmetric equilibrium, we have

$$\frac{\partial q(x_i^*, \dots, x_i^*)}{\partial x_i^k} v_i(m_i) = (MRS(x_i^*, v_i(m_i) q(x_i^*, \dots, x_i^*)))^{-1}. \tag{5.79}$$

Define $\hat{x}_i$ such that $q_{m_i}(x_i^*, \dots, x_i^*) = q_{\hat{m}_i}(\hat{x}_i, \dots, \hat{x}_i)$. A simplification in comparison to the contest model is that we do not need to consider the responses of other groups to a change in efforts after a change in group size. Instead, we face the difficulty that the RHS of the FOC under group size m_i given efforts x_i^* is not identical to the RHS of the FOC under group size $\hat{m}_i$ given efforts $\hat{x}_i$. To obtain an identical RHS we define $\breve{x}_i$ such that:

$$MRS(x_i^*, v_i(m_i) q(x_i^*, \dots, x_i^*)) = MRS(\breve{x}_i, v_i(m_i) q(\breve{x}_i, \dots, \breve{x}_i)). \tag{5.80}$$

We need to determine how $\breve{x}$ compares with $\hat{x}$. For this, define d_i such that $\breve{x}_i = d_i \left(\frac{v_i(m_i)}{v_i(\hat{m}_i)}\right)^{1/r_i} \hat{x}_i$. Note that due to homogeneity of q_i, we have that:

[22] u being linear in $w_i - x_i^k$ implies $d_i = 0$.

$v_i(m_i)q_{m_i}(x_i^*,\dots,x_i^*) = v_i(\hat{m}_i)q_{\hat{m}_i}\left(\frac{\breve{x}_i}{d_i},\dots,\frac{\breve{x}_i}{d_i}\right).$

Next, we have:

$$\frac{\breve{x}_i}{d_i} \gtreqless x_i^* \quad \Leftrightarrow \quad MRS\left(\frac{\breve{x}_i}{d_i}, v_i(\hat{m}_i)q_{\hat{m}_i}\left(\frac{\breve{x}_i}{d_i},\dots,\frac{\breve{x}_i}{d_i}\right)\right) \gtreqless MRS(x_i^*, q_{m_i}(x_i^*,\dots,x_i^*)), \tag{5.81}$$

since the second argument of the two MRS is identical and the MRS is strictly decreasing in the first argument. Since $MRS(x, v(\hat{m}_i)q_{\hat{m}_i}(x,\dots,x))$ is strictly decreasing in x and (5.80), we have that:

$$\frac{\breve{x}_i}{d_i} \gtreqless x_i^* \quad \Leftrightarrow \quad 1 \gtreqless d_i. \tag{5.82}$$

Using (5.32), which holds in virtue of Assumption 5.6, and our definition of d_i we can solve for the left condition as:

$$\frac{v_i(m_i)}{v_i(\hat{m}_i)} \frac{1}{s_i(m_i,\hat{m}_i)} \left(\frac{m_i}{\hat{m}_i}\right)^{r_i} \gtreqless 1 \quad \Leftrightarrow \quad 1 \gtreqless d_i. \tag{5.83}$$

Switching gears, we can now look at what determines whether $\breve{x}_i \gtreqless \hat{x}_i^*$. Since the RHS of the FOC is strictly increasing in x_i and the LHS is strictly decreasing in x_i, we have that:

$$\breve{x}_i \gtreqless \hat{x}_i^* \quad \Leftrightarrow \quad \frac{\partial q_{\hat{m}_i}(\breve{x}_i,\dots,\breve{x}_i)}{\partial x_i^k} v_i(\hat{m}_i) \gtreqless (MRS(\breve{x}_i, q_{m_i}(\breve{x}_i,\dots,\breve{x}_i)))^{-1} \tag{5.84}$$

Substituting the $MRS(\dots)$ term:

$$\breve{x}_i \gtreqless \hat{x}_i^* \Leftrightarrow \frac{\partial q_{\hat{m}_i}(\breve{x}_i,\dots,\breve{x}_i)}{\partial x_i^k} v_i(\hat{m}_i) \gtreqless \frac{\partial q_{m_i}(x_i^*,\dots,x_i^*)}{\partial x_i^k} v_i(m_i) \tag{5.85}$$

Making use of the results up to (5.34), we get via homogeneity of q_{m_i}:

$$\breve{x}_i \gtreqless \hat{x}_i^* \quad \Leftrightarrow \quad \frac{v_i(m_i)}{v_i(\hat{m}_i)} \gtreqless s_i(m_i,\hat{m}_i)^{1/r_i} \left(\frac{\breve{x}_i}{\hat{x}_i}\right)^{r_i} \tag{5.86}$$

Finally, cancelling terms:

$$d_i \hat{x}_i \left(\frac{v_i(m_i)}{v_i(\hat{m}_i)} \right)^{1/r_i} \gtreqless \hat{x}_i^* \quad \Leftrightarrow \quad 1 \gtreqless s_i(m_i, \hat{m}_i)^{1/r_i} (d_i)^{r_i} \tag{5.87}$$

By using (5.32) and homogeneity of q_i, we can simplify the left condition:

$$(d_i)^{r_i} v_i(m_i) q_{m_i}(x_i^*) \gtreqless v_i(\hat{m}_i) q_{\hat{m}_i}(\hat{x}_i^*) \quad \Leftrightarrow \quad 1 \gtreqless s_i(m_i, \hat{m}_i)^{1/r_i} (d_i)^{r_i} \tag{5.88}$$

□

Using a utility function u which is homogeneous in each argument, we could also obtain similar results for the case of heterogeneous valuations. Instead, for a variant of Proposition 5.3 as a voluntary contributions game we assume individuals maximize the following utility function:

Assumption 5.10. Individuals k maximize:

$$\pi_i^k(x_i^k, \vec{x}_{-x_i^k}) = g(q_{m_i}(\vec{x}_i)) v_i^k(m_i) - x_i^k \tag{5.89}$$

where g is twice continuously differentiable and $\frac{\partial g(x)}{\partial x} > 0$ and $\frac{\partial^2 g(x)}{\partial x^2} < 0$.

Note that this model is not covered by Assumptions 5.9 and 5.6, since $g(q(\vec{x}_i))$ is allowed to be non-homogeneous in efforts. However, it assumes linear costs instead. The model can be understood as a voluntary contributions game to some intermediate impact q_i, from which some final good $g(\dots)$ with value $v_i(m_i)$ is produced. A characterization of the group-size paradox just by the properties of q_i is helpful in case we have a clear idea how the intermediate good is produced (e.g. media impact of demonstrations), but not how the final good is produced (e.g. political influence).

Proposition 5.6. *Consider two voluntary contribution games fulfilling Assumptions 5.10, 5.2, 5.7 which differ only by the set of group members M_i and $\hat{M}_i = M_i \cup \Xi_i$. Let the equilibrium efforts in each equilibrium be $\vec{x}_i^* \geq 0$ and*

$\vec{\hat{x}}_i^*$, *respectively. Then:*

$$g(q_{m_i}(\vec{x}_i^*)) \gtreqless g(q_{\hat{m}_i}(\vec{\hat{x}}_i^*)) \quad \Leftrightarrow$$
$$\frac{\mathcal{M}(\vec{v}_{i,M_i}(m_i), \frac{\gamma_i}{1-\gamma_i})}{\left(\frac{m_i}{\hat{m}_i}\cdot\mathcal{M}(\vec{v}_{i,M_i}(\hat{m}_i), \frac{\gamma_i}{1-\gamma_i})^{\frac{\gamma_i}{1-\gamma_i}} + (1-\frac{m_i}{\hat{m}_i})\cdot\mathcal{M}(\vec{v}_{i,\Xi_i}(\hat{m}_i), \frac{\gamma_i}{1-\gamma_i})^{\frac{\gamma_i}{1-\gamma_i}}\right)^{\frac{1-\gamma_i}{\gamma_i}}} \gtreqless s_i(m_i, \hat{m}_i)^{1/r_i}. \tag{5.90}$$

It can easily be verified that the game has a unique Nash equilibrium. For ease of comparison, we again keep the index i for the group even though there is just one group in this game.

Proof. The first order conditions are for all k:

$$\left(g'(q_{m_i}(\vec{x}_i^*))m_i^{s_i+r_i-1} r_i \mathcal{M}(\vec{x}_i, \gamma)^{r_i-\gamma} v_i^k\right) = (x_i^{k*})^{1-\gamma} \tag{5.91}$$

Taking the γ mean over all x_i^k gives us:

$$\left(\sum_k \frac{1}{m_i}\left(g'(q_{m_i}(\vec{x}_i^*))m_i^{s_i+r_i-1} r_i \mathcal{M}(\vec{x}_i^*, \gamma)^{r_i-\gamma} v_i^k\right)^{\frac{\gamma}{1-\gamma}}\right)^{1/\gamma} = \mathcal{M}(\vec{x}_i^*, \gamma) \tag{5.92}$$

Cancelling terms and rearranging yields:

$$m_i^{s_i/r_i} r_i \mathcal{M}(\vec{v}_i, \frac{\gamma}{1-\gamma}) = q_{m_i}(\vec{x}_i^*)^{1/r_i-1}\left(g'(q_{m_i}(\vec{x}_i^*))\right)^{-1} \tag{5.93}$$

We now compare the production of the good between two groups consisting of members M_i and $M_i \cup \Xi_i$. Noting that the RHS of the above equation is strictly increasing in $q_{m_i}(\vec{x}_i^*)$ and $g(q_{m_i}(\vec{x}_i^*))$ is also strictly increasing in $q_{m_i}(\vec{x}_i^*)$, we have:

$$g(q_{m_i}(\vec{x}_i^*)) \gtreqless g(q_{\hat{m}_i}(\vec{\hat{x}}_i^*)) \Leftrightarrow m_i^{s_i/r_i}\mathcal{M}(\vec{v}_{i,M_i}(m_i), \tfrac{\gamma}{1-\gamma}) \gtreqless \hat{m}_i^{s_i/r_i}\mathcal{M}(\vec{v}_{i,M_i\cup\Xi_i}(\hat{m}_i), \tfrac{\gamma}{1-\gamma}) \tag{5.94}$$

Which using the definition of $s_i(m_i, \hat{m}_i)$ can be rewritten as:

$$g(q_{m_i}(\vec{x}_i^*)) \gtreqless g(q_{\hat{m}_i}(\vec{\hat{x}}_i^*)) \Leftrightarrow \frac{\mathcal{M}(\vec{v}_{i,M_i}(m_i), \frac{\gamma}{1-\gamma})}{\mathcal{M}(\vec{v}_{i,M_i \cup \Xi_i}(\hat{m}_i), \frac{\gamma}{1-\gamma})} \gtreqless s_i(m_i, \hat{m}_i)^{1/r_i} \tag{5.95}$$

□

Since the condition for the occurrence of the group-size paradox in this voluntary contribution game is identical to the one from the contest, it follows that also Proposition 5.4 continues to hold in the voluntary contribution game.

It should be noted that the interpretation of the RTS is not as straightforward as in the contest model, where it represented the discriminatory power of the contest. For example, the function g may be of the form $g(x) = x_i^t$ in which case the RTS of the overall model are $r_i + t_i$ instead of r_i. This result must therefore be understood as a decomposition property. If one can rewrite the production function of the collective good as a concave function g applied to the impact produced via a CES aggregate, then equation 5.90 determines whether the group-size paradox occurs. As mentioned above, such a decomposition may be helpful in many cases where we observe the impact of groups (e.g. media attention) but not final outcomes (e.g. political influence).

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Curriculum Vitae

Personal Details

Hendrik Rommeswinkel
Born 7 August 1985 in Herdecke, Germany.
Married to Pei-Hsuan Rommeswinkel, née Wu.

Education

2013-2014	VISITING STUDENT California Institute of Technology, United States of America
2011-2012	SWISS PROGRAM FOR BEGINNING DOCTORAL STUDENTS IN ECONOMICS Study Center Gerzensee, Switzerland
2010-2014	PH.D. IN ECONOMICS AND FINANCE University of St. Gallen, Switzerland
2008-2009	M.SC. IN ECONOMICS AND PHILOSOPHY London School of Economics, United Kingdom
2007-2008	VISITING STUDENT National Taiwan University, Taiwan
2005-2008	B.A. IN ECONOMICS University of St.Gallen, Switzerland