Portfolio Risk Forecasting - On the Predictive Power of Multivariate Dynamic Copula Models

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Summary

A large body of empirical evidence suggests that the dependence structure between financial variables is neither symmetric nor time-stable. Consequentially, forecasts of portfolio risks which neglect asymmetries and time variation in the interdependence of portfolio constituents might yield misleading results. This thesis investigates the impact of modeling such asymmetries and time variations by means of multivariate copulas firstly with regards to in-sample fit and secondly with regards to predictive power when employed to forecast the return distribution of multi-dimensional portfolios. To this end, univariate models which capture asymmetries in volatility and distribution are linked by both symmetric and asymmetric copula models to forecast the risk of investment portfolios. In order to investigate the adequacy of the models for portfolios with different risk/return characteristics, they are applied to an equity index portfolio, a portfolio of commodity futures indices and a multi asset classes index portfolio. To broaden the limited choice of multivariate copulas, scalable copulas are combined into mixture models. To account for time variation in the interdependencies of the portfolio constituents, regime switching and fully dynamic multivariate copulas are constructed. The models are employed in a comprehensive backtesting procedure covering out-of-sample one-week forecasts over 15 years and their predictive power is analyzed. A particular emphasis in the analysis is put on the models' performance during the last financial crisis. Overall, it is found that fully dynamic asymmetric copula models provide superior predictions for the lower tail of the portfolios' return distributions compared to both static and regime switching alternatives.

Zusammenfassung

Umfangreiche empirische Evidenz weist darauf hin, dass die Dependenzstruktur zwischen Finanzvariablen weder symmetrisch, noch zeitstabil ist. Dies lässt vermuten, dass Portfoliorisiko Vorhersagen unter Vernachlässigung von Asymmetrien und zeitlichen Variationen der Interdependenz irreführend sind. Diese Dissertation untersucht, wie sich diese Charakteristiken, wenn modelliert mit multivariaten Copula Modellen, auf die Passgenauigkeit und Vorhersagekraft der Modelle auswirken, wenn letztere zur Risikoprognose von multidimensionalen Portfolios eingesetzt werden. Zu diesem Zweck werden univariate Modelle, welche die Asymmetrien in der Volatilitäten und Randverteilungen abbilden, sowohl mit symmetrischen als auch asymmetrischen Copulas verbunden. Um die Adäquanz der Modelle für Portfolios mit unterschiedlichen Ausprägungen zu untersuchen, werden sie zur Prädiktion der Renditen eines Aktienportfolios, eines Rohstoff-Futures-Portfolios und eines Portfolios aus verschiedenen Anlageklassen eingesetzt. Zur Erweiterung der beschränkten Auswahl an multivariaten Copulas, werden skalierbare Copulas in Konvexkombinationen verbunden. Um der Zeitinstabilität in den Interdependenzen der Portfoliobestandteile Rechnung zu tragen, werden Regime-Switching- und volldynamische Copulas konstruiert. Die Modelle werden zur Vorhersage der einwöchigen Portfoliorenditen verwendet, deren Genauigkeit in einem 15 Jahre überspannenden Testverfahren überprüft und mit besonderem Fokus auf die Performance während der Finanzkrise analysiert wird. Insgesamt zeigt sich, dass die Voraussagen für das untere Ende der Rendite-Verteilung der volldynamischen asymmetrischen Copula Modelle sowohl denjenigen der statischen, als auch der Regime-Switching-Modelle überlegen sind.

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Chapter 1

Introduction

Financial risks emerge from the volatilities and the dependence structure of the assets comprised in a portfolio. Both elements are often estimated simultaneously in a historical covariance matrix, which in standard statistical approaches is mostly assumed to be constant over time. The interdependence is therewith captured by Pearson's linear correlation, which is the optimal measure in case each of the variables is normally distributed. However, there is strong evidence that the univariate distributions of many financial risk factors are non-normal and significantly fat-tailed (see e.g. Christoffersen, 2012; McNeil et al., 2005). Furthermore, there is basically no reason for different marginal variables to have the identical degree of fat-tailedness or even share the type of univariate distribution (Dias and Embrechts, 2010). Assuming a multivariate normal distribution in a non-elliptical world thus neglects or at least substantially underestimates the joint extreme events due to the symmetry and incapability of said distribution to capture tail dependence (Braun, 2011; Embrechts et al., 2002).

With regards to volatility, it is recognized in the literature that financial return series are often heteroscedastic showing alternating clusters of high and low volatility over time. Many scholars further provide evidence of volatility asymmetries, which means that negative news have a larger impact on volatility than positive news (see e.g. Brandt and Kang, 2004; Liu, 2007; Schwert, 1989). Choosing GARCH processes to model the univariate risk factor evolution ensures by construction that the conditional variances of the univariate distributions are time-varying (Bollerslev, 1986). In addition, extensions of the original GARCH specification allow to account for asymmetries in volatility as well as for excess kurtosis and skewness in the univariate distribution.

With regards to dependence, there is substantial evidence suggesting asymmetries in the dependence structure, as negative returns are found to be more dependent than positive returns. This phenomenon has been reported by many previous studies including Patton (2004), Emekter et al. (2009), Erb et al. (1994), Longin and Solnik (2001), Ang and Bekaert (2002b), Ang and Chen (2002) and Das and Uppal (2004). A large body of literature further suggests that the dependence structure of financial variables is not constant over time, but changes in shape and intensity (see e.g. Ang and Bekaert, 2002b; Christoffersen, 2009; Hamilton, 2008; Hsieh and Huang, 2012; Longin and Solnik, 2001; Patton, 2006a). These features are of prime concern for risk management, since inappropriate models for the relationships between financial variables have been identified as an important element of the last financial crisis (Financial Services Authority, 2009; Stöber and Czado, 2012). Yet, the best way to model asymmetries and time variations of the dependence structure is still an open question (Dias and Embrechts, 2010; Embrechts and Hofert, 2014).

The approach of this thesis to account for the outlined characteristics of the financial assets in a portfolio is based on copulas. The use of copula theory allows to separately specify the dependence structure and the marginals and provides entire freedom in combining different marginal distributions (Sklar, 1959). Accordingly, the marginals can be modeled to individually capture the features of each univariate series, while the copula is set to characterize the dependence among them. In contrast to Pearson's linear correlation, copula functions are further capable of capturing nonlinear dependencies and provide a more complete description of the dependence structure.

Even though copula theory dates back to the seminal work of Sklar (1959), the first applications to financial problems did not appear until the beginning of the 21$^{\text{st}}$ century. One of the most influential early publications on copulas in finance is that of Embrechts et al. (2002). Since then, the number of contributions on copulas in finance has increased tremendously. Some of the more extensive examples include Cherubini et al. (2004), McNeil et al. (2005) and Nelsen (2006). Most of the studies applying copulas to financial time series data however assume the dependence structure to be constant over time (Manner and Reznikova, 2012; Silva Filho et al., 2013). This seems unrealistic in the light of the empirical evidence of time-instable dependencies.

Patton (2006a) was among the first to address this issue by allowing copulas to be time-varying and therewith triggered a new and fast-growing line of research. Some of the recent contributions aimed at capturing time instabilities with copulas are Okimoto (2008), Ng (2008), Guégan and Zhang (2010), Dias and Embrechts (2010), Silva Filho et al. (2012) and De Lira Salvatierra and Patton (2013).

However, these studies and with them the majority of research on copulas are conducted on the bivariate level.[1] This may be attributed to the limited choice of multivariate copulas compared to the large number of available copula functions for two dimensions and the latter's ease of application. Numerical issues with bivariate copulas are mostly non-severe and their density function can still be graphically depicted without the loss of information.

Yet, most portfolios of financial assets contain more than two constituents. With higher dimensions, the situation drastically changes, as it becomes increasingly complicated to find a numerically tractable model that is flexible enough to capture real data behavior (Embrechts and Hofert, 2014). According to Patton (2009), the perhaps most difficult direction for research in this area is the extension of bivariate copula-based time series models to higher dimensions.

[1] Note that some authors refer to bivariate copulas as multivariate copulas. In this thesis, the latter designation is reserved for models exceeding two dimensions.

This thesis contributes to find multivariate copula models which are capable of accurately capturing and forecasting real portfolio dynamics. To broaden the limited choice of copulas for higher dimensions, available multivariate copulas are amalgamated into multivariate mixture copulas. The idea is based on Nelsen (2006) and Hu (2006), who created "new" dependence models with convex combinations of existing bivariate copulas. Uniting the diverse features of the enclosed copulas, these mixture models provide an increased flexibility to adapt to the data.

To account for time variations in the dependence structure, two different modeling approaches are followed in this thesis. The first approach is based on Chollete et al. (2009), who proposed a regime switching copula to capture the variations of the dependencies over time. In this setting, the copulas are static within one regime, but differ across the regimes. Since the switches between the regimes cannot be known in advance, they are assumed to be governed by a latent Markov process. The flexibility of this structure is further enhanced firstly by modeling regimes with multivariate mixture copulas and secondly by extending the two-state framework to include a third regime.

The second approach to model time-instable dependencies consists of fully dynamic copulas whose parameters are allowed to vary with every time step. To create dynamic elliptical copulas, the dynamic conditional correlation model (DCC) of Engle (2002) is applied to multivariate elliptical copulas. To build dynamic Archimedean copulas, this thesis follows Braun (2011) in extending Patton's (2006a) dynamic bivariate Archimedean copula model to higher dimensions. Finally, dynamic convex combinations of different dynamic copulas yield multivariate mixture models which are capable to vary the dependence structure in both intensity and shape over time.

In order to scrutinize the suitability of the models for portfolios with different risk/return characteristics, they are applied to a first portfolio of international equity indices, a second one consisting of commodity futures indices and a third portfolio containing indices from multiple asset classes. For each of these portfolios, the importance of accounting for asymmetries and time-variation in the dependence structure is analyzed with regards to fit and predictive power of the different models.

The empirical analysis is divided into two parts: in a first part, the appropriateness of static, regime switching and fully dynamic copulas is investigated in an in-sample setting by ranking the fit of the different structures to the data. In a second part, a comprehensive out-of-sample backtest is conducted to investigate the models' predictive power with regards to forecasting the portfolios' risks during the last 15 years. Finally, the focus of the analysis is narrowed to the performance of the models during the last financial crisis.

Thesis Structure

Taking advantage of copula theory, a two-step approach is followed in order to model the multivariate distribution function: in the first step, the univariate marginals are specified; in the second step, their dependence structure is modeled. The chapters are organized as follows:

The second chapter introduces the models for the univariate time series. Since the univariate models set the basis for the calibration of the multivariate copulas, their suitability for capturing the dynamics of each portfolio constituent plays an important role. To be capable of capturing the characteristics of every financial time series under consideration, three different GARCH processes and a flexible model for the residual distributions are introduced.

In the third chapter, copula theory is outlined and the models for the dependence structures are introduced. At first, the static elliptical and Archimedean copulas are presented as well as their convex combinations into static mixture models. Then, regime switching copulas as a first way to account for time variations in the dependence structure are described. Fully dynamic multivariate elliptical and Archimedean copulas are illustrated next before finally it is shown how these dynamic copulas may be combined into dynamic mixture copula models.

Chapter four lays out the multi-step estimation procedure and the computation of the model parameters' standard errors.

The fifth chapter introduces the three portfolios under consideration and contains the descriptives of the data sets under scrutiny. Furthermore, stationarity and heteroscedasticity of the different return series is investigated.

In the sixth chapter, the in-sample analysis is performed where firstly the univariate processes are estimated and their appropriateness is tested. Secondly, based on the resulting residuals, the static, regime switching and fully dynamic copula models are calibrated and their fit to the data is analyzed.

In chapter seven, the different models are employed to issue forecasts for the risks of the three portfolios whose accuracy is evaluated in a comprehensive backtesting procedure. Since the copula models used to forecast the portfolio profit and loss distribution for a specific time t are based on the same univariate models, differences in the forecast accuracy are only attributable to the differences of the copula functions. This allows for a direct comparison of the dependence models' impact on the forecast accuracy. To investigate the performance of the models in times of distress, their forecast accuracy during the last financial crisis is focused on in a separate section.

Finally, chapter eight draws conclusions on the findings and finalizes this thesis.

Chapter 2

Univariate Models

This chapter presents the models employed to capture the characteristics of the univariate time series. Bollerslev et al. (1994), Christoffersen (2012) and McNeil et al. (2005) point out that financial time series display a number of so-called stylized facts representing a collection of empirical observations and according implications that may be summarized as follows:

1. The time series tend to be uncorrelated although dependent.
2. The autocorrelation function of absolute or squared returns decays very slowly and volatility is stochastic.
3. The series are likely to be asymmetric and tend to have heavy tails. Outsized returns appear in clusters and materialize with a higher frequency than the normal distribution would expect.

Stylized fact number one proposes that the linear relation between consecutive observations is not large. Models which are traditionally utilized in time series analysis to capture (linear) serial dependence in the returns $y_t = \log(P_t) - \log(P_{t-1})$, such as AR(p) models of the form

$$y_t = \mu + \sum_{i=1}^{p} \phi_i y_{t-i} + \epsilon_t \tag{2.1}$$

and according extensions to capture movements of the average over time

$$y_t = \mu + \sum_{i=1}^{p} \phi_i y_{t-i} + \sum_{j=1}^{q} \xi_j \epsilon_{t-j} + \epsilon_t \tag{2.2}$$

are thus (as standalone models) not expected to perform well in terms of forecast accuracy. Stylized fact two indicates that the assumption of conditional homoscedasticity is too constricting as financial time series usually display clusters of volatility.

2.1 GARCH Processes

To model volatility clusters within the autoregressive models of equations (2.1) and (2.2), the residuals ϵ_t are decomposed such that

$$\epsilon_t = z_t \sigma_t. \tag{2.3}$$

To capture the dynamics of σ_t^2, Bollerslev (1986) introduced the GARCH class of models, where the acronym stands for Generalized Autoregressive Conditional Heteroscedasticity. "Autoregressive" points to a feedback mechanism that includes past observations to a certain degree into the present; "conditional" in the GARCH case means that variance is dependent on the immediate past, while "heteroscedasticity" stands for a volatility which is varying over time. In Bollerslev's GARCH(P,Q) model, the conditional variance is given by

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{P} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{Q} \beta_j \sigma_{t-j}^2. \tag{2.4}$$

Since its introduction in 1986, many different versions and extensions of the original GARCH have been developed.[1] Note that the model in equation (2.4) is symmetric, meaning that positive and negative news have identical in-

[1] See e.g. Bollerslev (2010) or Bauwens et al. (2006) for a survey on GARCH models.

fluence on volatility. However, broad empirical evidence suggests that positive (negative) innovations to volatility correlate with negative (positive) innovations to returns. The first to document this phenomenon were Black (1976) and Christie (1982). More recent empirical evidence can be found for example in Bekaert and Wu (2000), Nyström and Skoglund (2002), Selcuk (2005) and Hens and Steude (2009).

This thesis considers two extensions of Bollerslev's original model to account for this asymmetry. Glosten et al. (1993) proposed to introduce a Boolean indicator in the GARCH equation (2.4). The outcome is known as GJR-GARCH and results in

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{P} \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^{P} \gamma_i \psi\left(\epsilon_{t-i}\right) \epsilon_{t-i}^2 + \sum_{j=1}^{Q} \beta_j \sigma_{t-j}^2, \tag{2.5}$$

where $\psi\left(\epsilon_t\right) = \psi\left(z_t\right) = 1$ in case $z_t < 0$ and 0 if $z_t \geq 0$. In addition to the previous symmetrical model in (2.4), a shock is thus also captured by the term $\gamma_i \psi\left(\epsilon_{t-1}\right)$. Note that the original GARCH model is nested in the GJR model: if all coefficients γ_i are zero, then the GJR model collapses to the GARCH model. The GJR specification is the most common version of a threshold GARCH or T-GARCH with the threshold set at level zero (McNeil et al., 2005).

To ensure stationarity and positivity of the above GARCH specifications the parameters α_0, α_i, β_j and $(\alpha_i + \gamma_i)$ are constrained to be non-negative. Nelson and Cao (1992) argue that the non-negativity constraints are too restrictive, advocating Nelson's (1991) model to include the asymmetric volatility response to innovations with an exponential GARCH (EGARCH) setup:

$$\log(\sigma_t^2) = \alpha_0 + \sum_{i=1}^{P} \alpha_i |z_{t-i}| + \sum_{i=1}^{P} \gamma_i z_{t-i} + \sum_{j=1}^{Q} \beta_j \log(\sigma_{t-j}^2). \tag{2.6}$$

Since the logarithm of the conditional variance is being modeled, the EGARCH conditional variance is constrained to be non-negative regardless of the sign of the parameter estimates. Hence, the non-negativity constraints are not needed in the estimation of the EGARCH model. The asymmetry is captured with the parameters γ_i: with negative γ_i, negative innovations have a larger impact on volatility than positive innovations. To complete the univariate model specifications it remains to specify the marginal distributions of the standardized residuals, z_t.

2.2 Marginal Distributions

Chen and Fan (2006) propose to use the empirical distribution of the innovations. While the strength of their non-parametric approach lies in the fact that no assumption regarding the nature of the empirical distribution has to be made, its drawback is located in the tails. Since data in the tails is usually scarce, the resulting tail estimates are unsatisfactory and cannot be used to solve for out of sample quantiles. For risk forecasting purposes this handicap appears to be rather extensive.

Among others, McNeil and Frey (2000) and Kuester et al. (2006) advocate the use of a semiparametric approach. Accordingly, the center of the innovation distribution is modeled with the kernel smoothed empirical distribution while the tails are parametrized based on extreme value theory. However, Patton (2013) as well as De Lira Salvatierra and Patton (2013) emphasize that inference methods for models where nonparametric or semiparametric marginal distributions are combined with dynamic copulas are not yet available in the literature. The applicable univariate models for this thesis are thus constrained to contain fully parametric innovation distributions. To mitigate the impact of this constraint, a flexible model for the marginal distributions is required.

Assuming the distribution of z_t to be a normal distribution, still a widely observed practice, obviously stands in contradiction to stylized fact number three. Whether one considers the model residuals ϵ_t or the filtered residuals z_t, stylized fact number three remains valid; this means that there is neither empirical support for the symmetry of normal distributions nor for their exponentially decaying tails. While the latter can be accounted for by using the Student-t distribution, Hansen (1994) further generalizes the Student-t distribution to allow for skewness. In comparison to other generalizations aiming to capture skewness, Hansen's version stands out due to its simplicity and its proven track record of successfully modeling economic variables (see e.g. De Lira Salvatierra and Patton (2013); Fantazzini (2008); Jondeau and Rockinger (2003, 2006); Patton (2004, 2006a)).

Hansen's (1994) skewed Student-t distribution with mean zero and variance one is defined by

$$f(z|\nu,\lambda) = \begin{cases} bc\left(1+\dfrac{1}{\nu-2}\left(\dfrac{bz+a}{1-\lambda}^{2}\right)\right)^{-\frac{\nu+1}{2}} & \text{if } z < -a/b, \\ bc\left(1+\dfrac{1}{\nu-2}\left(\dfrac{bz+a}{1+\lambda}^{2}\right)\right)^{-\frac{\nu+1}{2}} & \text{if } z \geq -a/b, \end{cases} \tag{2.7}$$

where the constants are

$$a = 4\lambda c\frac{\nu-2}{\nu-1}, \quad b^2 = 1+3\lambda^2 - a^2, \quad c = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi(\nu-2)}\Gamma\left(\frac{\nu}{2}\right)}. \tag{2.8}$$

The density of the skewed-t distribution is meaningful for $\nu > 2$ and $-1 < \lambda < 1$. Given $\nu > 2$ the distribution is well defined and its second moment exists, while skewness exists if $\nu > 3$ and kurtosis is defined if $\nu > 4$.[2] The parameter λ controls for skewness: a positive λ indicates that the mode of the density is to the left of the mean and the variable thus positively skewed, while a negative λ corresponds to a left-skewed density signaling that there is more probability of observing large negative than large positive variables.

The skewed-t distribution has the advantage of being very flexible, nesting a large set of conventionally used distributions such as:

[2] For the empirical applications, the lower bound of ν in the estimation procedure is set to 2, such that the data itself indicates for a particular time period whether a specific moment exists or not.

- The traditional Student-t distribution if $\lambda = 0$.
- The skewed normal distribution for $\nu \to \infty$.
- The normal distribution if $\lambda = 0$ and $\nu \to \infty$.

Figure 2.1 gives a visual impression of the versatility of Hansen's skewed-t distribution depicting the density function subject to different parameters ν and λ. Modeling the dependence structure with copulas involves marginal cumu-

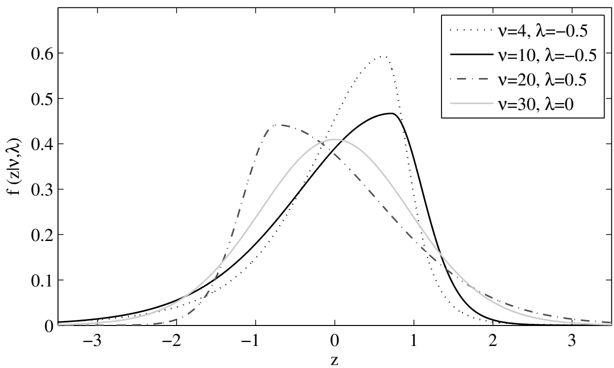

Figure 2.1: Hansen's skewed Student-t density subject to different ν and λ.

lative distribution functions rather than densities. To specify the cumulative distribution function of the skewed-t distribution recall that the traditional Student-t distribution is defined by

$$t(y,\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\pi\nu}} \left(1 + \frac{y^2}{\nu}\right)^{-\frac{\nu+1}{2}}. \tag{2.9}$$

With the cumulative distribution function of the traditional Student-t distribution with the degrees of freedom parameter ν given by

$$A(y,\nu) = \int_{-\infty}^{y} t(y)dy, \tag{2.10}$$

the cumulative distribution function of Hansen's (1994) skewed-t distribution is

$$F(z|\nu,\lambda) = \begin{cases} (1-\lambda)A\left(\dfrac{bz+a}{1-\lambda}\sqrt{\dfrac{\nu}{\nu-2}};\nu\right) & \text{if } z < -a/b, \\ (1-\lambda)A\left(\dfrac{bz+a}{1-\lambda}\sqrt{\dfrac{\nu}{\nu+2}};\nu\right) - \lambda & \text{if } z \geq -a/b. \end{cases} \tag{2.11}$$

Note that throughout this thesis, the cumulative distribution (cdf) of a random variable is denoted using an uppercase letter, and the corresponding density (pdf) using the lowercase letter.

The choice of Hansen's (1994) skewed Student-t distribution for the standardized innovations z_t completes the models for the univariate risk factor evolutions:

$$z_t = \frac{\epsilon_t}{\sigma_t} \overset{i.i.d.}{\sim} \textit{skewed-t}\ (\nu,\lambda). \tag{2.12}$$

The three univariate models considered in this thesis can thus be summarized as the ARMA specification in equation (2.2) combined with either the GARCH, the GJR-GARCH or the EGARCH model in equations (2.4), (2.5) respectively (2.6) with the standardized innovation distribution modeled with the skewed-t distribution. The next step to construct a valid multivariate distribution is to specify the dependence structure between the z_t of the different return series.

Chapter 3

Dependence Structure

While there are several methods to quantify and model dependence, traditionally the most widely used measure in theoretical finance is Pearson's correlation coefficient. This is partly based on the pivotal role of multivariate normal distributions in statistics and of the Capital Asset Pricing Model (CAPM) in finance. Pearson's linear correlation is a natural measure of dependence for elliptical distributions but comes with several limitations and pitfalls. Among other shortcomings, correlation is dependent on the marginal distributions and is not invariant under nonlinear strictly increasing transformations of the marginals. Furthermore, as a scalar measure of linear relations, correlation is unable to capture the entire dependence structure, misses non-linear relations and does not allow for dependency asymmetries.[1]

Copula functions are capable to overcome these limitations and provide an accessible way to model joint distributions, especially for non-normal variables. This chapter gives a short introduction to copula theory and presents the different copula models to be used in the empirical analysis.

[1] See Embrechts et al. (2002) or McNeil et al. (2005) for other limitations and pitfalls of Pearson's linear correlation.

3.1 Copula Fundamentals

The term *copula* is derived from the Latin word *copulare* which means to join respectively to connect. It was first mentioned in statistics literature by Sklar (1959) even though comparable ideas and outcomes may be tracked back to Fréchet (1935) and Hoeffding (1940). A number of contributions in the field of copula theory and applications have appeared in more recent literature. Nelsen (2006) and Joe (2001) are two reference works on copula theory, providing comprehensive introductions to copulas and dependence modeling, while emphasizing the statistical foundations. In McNeil et al. (2005) and Denuit et al. (2006), copula methods are presented from a risk management perspective. Cherubini et al. (2004) and Cherubini et al. (2012) use methods from mathematical finance to introduce copulas. Choroś et al. (2010) provide a succinct overview of parametric and nonparametric copula estimation methods for time series and i.i.d. data. Genest and Favre (2007) describe semiparametric inference methods for independent and identically distributed data including a detailed empirical illustration.

Jondeau and Rockinger (2002, 2006) were among the first to propose the combination of GARCH-models and copulas. Recent financial applications of the copula-GARCH model are for example Riccetti (2012), Cai et al. (2012) and Min and Czado (2012). Patton (2013) provides an overview of methods for economic forecasting with copulas also encompassing empirical examples and Patton (2012) reviews copula-based models for economic time series.

3.1.1 Sklar's Theorem

All copula theory is based on the contribution of Sklar (1959), who showed that a multivariate distribution can be divided into its d marginal distributions and a d-dimensional copula, which completely characterizes the dependence between the variables. His theorem provides an accessible way to build valid multivariate distributions from known marginals. Consider $F(y_1, \ldots, y_d)$ to be a continuous d-variate cumulative distribution function with univariate margins $F_i(y_i)$. Sklar's theorem states that there exists a function C named a copula, which maps $[0,1]^d$ into $[0,1]$ such that

$$F(y_1,\dots,y_d) = C(F_1(y_1),\dots,F_d(y_d)). \tag{3.1}$$

The product of the marginals and the copula density yields the joint density function

$$f(y_1,...,y_d) = \prod_{i=1}^{d} f_i(y_i) \frac{\partial C(F_1(y_1),...,F_d(y_d))}{\partial F_1(y_1)...\partial F_d(y_d)}. \tag{3.2}$$

This allows to define a d-dimensional copula as a cumulative distribution function with uniform [0,1] marginal distributions

$$C(u_1,...,u_d) = F(F_1^{-1}(u_1),\dots,F_d^{-1}(u_d)), \tag{3.3}$$

where $u_i = F_i(y_i)$, $i = 1,...,d$ are the probability integral transformations (PIT) of the marginal models. Copulas thus allow to map the univariate marginal distributions of d random variables, each supported in the $[0,1]$ interval, to their d-variate distribution, supported on $[0,1]^d$. This methodology is applicable regardless of the type and degree of dependence among the variables (Chollete et al., 2009).

Given any set of marginal distributions $(F_1,...,F_d)$ and any copula C, one may obtain a valid joint distribution with the given marginals using equation (3.1). An important implication of this result is that the marginal distributions are not required to be in any way similar to each other, nor does the choice of marginal distributions constrain the choice of copula (Patton, 2009).

A variety of copulas has been developed until today.[2] However, the majority of them are limited to a bivariate setting and since this thesis focuses on multivariate dependence models, the focus lies on copula models which are scalable to the multivariate level.[3] The eligible scalable models are rep-

[2] See e.g. Nelsen (2006) for a detailed compilation.
[3] Notably, pair-copula constructions (also known as vine copulas), which are constructed

resentatives of both of the major copula families: Archimedean copulas and elliptical copulas.

3.1.2 Conditional Copulas

Forecasting in a multivariate setting is based on an extension of Sklar's theorem (3.1) for conditional joint distributions presented in Patton (2006a). Considering some information set $\mathcal{F}_{t-1}$, Patton shows that the conditional distribution $F\left(y_1, \ldots, y_d | \mathcal{F}_{t-1}\right)$ can be decomposed into its conditional marginal distributions and the conditional copula such that

$$F\left(y_1, \ldots, y_d | \mathcal{F}_{t-1}\right) = C(F_1\left(y_1 | \mathcal{F}_{t-1}\right), \ldots, F_d\left(y_d | \mathcal{F}_{t-1}\right) | \mathcal{F}_{t-1}). \tag{3.4}$$

The d-dimensional conditional copula is:

$$C\left(u_{1,t}, ...u_{d,t} | \mathcal{F}_{t-1}\right) = F(F_1^{-1}\left(u_{1,t} | \mathcal{F}_{t-1}\right), \ldots, F_d^{-1}\left(u_{d,t} | \mathcal{F}_{t-1}\right)). \tag{3.5}$$

A valid conditional multivariate distribution based on Sklar's theorem and Patton's extension can thus be created by first estimating the models for each of the conditional marginal distributions, $F_i(y_i | \mathcal{F}_{t-1})$, $i = 1, .., d$, construct the probability integral transformed variables $u_{i,t} = F_i(y_{i,t} | \mathcal{F}_{t-1})$, $i = 1, .., d$ and then consider copula models for the joint distribution of these variables. In analogy to the construction of unconditional copulas, this procedure yields a valid d-dimensional model without the intricacy of a simultaneous specification and estimation.

Note that in equation (3.5) the identical information ($\mathcal{F}_{t-1}$) appears in each of the marginals and the copula. Fermanian and Wegkamp (2012) point out that using different information sets results in a function $F(\cdot|\cdot)$ which is not generally a joint distribution with the specified conditional marginal distributions. Patton (2013) however emphasizes that in empirical applications

by sequentially applying bivariate copulas, are therewith not covered - see Acar et al. (2012) for an important critique of vine copulas.

not the entire information set $\mathcal{F}_{t-1}$ might be required for every marginal distribution. For example, let $\mathcal{F}_{t-1}^{(1)}$ denote the information generated by $(y_{1,t-1}, ...$ $, y_{1,t-n})$ and $\mathcal{F}_{t-1}$ be the information set generated by $(y_{i,t-1}, ..., y_{i,t-n})$, $i = 1, ..., d$. For some processes, one may find that each variable uniquely depends upon its own lags, i.e. $y_{i,t}|\mathcal{F}_{t-1} \stackrel{d}{=} y_{i,t}|\mathcal{F}_{t-1}^{(i)}$. Models for marginal distributions are thus not explicitly required to make use of the entire information set, as long as they comply with the restriction that all marginal models and the copula utilize the same information set.

3.2 Static Copulas

This section presents the static multivariate copulas employed in this thesis including the according simulation algorithms.

3.2.1 Elliptical Copulas

Sklar's theorem (3.1) indicates that typical multivariate distributions describe central dependence structures which is why elliptical copulas are also known as implicit copulas: A multivariate normal distribution entails a Gaussian copula C^{Ga} whereas a multivariate t-distribution entails a Student-t copula C^t. The advantage of implicit copulas is their ability to easily extend to multiple dimensions and their parameter plurality (Nelsen, 2006).

Gaussian Copula

Based on (3.3), the d-dimensional Gaussian copula for a correlation matrix Σ is given by

$$C_{\Sigma}^{Ga}(u_1, ...u_d) = \Phi_{\Sigma}(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d))$$

$$= \int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_d)} \frac{1}{(2\pi)^{\frac{d}{2}} |\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2} y' \Sigma^{-1} y\right) dy_1 \cdots dy_d, \quad (3.6)$$

where Φ is the cumulative distribution function of a standard normal distribution whereas Φ_Σ is the cumulative distribution function of the multivariate normal distribution having a mean of zero and a covariance matrix Σ. The density of any copula which proves to be adequately differentiable may be computed with

$$c\left(u_1, ...u_d\right) = \frac{f(F_1^{-1}\left(u_1\right), \ldots, F_d^{-1}\left(u_d\right))}{f_1(F_1^{-1}\left(u_1\right)), ..., f_d(F_d^{-1}\left(u_d\right))}, \tag{3.7}$$

where f is the joint density and $f_1, ..., f_d$ are the marginal densities (McNeil et al., 2005; Schmidt, 2007).

Simulation from the Gaussian copula is straightforward. Step number one is to generate random variables according to the underlying multivariate distribution. Step two then consists of transforming them to uniform marginal distributions by quantile transformation. The resulting algorithm is the following (McNeil et al., 2005; Schmidt, 2007):

1. Obtain the correlation matrix Σ from any covariance matrix Σ by scaling each component to variance 1.
2. Compute the Cholesky-decomposition $\Sigma = A'A$
3. Generate independent and identically distributed standard normal random variables $\tilde{X}_1, \ldots, \tilde{X}_d$
4. From $\tilde{X} = (\tilde{X}_1, \ldots, \tilde{X}_d)'$ calculate $(X_1, \ldots, X_d)' = X = A\tilde{X}$.
5. Return $U = (\Phi(X_1), \ldots, \Phi(X_d))'$ where Φ equals the cumulative standard normal distribution function.

Student-t Copula

Identical to the multivariate normal distribution one may obtain an implicit copula from any other distribution with continuous marginal distribution functions. For t-distributions the d-dimensional t-copula with ν degrees of freedom is given by

$$C^t_{\nu,\Sigma}\left(u_1, ..., u_d\right) = t_{\nu,\Sigma}(t^{-1}_\nu\left(u_1\right), \ldots, t^{-1}_\nu\left(u_d\right))$$

$$= \int_{-\infty}^{t_\nu^{-1}(u_1)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_d)} \frac{\Gamma\left(\frac{\nu+d}{2}\right)|\Sigma|^{-\frac{1}{2}}}{\Gamma\left(\frac{\nu}{2}\right)(\nu\pi)^{\frac{d}{2}}} \left(1 + \frac{1}{\nu} y' \Sigma^{-1} y\right)^{-\frac{\nu+d}{2}} dy_1 \cdots dy_d, \tag{3.8}$$

where Σ is a correlation matrix, t_ν is the cumulative distribution function of the one dimensional t_ν-distribution and $t_{(\nu,\Sigma)}$ is the cumulative distribution function of the multivariate $t_{(\nu,\Sigma)}$-distribution.
The density of the multivariate Student-t copula has the form

$$c^t_{\nu,\Sigma}(u) = \frac{f_\nu, \Sigma\left(t^{-1}_\nu(u_1), ..., t^{-1}_\nu(u_d)\right)}{\prod_{i=1}^d f_\nu\left(t^{-1}_\nu(u_i)\right)}, \tag{3.9}$$

where $u \in (0,1)^d$, $f_{\nu,\Sigma}$ is the joint density of a $t_d(\nu, 0, \Sigma)$-distributed random vector and f_ν is the density of the univariate standard Student-t distribution with ν degrees of freedom (Demarta and McNeil, 2005).

Simulation from the Student-t copula can be done using the following algorithm (McNeil et al., 2005):

1. Use steps 1 to 4 of the algorithm for the Gaussian copula to generate multivariate normal X with covariance Σ.
2. Generate independent $\xi\tilde{\chi}^2_\nu$ by e.g. using $\xi = \sum_{i=1}^\nu Y_i^2$, where Y_i are independent and identically $N(0,1)$ distributed.
3. Return $U = (t_\nu(X_1/\sqrt{(\xi/\nu)}), ..., t_\nu(X_d/\sqrt{(\xi/\nu)}))'$ where t_ν denotes the cumulative distribution function of a univariate t-distribution with ν degrees of freedom.

While both the Gaussian and the Student-t copula are dependence structures implied by elliptical distributions, they differ with regards to tail dependence. Lower (upper) tail dependence refers to the density in the lower (upper) tail of the copula function and represents the probability of observing joint negative (positive) extremes (see section 6.2.1). The Student-t copula is capable of capturing equal lower and upper tail dependence, whereas the Gaussian copula has no tail dependence implying independence of the extreme realizations. The level of tail dependence of the Student-t copula is governed by ν: the greater the degrees of freedom, the lower the level of tail dependence, converging in the limit $\nu \to \infty$ to the Gaussian copula.

3.2.2 Archimedean Copulas

The name *Archimedean* refers to the copulas' algebraic property which resembles the Archimedean axiom for real numbers. In contrast to elliptical copulas, Archimedean copulas are given explicitly and capture all information about the dependence structure in the univariate generator function ϕ. There exists a wide selection of different bivariate Archimedean copulas, however, choices on the multivariate level are limited. While Schweizer and Sklar (1983) prove that the generator ϕ creates a bivariate copula if and only if it is convex, McNeil and Něshlehovà (2009) show that in case $\phi : [0,1] \to [0,\infty]$ is a strict Archimedean copula generator, then

$$C(u_1, ..., u_d) = \phi^{-1}(\phi(u_1) + ... + \phi(u_d)) \tag{3.10}$$

induces a copula in any dimension d if and only if the generator inverse $\phi^{-1} : [0,\infty] \to [0,1]$ is d-monotone. Accordingly, the strictly decreasing function ϕ has to be continuous on $[0,\infty]$, admit derivatives up to the order $d-2$ and satisfy

$$(-1)^k \frac{d^k}{dt^k}\phi(t) \geq 0, \quad k \in \{0, ..., d-2\}, t \in (0,\infty). \tag{3.11}$$

Mostly, it is assumed, that ϕ is completely monotonic, meaning that $k \in \mathbb{N}_0$ (Hofert et al., 2013).

With the Clayton and the Frank copula, this thesis includes two copulas of the Archimedean family. Both have completely monotonic generators and are thus well-defined for multiple dimensions. While the Frank copula is symmetric exhibiting no tail dependence, the asymmetric Clayton copula is particularly interesting for risk forecasting purposes, since it is capable of modeling lower tail dependence.

Clayton Copula

The generator function of the Clayton copula is defined as

$$\phi_{Cl}(u) = \frac{1}{\theta}\left(u^{-\theta} - 1\right), \tag{3.12}$$

where the permissible parameter range is $\theta \in (0, \infty)$. A d-dimensional Clayton copula is given by

$$C\left(u_1, ..., u_d\right) = \left(\sum_{i=1}^{d} u_i^{-\theta} - d + 1\right)^{-\frac{1}{\theta}}. \tag{3.13}$$

As the copula parameter θ tends to infinity, the dependence becomes maximal while the limiting case $\theta = 0$ should be interpreted as the d-dimensional independence copula (McNeil et al., 2005).
The density of the multivariate Clayton copula is

$$\frac{\partial^d C}{\partial u_1 ... \partial u_d} = \theta^d \frac{\Gamma\left(\frac{1}{\theta} + d\right)}{\Gamma\left(\frac{1}{\theta}\right)} \left(\sum_{i=1}^{d} u_i^{-\theta} - d + 1\right)^{-\frac{1}{\theta} - d} \left(\prod_{i=1}^{d} u_i^{-\theta - 1}\right), \tag{3.14}$$

where Γ denotes the usual Euler Γ function.

The contribution of Marshall and Olkin (1988) can be used to elegantly simulate from multivariate Archimedean copulas. The simulation algorithm exploits the fact that every completely monotonic function mapping from $[0, \infty]$ to $[0, 1]$ can be expressed in terms of Laplace-Stieltjes transforms of distribution functions on $\mathbb{R}^+$ and therewith provide a way of describing mul-

tivariate Archimedean copulas.
Let G be a distribution function on $\mathbb{R}^+$ which satisfies $G(0) = 0$ with Laplace-Stieltjes transform

$$\hat{G}(t) = \int_0^{+\infty} e^{-tx} dG_\gamma(x). \tag{3.15}$$

For $\hat{G}(\infty) := 0$ it is clear that $\hat{G} : [0, \infty] \to [0, 1]$ is a continuous, strictly decreasing function with the property of complete monotonicity (McNeil et al., 2005). It can thus serve as a candidate of a multivariate Archimedean copula generator inverse which leads to the following simulation procedure for the Clayton copula (Hofert, 2008; McNeil et al., 2005):

1. Generate a gamma variable $\gamma \sim Gamma(1/\theta, 1)$ with $\theta \in (0, \infty)$. The distribution function of γ thus has Laplace transform $\hat{G}(t) = (1+t)^{-\frac{1}{\theta}}$
2. Generate independent uniform variates $(X_1, ..., X_d) \sim U[0, 1]$.
3. Return $(U_1, ..., U_d) = (\hat{G}(-\log(X_1)/\gamma), ..., \hat{G}(-\log(X_d)/\gamma))'$

Frank Copula

The Frank copula generator is given by

$$\phi_{Fr}(u) = \log\left(\frac{\exp(-\theta u) - 1}{\exp(-\theta) - 1}\right), \tag{3.16}$$

hence

$$\phi_{Fr}^{-1}(u) = \frac{1}{\theta}\log\left(1 + e^u(e^{-\theta} - 1)\right) \tag{3.17}$$

is completely monotonic if $\theta \in (0, \infty)$. The multivariate Frank copula is

$$C(u_1, ..., u_d) = -\frac{1}{\theta}\log\left(1 + \frac{\prod_{i=1}^d (e^{-\theta u_i} - 1)}{(e^{-\theta} - 1)^{d-1}}\right). \tag{3.18}$$

The independence copula is attained for $\theta = 0$ whereas with $\theta \to \infty$ maximal dependence is achieved. The density of the multivariate Frank copula is given by

$$\frac{\partial^d C}{\partial u_1 ... \partial u_d} = \left(\frac{\theta}{1-e^{-\theta}}\right)^{d-1} \mathrm{Li}_{-(d-1)}(h_\theta(u)) \frac{\exp(-\theta \sum_{i=1}^d u_i)}{h_\theta(u)}, \tag{3.19}$$

where $\mathrm{Li}_{-s}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$ is the polylogarithm of order s at z and $h_\theta(u) = (1-e^{-\theta})^{1-d} \prod_{i=1}^d (1-\exp(-\theta u_i))$ (Hofert et al., 2013). The following algorithm describes the simulation procedure for the Frank copula:

1. Generate a discrete variable V with probability mass function $p(k) = P(V=k) = (1-exp(-\theta))^k/(k\theta)$ for $k = 1, 2, ...n$ and $\theta \in (0, \infty)$.
2. Generate independent uniform variates $(X_1, ..., X_d) \sim U[0,1]$.
3. Return $(U_1, ..., U_d) = (\hat{G}(-\log(X_1)/V), ..., \hat{G}(-\log(X_d)/V))'$

As for the Clayton copula simulation, the algorithm is especially efficient in large dimensions, as only d + 1 random numbers are required for the generation of a d-dimensional observation.

3.2.3 Mixture Copulas

Nelsen (2006) demonstrates that a convex combination of different copulas is yet again a copula. Mixing copulas with different dependence features provides an appealing flexibility in constructing dependence characterizations and broadens the range of copula structures applicable on the multivariate level. Mixture copulas were applied e.g. by Hu (2006), Dias and Embrechts (2010), Weiss (2011) and Ruenzi and Weigert (2013) in a bivariate setting and by Braun (2011) in a multivariate context. This thesis constructs dependence structures which are capable of capturing asymmetries by amalgamating the lower tail dependence feature of the Clayton copula with the symmetric dependence structure of the other copulas under consideration by combining them into mixture structures. The applied static mixture copulas consist of two different copulas but of d dimensions and have distribution functions of the form

$$C\left(u_1,...,u_d;w,\theta\right) = wC_1(u_1,...,u_d,\theta_1) + (1-w)C_2(u_1,...,u_d,\theta_2) \tag{3.20}$$

where θ_1 and θ_2 are the parameter sets of the different copulas. The density of the static mixture construct is simply the convex combination of the copula densities involved in the mixture:

$$c\left(u_1,...,u_d;w,\theta\right) = wc_1(u_1,...,u_d,\theta_1) + (1-w)c_2(u_1,...,u_d,\theta_2). \tag{3.21}$$

While the degree of dependence is carried by the parameters of the copulas within the mixture, the shape of the dependence is summarized by the weight of each individual copula. Compared to the standalone static copulas, the mixture copula provides increased flexibility to adapt to different dependence structures, since it unites the diverse features of the enclosed copulas. Combining an elliptical copula with an Archimedean copula means that two different copulas describe the dependence structure of the data according to their proportion in the mixture construct. As a consequence, each copula is adjusted solely to the share of the data set it represents. Such a partition of the dependence structure should allow a more precise calibration of the copula parameters, as they only have to accommodate their fraction of the data set. In contrast, the parameters of standalone copulas are required to cover the entire data set which may entail larger compromises and inaccuracies in the calibration. Figure 3.1 gives a visual illustration of the mixture concept on the bivariate level. The density of the Gaussian copula is depicted on the left, the Clayton copula density on the right and the plot in the center shows the density of the combination of these two copulas in a mixture copula with $w = 0.5$. This mixture copula has the advantage of accommodating the parameter plurality of the Gaussian copula (with increasing number of dimensions) and the asymmetric dependence feature of the Clayton copula. While static mixture copulas aim at a more precise representation of the dependence structure compared to standalone copulas, they are not capable of capturing dependence shifts over time, since their shapes, parameters and mixture weights remain stable over time.

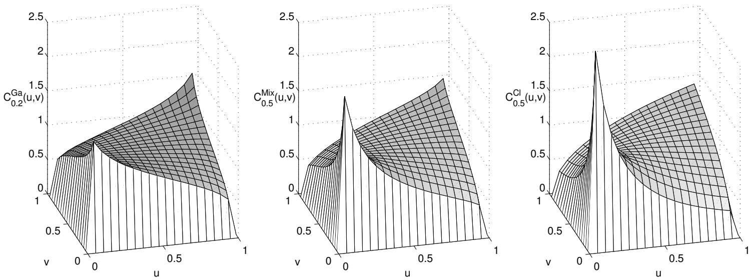

Figure 3.1: Gaussian (left), Gauss-Clayton mixture with equal weights (center) and Clayton copula density (right).

3.3 Regime Switching Copulas

The copulas laid out so far assume the dependence structure to be constant over time. Even though copulas provide a more general specification of the dependence structure than Pearson's linear correlation, the assumption of time invariant dependence appears to be rather unrealistic, given the broad empirical evidence of time-varying correlations (see e.g. Boyer et al., 1999; Engle, 2002; Loretan and Phillips, 1994; Pritsker, 2006; Ramchand and Susmel, 1998; Tse and Tsui, 2002). The Basel Committee on Banking Supervision (2011, p.10) concludes that for calculating value-at-risk-measures "time-varying correlations should be taken into account."

Stöber and Czado (2012) and Hsieh and Huang (2012) show that there are structural breaks in the dependence structure of financial variables similar to the clusters in univariate volatilities. In times of crises, dependence between the assets seems to be significantly increased, compared to "normal" times. One approach to account for the different levels of dependence is to switch between different copula models of dependence. Since the switches between the regimes cannot be known in advance they are assumed to be governed by a latent Markov process.

Allowing for regime switches, this thesis follows a long tradition in economics. Since their introduction by Hamilton (1989), regime switching models have experienced numerous applications in the field of finance. Among others Ang and Bekaert (2002a), Guidolin and Timmermann (2006a) and Guidolin and Timmermann (2006b) use regime switching models for interest rates. Ang and Bekaert (2002b) and Guidolin and Timmermann (2008) employ a regime switching model for international financial returns.

A few years ago, combinations of regime switching models with bivariate copulas were proposed for example by Okimoto (2008), Rodriguez (2007) and Silva Filho et al. (2012), who estimate regime-switching copulas for bivariate international stock index data. While Rodriguez focuses on index pairs of Latin American as well as Asian countries, Okimoto investigates a US-UK pair and Silva Filho et al. concentrate on pairs between a US, a UK and a Brazilian stock market index.

However, the thesis at hand covers dependence structures exceeding the bivariate level. The methodology of Chollete et al. (2009), Garcia and Tsafack (2011) and Braun (2011) is thus chosen as a basis for the thesis at hand. These authors employ two dependence regimes, which are different in intensity and/or shape. The marginal distributions are modeled separately from the dependence structure and are thus not dependent on the regime. This approach allows to model separate copulas for different dependence regimes. Accordingly, the parameters and the families of the copulas remain constant within a regime but differ across the regimes. Switching between the regimes is governed by a latent Markov process which determines the regime probabilities.

Applications of the multivariate regime switching copula models in literature to date are limited to two regimes. While covering different two regime cases, this thesis further extends the model to three regimes. The regime switching copula model thus contains as many different copulas as regimes, where the copulas differ in intensity and/or shape. However, the variation over time is limited to the probabilities of the copula to describe the dependence, while the copulas and their according parameters remain constant. As a consequence, since different parts of the time series are modeled by different copulas, the shapes of regime switching copulas change over time.

3.3.1 Hamilton Filter

To model the dynamics of the data, this thesis follows Hamilton (1989), who switched between different density functions. While Hamilton considered univariate time series, this thesis focuses on the joint density of multiple time series as described by the copula functions. Since the modeled copulas only diverge with regards to their dependence characteristics, the impact of the different regimes is concentrated on the dependence structure. The model thus expresses different fractions of the joint data density by separate copula functions.
Conditional on being in regime j, the data density is

$$f\left(Y_t|Y_{t-1,s_t=j}\right) = c^j\left(F_1 y_{1,t},\ldots,F_d y_{d,t};\theta_c^{(j)}\right)\prod_{i=1}^{d} f_i\left(y_{i,t};\theta_{m,i}\right), \tag{3.22}$$

where $Y_t = (y_{1,t},...,y_{d,t})$, s_t represents the state variable for the regime, $c^{(j)}(.)$ is the copula density function in the regime j with the according parameter set $\theta_{m,i}$, F_i is the distribution and f_i the according density function of the marginal y_t with the parameters $\theta_{m,i}$. Note that j is an index of the copula, but not of the marginal densities. The model assumes the unobserved state variable to be governed by the transition probability matrix

$$P = Pr(s_t = i|s_{t-1} = j) = p_{i|j}, \tag{3.23}$$

where $p_{i|j}$ derives the probability that state j will be followed by state i. As the Markov chain is latent and thus not observable, Hamilton's (1989) filter is applied. Accordingly, the transition probability matrix drives the regime probabilities which in turn define the density function of the complete dataset. Explicitly, the filtered process for k regimes obeys

$$\xi_{t|t} = \frac{\xi_{t|t-1} \bigodot \delta_t}{1'(\xi_{t|t-1} \bigodot \delta_t)'} \tag{3.24}$$

$$\xi_{t+1|t} = P'\xi_{t|t} \tag{3.25}$$

$$\delta_t = \begin{pmatrix} c^{(1)}(F_1\left(y_{1,t}|y_1^{t-1}\right), \ldots, F_d\left(y_{d,t}|y_d^{t-1}\right); \theta_c^{(1)} \\ \vdots \\ c^{(k)}(F_1\left(y_{1,t}|y_1^{t-1}\right), \ldots, F_d\left(y_{d,t}|y_d^{t-1}\right); \theta_c^{(k)} \end{pmatrix} \tag{3.26}$$

where $\xi_{t|t}$ is a (k x 1) vector with all the regime probabilities at time t, conditional on the observations until time t; 1 is a (k x 1) vector of ones and $\bigodot$ stands for the Hadamard product. The regime probabilities $\xi_{t+1|t}$ at time $t+1$ conditional on all information until time t are captured by the transition probability matrix P. The copula densities at time t, conditional on being in each one of the two regimes are contained in the vector δ_t. While equation 3.24 represents a Bayesian updating of the probability to be in a specific regime given all observations δ_t up to the current time, equation 3.25 comprises one forward iteration of the Markov chain. With this recursive procedure it is straightforward to forecast the regime probabilities ($\xi_{t+1|t}$).

The filtered system needs initial values for the regime probabilities $\xi_{1|0}$ from which the optimization procedure is started.[4] Iterations over the two equations 3.24 and 3.25 yield the likelihood value

$$\log \mathcal{L}(\theta) = \sum_{t=1}^{T} \log(1'(\xi_{t|t-1} \bigodot \delta_t)). \tag{3.27}$$

[4]Gray (1996) points out that for a long enough data set the particular choice of initial values for the regime probabilities becomes irrelevant.

Naturally one would like to test the null hypothesis that there are k regimes versus the alternative of $k+1$ regimes. Using for example $k = 1$ would answer whether there are any regime switches at all while using $k = 2$ would determine whether the existence of more than two regimes is supported by the data. However, Hamilton (2008) points out that likelihood ratio tests of these hypotheses do not comply with the usual regularity conditions. Given for example that there is truly only one regime, the maximum likelihood estimate for the probability of staying in regime one fails to converge to a well-defined population value. The likelihood ratio test does therewith not have the χ^2 limiting distribution. As a solution, Hamilton (2008) proposes to establish model comparisons based on their ability to forecast, which is the approach adopted in this thesis.

3.3.2 Kim Filter

With the estimated transition probabilities, one can form an inference about the dependence regime at date t based on the realized observations at a later date T. In order to calculate these inferences for the regime probabilities, the Kim filter may be used, which represents a combination of the Kalman filter and the Hamilton filter, particularly designed for Markov-switching models (Hamilton, 1988, 1989, 1994).

Accordingly, the inference of the state variable $\xi_{t|T}$ is performed by considering the entire data obtained until date T ($\xi_{t|T}$). When $T < t$, a forecast about the regimes in the future is made, but when $T > t$, the probabilities for the regime at time t are ex post probabilities. According to Kim (1994), these inferences may be calculated with the following iterative algorithm

$$\xi_{t|T} = \xi_{t|t} \bigodot \left(P' \times [\xi_{t+1|T}(\div)\xi_{t+1|t}]\right), \tag{3.28}$$

where $\bigodot$ and $(\div)$ stand for the Hadamard multiplication respectively division. Initiating with the probability $\xi_{T|T}$, obtained from equation (3.24) for $t = T$, the process iterates backwards on equation (3.28). This procedure is valid only with first-order Markov chains such as the one at hand. With the estimated transition probability matrix (3.23), the inference may be computed based on the entire information in the sample.

3.4 Dynamic Copulas

The restriction to a limited number of different static dependence structures as put forward with the regime switching copulas may still be too constricting. To increase the adaptability of the dependence specification one might think of simply increasing the number of regimes. However, a more flexible approach consists in allowing the dependence structure to be fully dynamic i.e. vary with every discrete time step. Engle and Sheppard (2001) and Engle (2002) established the basis for models with dynamic dependence by introducing dynamic correlation coefficients. In the field of copulas, the seminal work of Patton (2006a) was among the first to allow copulas to be time-varying and therewith opened a new and fast-growing line of research. Some of the contributions to this field on the bivariate level include Ng (2008), Fantazzini (2008), Guégan and Zhang (2010), Dias and Embrechts (2010) and De Lira Salvatierra and Patton (2013) and on the multivariate level Jin and Lehnert (2011), Braun (2011) and Christoffersen et al. (2012).

This section firstly introduces the dynamic versions of the elliptical copulas before outlining the dynamic multivariate Archimedean copulas and finally describing the dynamic multivariate mixture copulas employed in the empirical applications.

3.4.1 Dynamic Elliptical Copulas

Since the elliptical copulas presented in section 3.2.1 are specified by a correlation matrix (and the degrees of freedom parameter for the Student-t copula), the dynamic conditional correlation (DCC) specification proposed by Engle (2002) sets a well defined basis to model the changes of the correlations over time. Transferring the DCC approach to multivariate copulas is a research topic of current interest. While the applications of Fantazzini (2009), Jin and Lehnert (2011) and Christoffersen et al. (2012) aim at handling observations from heavy-tailed stock market return distributions, Braun (2011) employs dynamic copulas to estimate the evolution of stocks and bond yields. Ignatieva et al. (2010) and Diks et al. (2010) utilize dynamic copulas to model the dependence structure of currencies and Ignatieva and Trück (2012) are working on the application of time varying copulas to model the spot price dependence in Australian electricity markets. Whilst very recently, the DCC specification has been criticized to cause spurious conditional correlation dynamics (see Füss et al., 2012), the model is still regarded as state-of-the-art approach in empirical finance to capture correlation behavior over time.

Dynamic Gaussian Copula

Based on Engle (2002), the correlation matrix Σ_t of the dynamic Gaussian copula is set to evolve through time as follows:

$$Q_t = (1 - \alpha - \beta) \times \bar{Q} + \alpha z_{t-1} \times z'_{t-1} + \beta \times Q_{t-1} \tag{3.29}$$

$$\Sigma_t = \tilde{Q}_t^{-1} Q_t \tilde{Q}_t^{-1}, \tag{3.30}$$

where z_t is the vector of transformed standardized residuals $z_{i,t} = skewed - t_{\nu,\lambda}^{-1}(u_{i,t})$, $\bar{Q}$ is the sample correlation of z_t and $\tilde{Q}_t = [\tilde{q}_{ii,t}] = [\sqrt{q_{ii,t}}]$ is the diagonal square matrix with the square root of the ith diagonal element of Q_t on its ith diagonal position. The constraints for the parameters α and β are $\alpha + \beta < 1$, with $\alpha, \beta \in (0, 1)$.

The dynamic Gaussian copula is therewith defined as:

$$C_{\Sigma_t}^{Ga}(u_1, ...u_d) = \Phi_{\Sigma_t}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)). \tag{3.31}$$

Dynamic Student-t Copula

The Student-t copula parameters are the correlations and the degrees of freedom, ν. The dynamic process which drives the correlations is identical to the one defined for the Gaussian copula in equations (3.29) and (3.30). This thesis further allows the degrees of freedom parameter to vary over time.[5] The Student-t copula is therewith not only provided with the capability to adapt the level of dependence, but also the strength of tail dependence over time. Fantazzini (2008) proposes to model the evolution of the degrees of freedom parameter of a bivariate Student-t copula as

$$\nu_t = \Lambda(\varsigma + \varphi \times |u_{t-1} - v_{t-1}|), \tag{3.32}$$

where Λ is a logistic transformation designed to keep the conditional degrees of freedom in $(2, 100)$ at all times. Recalling that the Student-t converges in distribution to a normal distribution when $\nu \to \infty$, the Student-t copula nests the Gaussian copula when $\nu \to \infty$. However, it is hard to distinguish a Student-t distribution with $\nu > 30$ from a normal distribution, such that a Student-t copula $C^t_{\nu,\Sigma}$ with $\nu = 100$ is very close to a Gaussian copula C^{Ga}_Σ. Note that the Student-t copula with constant degrees of freedom is nested with $\varphi = 0$. The specification (3.32) is limited to the bivariate setting through the absolute difference term $|u_{t-1} - v_{t-1}|$. As this thesis focuses on multivariate models, an extension of equation (3.32) is called for.

For the extension to the multivariate setting this thesis follows Braun (2011), who in a similar situation identified the K-Means clustering algorithm as being able to provide a valid scalar measure for the absolute difference in the multidimensional space. Generally, the K-Means clustering algorithm is used to partition N points into k clusters by minimizing the sum of point-to-centroid distances, aggregated over all clusters. While there are a number

[5] To the best of the author's knowledge, the only other work allowing the degrees of freedom parameter of a multivariate dynamic Student-t copula to be time varying is Jin and Lehnert (2011).

of distance measures, the closest to a multivariate analogon of the bivariate absolute distance used in (3.32) is the ℓ_1-norm where the distance between point and centroid equals the sum of the absolute differences of their Cartesian coordinates. Each centroid is defined as the component-wise median of the points in that cluster.

The aim for the application at hand, however, is not to partition the observations into multiple clusters but to compute the absolute distance (AD) between all observations in time t. Therefore, the number of clusters is set to $k = 1$, which means that the ℓ_1-norm is the sum of absolute differences between the observations u_t and their median $\tilde{u}_t$ in time t

$$AD_{\ell_1} = \sum_{j=1}^{k}\sum_{i=1}^{d} |u_{i,t} - \tilde{u}_{j,t}| = \sum_{i=1}^{d} |u_{i,t} - \tilde{u}_t|. \tag{3.33}$$

Replacing the bivariate absolute difference in equation (3.32) with the multivariate absolute difference AD_{ℓ_1} in (3.33) yields the dynamic process of the degrees of freedom of a multivariate Student-t copula

$$\nu_t = \Lambda(\varsigma + \varphi \times \sum_{i=1}^{d} |u_{i,t-1} - \tilde{u}_{t-1}|). \tag{3.34}$$

The mapping into the authorized domain $(L_\nu = 2, U_\nu = 100)$ is ensured by the logistic transformation

$$\Lambda(x) = L_\nu + \frac{(U_\nu - L_\nu)}{1 + e^{-x}}. \tag{3.35}$$

Even if x is permitted to vary over the entire real line, $\Lambda(x)$ will be constrained to lie in the domain $[L_\nu, U_\nu]$. The dynamic multivariate Student-t copula is therewith defined as

$$C^t_{\nu_t,\Sigma_t}(u_1, ..., u_d) = t_{\nu_t,\Sigma_t}(t^{-1}_{\nu_t}(u_1), ..., t^{-1}_{\nu_t}(u_d)), \tag{3.36}$$

with the dynamics of Σ_t given in (3.29) respectively (3.30) and the evolution of ν_t modeled with (3.34) and (3.35).

3.4.2 Dynamic Archimedean Copulas

Patton (2006a) adapts the idea of Engle (2002) to model the dynamics of bivariate Archimedean copulas with an ARMA-type process. He assumes that the functional form of the copula stays fixed over the sample, whereas the transformed copula parameter as Kendall's tau varies according to the evolution equation

$$\rho_{\tau_t} = \Lambda\left(\omega + \beta \times \rho_{\tau_{t-1}} + \alpha \times \frac{1}{10}\sum_{j=1}^{10} |u_{t-j} - v_{t-j}|\right) \tag{3.37}$$

where $\Lambda(x) = (1 + e^{-x})^{-1}$ is the logistic transformation to keep $\rho_{\tau_t} \in [0, 1]$ at all times and (u_t, v_t) are two observations at time t.[6] The Clayton respectively the Frank copula parameter in time t can then be obtained using the functional relationship between Kendall's tau and the Archimedean copula parameter

Copula	ρ_τ
C_θ^{Cl}	$\theta/(\theta + 2)$
C_θ^{Fr}	$1 - 4\theta^{-1}(1 - D_1(\theta))$

(3.38)

where $D_1(\theta)$ is the Debye function of order one $D_1(\theta) = \theta^{-1} \int_0^\theta t/(\exp(t)-1)dt$ (Hofert et al., 2013).

The dynamic process of Patton (2006a) in (3.37) is yet again limited to bivariate applications through the absolute difference term $|u_{t-1} - v_{t-1}|$. To extend (3.37) to the multidimensional world, this difference term is substituted with the multivariate absolute distance AD_{ℓ_1} in (3.33). This yields a

[6] Kendall's tau is the rank correlation for two vectors of random variables Y_1 and Y_2, defined as $\rho_\tau = E(\text{sign}((Y_1 - \tilde{Y}_1)(Y_2 - \tilde{Y}_2)))$ where $(\tilde{Y}_1, \tilde{Y}_2)$ is an independent copy of (Y_1, Y_2) (McNeil et al., 2005).

multivariate extension of Patton's (2006a) parameter evolution process

$$\rho_{\tau_t} = \Lambda\left(\omega + \beta \times \rho_{\tau_{t-1}} + \alpha \times \frac{1}{10}\sum_{j=1}^{10}\sum_{i=1}^{d}|u_{i,t-j} - \tilde{u}_{t-j}|\right), \tag{3.39}$$

where $\tilde{u}_t$ is the median of $u_1, ..., u_d$ in time t and $\Lambda(x) = (1 + e^{-x})^{-1}$. With (3.39), the parameter of the multivariate Clayton copula in time t, θ_t^{Cl}, is then given in closed form through (3.38). The Frank copula parameter θ_t^{Fr} in terms of Kendall's tau however is not available in closed form but has to be determined numerically. This is computationally burdensome since the numerical optimization has to be done for every time step t in every iteration of the overall calibration procedure of the dynamic Frank copula. To achieve an efficient calibration of the dynamic Frank copula, this thesis directly models the dynamics of θ_t^{Fr} as

$$\theta_t^{Fr} = \omega + \beta \times \theta_{t-1}^{Fr} + \alpha \times \frac{1}{10}\sum_{j=1}^{10}\sum_{i=1}^{d}|u_{i,t-j} - \tilde{u}_{t-j}|, \tag{3.40}$$

where the constraint $\theta_t^{Fr} \geq 0$ ensures that the parameter remains in the permissible range. Stationarity and invertibility is accounted for with the constraints $|\alpha| < 1$ and $|\beta| < 1$.

3.4.3 Dynamic Mixture Copulas

Ng (2008) adopts the dynamic process of Patton (2006b) to create a time varying specification of the weight in the mixture copula depending on the natural filtration of the process. He suggests a dynamic bivariate mixture copula model, where the parameters of the copulas are constant, but the weighting parameter is stochastic, following an ARMA-type model for the mixture weight:

$$w_{i,t} = \omega_i + \alpha_i \times h_{i,t-1}(.) + \beta \times w_{i,t-1}. \tag{3.41}$$

He therewith establishes a linear relationship between the mixture weight w_i in time t and the according lagged value in $t-1$ and $h(.)$, which is a stochastic explanatory variable or a special function. In particular, Ng (2008) proposes to model $w_{i,t}$ with the special function being

$$h_{i,t-1}(.) = \frac{1}{10}\sum_{p=1}^{10} |u_{t-p} - v_{t-p}|. \tag{3.42}$$

However, also this model is limited to the bivariate setting due to the absolute distance measure $|u_{t-p} - v_{t-p}|$. Braun (2011) suggests an extension of Ng's (2008) concept to higher dimensions by replacing the absolute distance measure with the copula density relative to the sum of all copula densities during the lag period. This results in a special function of the following type:

$$h_{i,t-1}(.) = \frac{1}{10}\sum_{p=1}^{10}\left(\frac{c_i(u_{1,t-p},...,u_{d,t-p};\theta_i)}{\sum_{j=1}^{n} c_j(u_{1,t-p},...,u_{d,t-p};\theta_j)}\right). \tag{3.43}$$

To generate weight forecasts with equation 3.43 plugged into 3.41, Braun (2011) has to impose six different constraints on the weight process parameters in order to keep the resulting weights $w_{i,t}$ within the unit interval. In contrast, this thesis makes use of the logistic function $\Lambda(x) = (1+e^{-x})^{-1}$ which in combination with 3.43 and 3.41 results in:

$$w_{i,t-1} = \Lambda\left(\omega_i + \alpha_i \times \frac{1}{10}\sum_{p=1}^{10}\left(\frac{c_i(u_{1,t-p},...,u_{d,t-p};\theta_i)}{\sum_{j=1}^{n} c_j(u_{1,t-p},...,u_{d,t-p};\theta_j)}\right) + \beta_i \times w_{i,t-1}\right) \tag{3.44}$$

The weight parameters are therewith bounded on the unit interval without the need to impose any constraints on the parameters. Note that equation (3.44) also nests the static mixture copula with $\alpha = \beta = 0$. Employing the dynamic weights of equation 3.44 in the mixture copula construct in equation

3.20 yields the complete multivariate dynamic mixture copula model:

$$C\left(u_1, ..., u_d; w_{1,t}, ..., w_{n,t}; \theta_1, ..., \theta_n\right) = \sum_{j=1}^{n} [w_{j,t}, C_j(u_1, ..., u_d, \theta_j)]. \quad (3.45)$$

It has to be emphasized that the parameters θ_j and $w_{j,t}$ have different functions within the mixture copula construct, allowing a very flexible way of modeling dependence structures. While the association parameter θ controls the degree of dependence, the weight parameter w determines the structure of the dependence.

The advantage of linking the weight parameter to the copula densities lies in the difference of the copula density functions. The Clayton copula for example is capable of modeling lower tail dependence and exhibits its largest density in the lower tail (see figure 3.1). Thus, the weight parameter in the dynamic mixture structure is directly coupled with the capabilities of the mixture copula constituents to describe the dependence structure during the lag period.

Shifts in the dependence structure are expected to have an immediate effect on the dynamic weights. A rise of one copula's relative density signalizes its enhanced fit to the current dependence pattern. Through the dynamic weight process in 3.44, this copula's weight in the mixture construct and its impact on the overall mixture density therewith extends. Calibrating this model using maximum likelihood estimation ensures that the parameters of each copula in the mixture are fitted most accurately to those data fractions, where the dependence structure naturally concurs with the copula's characteristics. Every individual copula thus only captures the dependence in a specific part of the data set in a optimal way, but merging the copulas into a mixture structure governed by the dynamic weight process yields an overall accurate and flexible dependence model.

To employ the introduced static, regime switching and fully dynamic multivariate copulas in empirical applications, the model parameters have to estimated. To this end, this thesis makes use of a multi-stage maximum likelihood estimation procedure whose description is the subject of the following chapter.

Chapter 4

Multi-Stage Maximum Likelihood Estimation

The first part of this chapter outlines the estimation procedure for the model parameters and the second part explains how the according standard errors are computed.

4.1 Estimation Procedure

The model estimation takes advantage of the fact that the copula is independent of the marginal distributions and separates the procedure into different stages (see e.g. Chollete et al., 2009; Joe, 2001; McNeil et al., 2005).
The overall log likelihood depends on all the data $Y = (Y_1', ..., Y_T')$, and is given by

$$\log \mathcal{L}(Y; \theta_m, \theta_c) = \sum_{t=1}^{T} \log f(Y_t | Y^{t-1}; \theta_m; \theta_c), \tag{4.1}$$

where $Y^{t-1} = (Y_1, ..., Y_t)$ represents the history of the entire process. The likelihood can thus be decomposed into one part $\log \mathcal{L}_m$ containing the marginal densities and a second part $\log \mathcal{L}_c$ which contains the copula densities

$$\log \mathcal{L}(Y; \theta_m, \theta_c) = \log \mathcal{L}_m(Y; \theta_m) + \log \mathcal{L}_c(Y; \theta_m, \theta_c) \tag{4.2}$$

$$\log \mathcal{L}_m(Y;\theta_m) = \sum_{t=1}^{T}\sum_{i=1}^{d} \log f_i(y_{i,t}|y_i^{t-1};\theta_{m,i}) \tag{4.3}$$

$$\log \mathcal{L}_c(Y;\theta_m,\theta_c) = \sum_{t=1}^{T} \log c(F_1(y_{1,t}|y_1^{t-1};\theta_{m,1}),...,F_d(y_{d,t}|y_d^{t-1},\theta_{m,d});\theta_c), \tag{4.4}$$

where $y_i^{t-1} = (y_{i,1},...,y_{i,t})$ is the entire history of variable i.
The marginal models' likelihood $\log\mathcal{L}_m$ is a function of the parameter vector $\theta_m = (\theta_{m,1},...,\theta_{m,d})$ which collects the parameters for each of the d marginal density functions f_i. The likelihood of the copula $\log\mathcal{L}_c$ directly depends on vector θ_c. For the estimation of singular static copulas, θ_c contains the copula parameters. In case of the static mixtures, $\theta_c = (\theta_c^{(1)},\theta_c^{(2)},w)$ comprises the parameters of both copulas in the mixture copula and the mixture weight. For the estimation of regime switching copula containing singular copulas, the vector $\theta_c = (\theta_c^{(1)},...,\theta_c^{(k)},P)$ collects the copula parameters over all regimes plus the parameters of the transition probability matrix P while regime switch copulas comprising mixtures, the mixture weight parameter w is also collected $\theta_c = (\theta_c^{(1a)},\theta_c^{(1b)},\theta_c^{(2)},P,w_{1a})$. Through the distribution function F_i this parameter vector θ_c also indirectly depends on the parameters of the marginal densities, since F_i transforms the observations into uniform $[0,1]$ variables based on which the copula is estimated.

The models in this thesis accommodate a number of parameters such that a full single-step likelihood maximization is numerically rather intricate and time consuming. Maximizing the parameters separately for the margins and the copula is also referred to as "inference functions for margins" or IFM (see e.g. Joe, 2001; McNeil et al., 2005), though more generally it is known as multi-stage maximum likelihood estimation (MSMLE) (Patton, 2013). Compared to a one-stage estimation it represents a much more tractable procedure whose properties have been studied by Chen and Fan (2006); Joe (2005) and Patton (2006b) and which has repeatedly been applied in the context of copulas (see e.g. Chollete et al., 2009; Dias and Embrechts, 2010; Garcia and Tsafack, 2011;

Patton, 2006b). While the multi-stage estimation generally entails some loss of efficiency in comparison to estimating the entire joint distribution in one single step, it substantially simplifies the computational burden and comes with a low loss of efficiency. (Joe, 2005; Patton, 2006b)

In the first step, it is assumed that the marginals are independent from each other and depend only on their own history:

$$\hat{\theta}_m = \underset{\theta_m}{\operatorname{argmax}} \mathcal{L}_m(Y; \theta_m). \tag{4.5}$$

This estimation can further be simplified since the univariate model parameters of each time series can be calibrated separately:

$$\hat{\theta}_{m,i} = \underset{\theta_{m,i}}{\operatorname{argmax}} \sum_{t=1}^{T} \log f_i(y_{i,t}|y_i^{t-1}; \theta_{m,i}). \tag{4.6}$$

For the second step, the marginal parameters are bundled in a vector $\hat{\theta}_m = (\hat{\theta}_{m,i}, ..., \hat{\theta}_{m,n})$ and are considered as given in order to calibrate the copula:

$$\hat{\theta}_c = \underset{\theta_c}{\operatorname{argmax}} \mathcal{L}_c(Y; \hat{\theta}_m, \theta_c). \tag{4.7}$$

This procedure tremendously reduces the computational effort and time, firstly because a numerical optimization with lots of parameters is much more time consuming compared with several numerical optimizations, each with fewer parameters. Secondly, the multi-stage method reduces the overall computational burden since different copulas are estimated given the same univariate models, which means that the parameter vector $\hat{\theta}_m$ can be reused.

To guarantee positive semi-definiteness of the correlation matrices Σ contained in the elliptical copula models, the numerical optimization is effectively carried out for the lower triangular matrix A of Cholesky factors, such that $\Sigma = AA'$. All maximum likelihood estimation procedures require starting values for the optimization to initialize. With the plurality of the parameters to be estimated in the more elaborated dependence models, the choice of initial values can have a direct impact on the overall estimation time. Naturally the closer the starting values are to the final parameters, the faster the optimization algorithm converges. For the correlation parameters of the Gaussian copulas, the Cholesky factors of the quantile transformed uniform values' correlation matrix are used as initial values. For the Student-t copula's correlation parameters, the fact that sample rank correlations can be used to partially calibrate Student-t copulas is exploited (see McNeil et al., 2005). The relationship between Kendall's tau and the t-copula's correlation is

$$\rho_\tau(y_i, y_j) = \frac{2}{\pi} \arcsin \rho^c_{i,j}, \tag{4.8}$$

such that a possible estimator of the Student-t copula correlation matrix is given by a matrix with the components

$$\rho^c_{i,j} = \sin(\frac{1}{2}\pi\rho^\tau_{i,j}). \tag{4.9}$$

For the Archimedean copulas, in a first step the $\binom{d}{2}$ pairwise Kendall's tau estimators are computed. Using the arithmetic average over all these estimators, the functional relationship between Kendall's tau ρ_τ and the Archimedean copula parameter θ presented in (3.38), is exploited. The usage of these initial values was found to decrease estimation time supporting the findings of Hofert et al. (2013), who in contrast to some statements in literature (see e.g. Berg and Aas (2009) or Weiss (2010)) find that maximum-likelihood estimation is feasible in higher dimensions and performs well.

4.2 Standard Error Computation

To calculate the standard errors within the MSMLE framework, Joe (2001) and Durrleman et al. (2000) show that the vector of parameter estimates, $\hat{\theta}_{MSMLE} = [\hat{\theta}_{m,1}, ..., \hat{\theta}_{m,d}, \hat{\theta}_c]$ verifies the property of asymptotic normality such that

$$\sqrt{T}(\hat{\theta}_{MSML} - \theta_0) \xrightarrow{d} \mathcal{N}(0, \mathbf{G^{-1}}(\theta_\mathbf{0})) \quad \text{as} \quad T \to \infty, \tag{4.10}$$

where $\mathbf{G}(\theta_\mathbf{0})$ is the asymptotic variance-covariance matrix known as the information matrix of Godambe. It is based on the theory of inference functions which imposes optimality criteria on the score functions of the estimating equations rather than on the estimators received from them (Godambe, 1960, 1976, 1991).
Following Joe (2001) in defining a score function $\mathbf{g}$:

$$\mathbf{g}(\theta) = (\partial_{\theta_\mathbf{1}} \log \mathcal{L}_{\mathbf{m},\mathbf{1}}, ..., \partial\theta_{\mathbf{m},\mathbf{d}} \log \mathcal{L}_{\mathbf{m},\mathbf{d}}, \partial\theta_\mathbf{c} \log \mathcal{L}_\mathbf{c}), \tag{4.11}$$

the Godambe information matrix takes the form

$$\mathbf{G}(\theta_\mathbf{0}) = \mathbf{D^{-1}M(D^{-1})'}, \tag{4.12}$$

where

$$\mathbf{D} = \mathbf{E}\left[\frac{\partial \mathbf{g}(\theta)'}{\partial\theta}\right], \quad \mathbf{M} = \mathbf{E}\left[\mathbf{g}(\theta)'\mathbf{g}(\theta)\right]. \tag{4.13}$$

For the models at hand, the estimation of the Godambe information matrix requires the computation of a number of derivatives (i.e. score functions for likelihood functions) which can be computationally demanding. Moreover, the main issue in implementing the multi-stage estimation method compared to a one-step maximum likelihood method is its loss of performance in the estimation of the parameters since the marginal estimation procedure in the first step neglects the possible dependence between $\theta_{m,1}, ..., \theta_{m,d}$ when estimating θ_c (Liu and Luger, 2009).

In order to adequately compute standard errors for the models, this thesis therefore follows Silva Filho et al. (2012) and De Lira Salvatierra and Patton (2013) in employing a bootstrap approach to calculate the covariance matrix of the parameters. To asymptotically preserve the cross-sectional dependence in the time series data, the block bootstrap method of Politis and Romano (1994) is applied.[1] The optimal block size is determined based on the automatic block-length selection of Politis and White (2004) and Patton et al. (2009). The standard error computation procedure is as follows:

1. Estimate parameters $\hat{\theta}$ as described in section 4.1.
2. Generate a block bootstrap sample of the same size as the data set.
3. Estimate the parameters $\hat{\theta}^{(r)}$ on the simulated data.
4. Repeat steps (2) and (3) R times.
5. Compute standard errors using the covariance matrix

   $$R^{-1} \sum_{r=1}^{R} (\hat{\theta}^{(r)} - \hat{\theta})(\hat{\theta}^{(r)} - \hat{\theta})',$$

   where $\hat{\theta}^{(r)}$ is the column vector of the estimated parameters for every replication r.

Before deploying the multi-step estimation and standard error computation procedures for the models presented in chapter 2 and 3, the next chapter presents the data sets under investigation.

[1] See Gonçalves and White (2004) for theoretical justification.

Chapter 5

Data Description

This chapter introduces the three data sets used for the empirical analysis and presents descriptive statistics. With either purely equities or commodities, the first two data sets focus on portfolios of assets from the same asset class, while the third set contains assets from multiple asset classes. Each portfolio is therewith expected to reflect different characteristics in terms of risk and return and allows to investigate the forecast accuracy of the different copula models for different types of portfolios. All portfolios contain only indices as opposed to individual stocks. Based on the central limit theorem, such portfolios are more likely to exhibit elliptical dependence than are individual stocks.[1] Hence, the analysis is biased against the empirical tests of return forecasts incorporating asymmetric dependence. The data have been collected from Bloomberg. All returns are computed with $\log(P_t/P_{t-1})$ where P_t is the value of the index at time t.

5.1 Equity Index Portfolio

With the first data set, this thesis investigates the interactions between international equity indices. The utilized sample reaches back to the initialization of the Swiss Market Index (SMI) and covers the period from June 30, 1988

[1] Note that the prices of some indices in the data sets are market capitalization-weighted averages while the classical Central Limit Theorem is based on equally weighted averages.

until June 5, 2013, yielding 1300 weekly returns. Besides the Swiss index, the data covers the following indices: the German DAX 30, the French CAC 40, the British FTSE 100, the US S&P 500, the Canadian TSX 60, the Chinese Hang Seng and the Japanese Nikkei 225. To take into account the fact that stock exchanges around the globe have different trading hours and to avoid introducing artificial dependence due to the differences in closing times, the data consists of weekly returns from Wednesday to Wednesday. For the analyses in this thesis, a currency hedged USD investor is assumed in order to eliminate exchange rate effects and concentrate on the interactions of the assets.

The descriptive statistics of the weekly returns of each equity index under consideration are summarized in table 5.1. Each of the equity index return series displays a negative skewness, indicating that the tail on the left side of the probability density function is longer compared to the right side and the majority of the values are located to the right of the mean. Every series further shows a leptokurtic distribution, with excess kurtosis ranging from 2.8 to 4.4. Unsurprisingly, the test of Jarque and Bera (1987) rejects the null hypothesis that the sample returns come from a normal distribution for every continuous return series on the significance level of 1%. The largest unconditional correlation occurs between the DAX 30 and the CAC 40 (0.851) while the smallest correlation is between the SMI and the Nikkei 225 (0.414). The test of Leybourne and McCabe (1999) (LMC) and the Augmented Dickey-Fuller (ADF) test (Dickey and Fuller, 1979) both indicate that each of the return series is stationary. The first by not rejecting the null hypothesis of a stationary AR(k) against a nonstationary ARIMA(k,1,1) and the latter by rejecting the null hypothesis of a unit-root against a trend-stationary alternative augmented with k lagged difference terms on the usual significance levels for all orders $k = (1, ..., 20)$.[2]

To identify whether conditional heteroscedasticity effects are present in

[2] To conserve space, table 5.1 only reports the test statistics for the orders one, five and ten.

the data, an indirect and a direct test is conducted. The indirect approach is the portemanteau test of Ljung and Box (1978) applied to the squared residuals of the de-meaned return series. Engle's (1982) Lagrange multiplier statistic LM(k) on the other hand directly assesses the significance of the ARCH effects by testing whether the squared returns are serially correlated up to lag k. Both statistics clearly indicate that ARCH effects are likely to be found in all of the equity index returns series. The univariate models for the equity index returns thus have to account for conditional heteroscedasticity.

5.2 Commodity Futures Index Portfolio

The second data set consists of commodity futures indices. In the past decade, the number of open contracts in commodity exchanges almost doubled resulting in volumes of exchange-traded derivatives of 20 to 30 times the physical production for many commodities (Silvennoinen and Thorp, 2013). Whilst hedge fund activity tripled between 2004 and 2007, capital flows from institutional investors increased thirteenfold. (Commodity Futures Trading Commission, 2008)

Commodities exhibit certain risk characteristics which differ from traditional assets such as equities. On the one hand, some commodities like agricultural products are not storable while others such as energy and livestock commodities may only be stored at very high costs. Changes in demand or supply thus translate directly into price changes which results in greater volatility of the commodity investments in comparison to investments in traditional assets. On the other hand, shocks of demand and supply are more extensive and observed more often as frost, drought or natural disasters have an immediate effect on some commodity prices. Additionally, the political instability of oil-exporting countries and the lack of governmental control account for further variation in commodity prices. (Füss et al., 2010) An appropriate model to forecast commodity portfolio risks must be capable to capture these characteristics.

Table 5.1
Summary statistics: Equity portfolio

	SMI	DAX	CAC	UKX	SPX	TSX	HSI	NKY
Mean	0.001	0.002	0.001	0.001	0.001	0.001	0.002	-0.001
Std	0.026	0.031	0.030	0.023	0.023	0.022	0.035	0.031
Max	0.148	0.172	0.166	0.136	0.102	0.087	0.156	0.148
Min	-0.140	-0.168	-0.148	-0.127	-0.165	-0.141	-0.210	-0.211
Skew	-0.469	-0.715	-0.376	-0.250	-0.615	-0.625	-0.592	-0.398
Kurt	7.022	6.640	6.024	6.351	7.412	5.887	5.847	6.204
JB	902.6^a	808.3^a	514.2^a	607.3^a	1109.0^a	522.4^a	502.9^a	576.0^a
Correlations								
SMI		0.764	0.770	0.750	0.660	0.539	0.445	0.414
DAX			0.851	0.748	0.709	0.597	0.504	0.475
CAC				0.798	0.705	0.599	0.499	0.477
UKX					0.724	0.614	0.542	0.466
SPX						0.760	0.489	0.453
TSX							0.464	0.419
HSI								0.448
Stationarity Tests								
LMC (1)	0.084^c	0.063^c	-0.248^c	0.054^c	0.081^c	0.178^a	0.037^c	0.042^c
LMC (5)	0.087^c	0.057^c	0.049^c	0.056^c	0.077^c	0.040^c	0.036^c	0.042^c
LMC (10)	0.109^c	0.071^c	0.073^c	0.089^c	0.091^c	0.033^c	0.035^c	0.037^c
ADF (1)	-38.6^a	-39.6^a	-41.0^a	-39.0^a	-38.9^a	-36.2^a	-35.5^a	-36.4^a
ADF (5)	-15.7^a	-15.5^a	-15.9^a	-16.3^a	-15.8^a	-15.1^a	-15.8^a	-15.5^a
ADF (10)	-11.0^a	-10.7^a	-11.3^a	-12.2^a	-11.3^a	-10.4^a	-11.1^a	-11.1^a
Heteroscedasticity Tests								
LBQ (1)	167.7^a	74.9^a	113.6^a	109.5^a	54.3^a	87.0^a	51.0^a	18.6^a
LBQ (5)	228.2^a	174.6^a	225.8^a	188.1^a	181.3^a	293.1^a	212.8^a	152.9^a
LBQ (10)	292.3^a	238.5^a	302.6^a	284.4^a	264.0^a	398.8^a	311.0^a	162.0^a
ELM (1)	164.4^a	74.7^a	111.9^a	105.8^a	52.7^a	84.5^a	51.1^a	18.8^a
ELM (5)	179.0^a	116.6^a	152.1^a	129.6^a	114.8^a	170.9^a	120.6^a	110.4^a
ELM (10)	191.4^a	124.7^a	162.1^a	150.3^a	149.4^a	185.4^a	142.3^a	119.8^a

Summary statistics of the weekly returns over the full sample period from June 30, 1988 to June 5, 2013 for the SMI, DAX 30, CAC 40, FTSE 100 (UKX), S&P 500 (SPX), S&P/TSX 60, Hang Seng (HSI) and Nikkei 225 (NKY) index. Mean, Std, Skew and Kurt denote the mean, standard deviation, skewness and kurtosis. JB is the test statistic of the Jarque-Bera test for normality of the unconditional distribution of the returns. The correlations report Pearson's linear unconditional sample correlations between the weekly returns over the full sample period. LMC(k) is the statistic of Leybourne and McCabe's (1999) test assessing the null hypothesis of a trend stationary AR(k) process against the alternative of a nonstationary ARIMA(k,1,1) process. ADF(k) is the statistic of the augmented Dickey-Fuller (1979) test for a unit root against a trend-stationary alternative augmented with k lagged difference terms. LBQ (k) is the statistic of the Ljung-Box (1978) portmanteau Q-test assessing the null hypothesis of no autocorrelation in the squared (mean-subtracted) residuals at k lags. ELM (k) is Engle's (1982) Lagrange multiplier statistic for heteroscedasticity obtained by regressing the squared returns on k lags. Significance is denoted by superscripts at the 1% (^{a}), 5% (^{b}) and 10% (^{c}) levels.

The growing importance of commodities as an asset class along with its particularities have attracted the interest of scholars in the last few years. Recent contributions applying copula models include Weiss (2011), who investigates the use of goodness-of-fit tests for static copulas applied to stocks, commodities and FX futures. Gronwald et al. (2011) study the dependence structure between carbon emission allowances and commodities, equity and energy indices. Constrained to bivariate copulas, Delatte and Lopez (2013) investigate the linkages between commodity and equity markets. To the best of the author's knowledge, this thesis is the first work to apply higher dimensional dynamic copula models to a commodity portfolio.

Financial investors primarily take positions in commodity futures since their interest is not to own the physical asset, but rather to gain exposure to commodity risk (Gonzalez-Pedraz et al., 2012). The most popular strategy to invest in commodities is to invest in given commodity futures indices (Tang and Xiong, 2012). The commodity data set used in this thesis consists of subindices of the Standard & Poors Goldman Sachs Commodity Index (SPGSCI), which together with the DJ-UBSCI is by far the most influential commodity index. The series were chosen based on the length of their data history and their weight in the main index. The data set contains the following commodity futures excess return indices: crude oil, heating oil, unleaded gasoline, gold, silver, copper, wheat as well as corn and therewith covers the commodity sectors energy, precious metals, industrial metals and agriculture. The excess return measures the return from investing in nearby commodity futures and rolling them forward each month to avoid the cost of holding physical commodities. In this way, the selected commodity indices yield returns comparable to passive long positions in listed commodity futures contracts.[3]

To ensure consistency with the utilized equity index data, Wednesday to

[3]More precisely, the return consists of a spot and a roll element. The spot return is the percentage change in the near-month futures contract. To keep a long future position the futures contracts are rolled forward to the next-month futures contract. The roll return is positive when the market is in backwardation and negative when the market is in contango. The roll return therewith comes from rolling up or down the term structure of futures prices.

Wednesday weekly returns over the identical period ranging from June 30, 1988 until June 5, 2013 are used, yielding 1300 returns. Table 5.2 presents the descriptive statistics of the commodity future data. The energy sector index returns display the largest standard deviations while gold returns have the smallest standard deviation among the commodity indices. With the exception of wheat and corn, all commodity returns are negatively skewed. All series display excess kurtosis ranging from 2.399 to 4.638. The Jarque-Bera test clearly rejects the hypothesis of a normal distribution for all commodity index returns.

The largest unconditional sample correlations are detected between the fossil fuel returns, followed by the correlation between the agricultural returns. For all lags $k = (1, ..., 20)$, the test of Leybourne and McCabe cannot reject the null hypothesis of a stationary AR(k) process. Stationarity of all the series is confirmed by the results of the augmented Dickey-Fuller tests which reject the null of a unit-root against a trend-stationary alternative for all $k = (1, ..., 20)$. Engle's (1982) Lagrange multiplier test rejects the null hypothesis of no conditional heteroscedasticity for every commodity futures index such that one may conclude that there are significant ARCH effects in all the commodity return series.

5.3 Multi Asset Classes Index Portfolio

Based on modern portfolio theory, investors should not allocate the entire capital in one asset class such as equities but seek diversification. According to Lombardi and Ravazzolo (2013), most fund managers have begun to advise their customers to allocate a share of their portfolios to commodity-related products as a part of the diversification strategy. This is frequently motivated by studies of e.g. Gorton and Rouwenhorst (2006), Büyükşahin et al. (2010) or Cheung and Miu (2010), who show that commodities display low correlation with other asset classes, specifically with equities. Following this advice, two commodity indices are added to a selection of four international equity indices.

Table 5.2
Summary statistics: Commodity portfolio

	OIL	HOL	GAS	GLD	SLV	CPP	WHT	CRN
Mean	0.001	0.001	0.002	0.000	0.000	0.002	-0.002	-0.002
Std	0.046	0.044	0.047	0.022	0.039	0.035	0.037	0.035
Max	0.232	0.205	0.241	0.129	0.148	0.170	0.190	0.153
Min	-0.318	-0.276	-0.266	-0.132	-0.295	-0.171	-0.177	-0.169
Skew	-0.498	-0.278	-0.286	-0.242	-0.660	-0.109	0.297	0.001
Kurt	6.774	5.667	5.891	7.518	7.638	5.399	4.734	5.411
JB	825.0^a	401.9^a	470.6^a	1118.2^a	1259.8^a	314.4^a	182.0^a	315.0^a
Correlations								
OIL		0.882	0.858	0.261	0.264	0.246	0.113	0.163
HOL			0.833	0.243	0.244	0.225	0.123	0.164
GAS				0.212	0.217	0.231	0.104	0.140
GLD					0.731	0.284	0.157	0.193
SLV						0.331	0.164	0.220
CPP							0.174	0.171
WHT								0.594
Stationarity Tests								
LMC(1)	0.067^c	0.059^c	0.047^c	0.050^c	0.050^c	0.144^b	0.051^c	0.062^c
LMC(5)	0.043^c	0.043^c	0.036^c	0.090^c	0.058^c	0.112^b	0.054^c	0.059^c
LMC(10)	0.038^c	0.057^c	0.039^c	0.196^a	0.073^c	0.082^c	0.060^c	0.045^c
ADF(1)	-37.2^a	-36.4^a	-36.9^a	-36.7^a	-37.1^a	-36.3^a	-36.2^a	-36.5^a
ADF(5)	-14.3^a	-15.1^a	-14.5^a	-18.6^a	-17.6^a	-14.2^a	-16.0^a	-15.5^a
ADF(10)	-9.6^a	-10.1^a	-11.2^a	-13.2^a	-12.7^a	-9.9^a	-11.5^a	-10.0^a
Heteroscedasticity Tests								
LBQ(1)	67.8^a	72.5^a	107.4^a	31.5^a	11.5^a	69.3^a	60.6^a	29.4^a
LBQ(5)	140.0^a	154.7^a	184.2^a	181.2^a	75.5^a	356.8^a	239.3^a	129.3^a
LBQ(10)	312.9^a	237.1^a	297.9^a	272.4^a	132.9^a	463.3^a	288.8^a	241.0^a
ELM(1)	67.8^a	72.5^a	107.4^a	31.5^a	11.5^a	69.3^a	60.6^a	29.4^a
ELM(5)	140.0^a	154.7^a	184.2^a	181.2^a	75.5^a	356.8^a	239.3^a	129.3^a
ELM(10)	312.9^a	237.1^a	297.9^a	272.4^a	132.9^a	463.3^a	288.8^a	241.0^a

Summary statistics of the weekly returns over the full sample period from June 30, 1988 to June 5, 2013 for the Standard & Poors Goldman Sachs Commodity excess return subindices: crude oil (OIL), heating oil (HOL), unleaded gasoline (GAS), gold (GLD), silver (SLV), copper (CPP), wheat (WHT) and corn (CRN). Mean, Std, Skew and Kurt denote the mean, standard deviation, skewness and kurtosis. JB is the test statistic of the Jarque-Bera test for normality of the unconditional distribution of the returns. The correlations report Pearson's linear unconditional sample correlations between the weekly returns over the full sample period. LMC(k) is the statistic of Leybourne and McCabe's (1999) test assessing the null hypothesis of a trend stationary AR(k) process against the alternative of a nonstationary ARIMA(k,1,1) process. ADF(k) is the statistic of the augmented Dickey-Fuller (1979) test for a unit root against a trend-stationary alternative augmented with k lagged difference terms. LBQ (k) is the statistic of the Ljung-Box (1978) portmanteau Q-test assessing the null hypothesis of no autocorrelation in the squared (mean-subtracted) residuals at k lags. ELM (k) is Engle's (1982) Lagrange multiplier statistic for heteroscedasticity obtained by regressing the squared returns on k lags. Significance is denoted by superscripts at the 1% (^{a}), 5% (^{b}) and 10% (^{c}) levels.

As for the portfolio in section 5.2, the added commodity futures indices are part of the Standard & Poors Goldman Sachs Commodity Index family. With the S&P GSCI Non Energy Index and the S&P GSCI Energy Index, the two selected indices for the multi asset classes portfolio are on a higher level of aggregation. The two indices cover all commodity sectors while accounting for the different characteristics of energy and non-energy commodities.

Along with commodities, an important non-traditional asset class expected to provide a defensive component to a stock market portfolio is real estate (Chang et al., 2011). Ghysels et al. (2013) emphasize that the importance of real estate as an asset class cannot be overstated: at the end of 2011 the total value of this asset class in the United States was about USD 25 trillion, while the capitalization of the U.S. stock market amounted to roughly USD 18 trillion. With the expansion of the modern real estate investment trust (REIT) in the 1990s, investors became increasingly able to add a wider choice of property assets to their portfolio, which overcome many drawbacks of direct real estate investment. (Chang et al., 2011) Whilst the residential housing market decline and the collapse of subprime mortgage derivatives questioned the expectation of a diversification effect, the recent study of Chang and Chen (2014) of the period from 2006 to 2010 shows that REITs are an effective method of international diversification. The addition of a real estate exposure to a portfolio of equities and commodities should thus enhance the risk-return profile and create a multi-faceted dependence structure.

Aiming to test the predictive power of the presented models for an accordingly diversified portfolio, the third data set utilized for the empirical analysis consists of a portfolio of four equity indices enriched with the two commodity indices introduced above and two real estate indices. With the FTSE EPRA/NAREIT North America Index and the FTSE EPRA/NAREIT Europe Index, the choice of REITs covers two distinctly different parts of the world, taking into account the contribution of Adams et al. (2014) who highlight the importance of geographical diversification when investing in REITs. The indices of the European Public Real Estate Association (EPRA) and

National Association of Real Estate Investment Trusts (NAREIT) in collaboration with the Financial Times Stock Exchange (FTSE) are considered to be the leading benchmarks for listed real estates and serve as the basis for many investment products, such as derivatives and Exchange Traded Funds (ETFs).

In analogy with the previous two portfolios, weekly returns from Wednesday to Wednesday are used. The available data history for the chosen real estate indices is almost as extensive as for the equities and commodities. The data series for the multi asset classes portfolio contain 1222 weekly returns covering the time frame from January 3, 1990 to June 5, 2013. Table 5.3 presents the summary statistics of the multi asset classes portfolio. The real estate investment trust indices reveal similar mean and standard deviations compared to the equity indices. The REIT index series are left skewed - the NAREIT North America Index even shows the most skewed return distribution of all considered return series in this thesis. The REITs further show excess kurtosis of more than 7 (EPR) and 12 (NAR). The Jarque-Bera test rejects the null hypothesis of normally distributed REIT returns with the largest JB test statistics of all series under consideration.

The combination of indices of three different asset classes results in a diverse unconditional sample correlation matrix containing values from 0.002 to 0.776. The highest correlations are observed between the European equity index returns, followed by the correlations between the S&P 500 and the European equity indices. The NAREIT North America Index returns correlate more with the S&P 500 returns than with the returns of the European REIT index. The commodity indices clearly depict the lowest correlation to both other asset classes. The S&P GSCI Energy index seems to be almost uncorrelated to the returns of the continental European equity index returns. Note that the overall sample of the multi asset classes portfolio is one and a half years shorter than both the commodities and the equities sample. The small differences in the statistics for the equity indices in table 5.3 compared to the values in table 5.1 are attributable to the somewhat different lengths of the data sets. Failing to reject the null of a stationary AR(k) process, the tests of Leybourne and McCabe suggest that all series in the multi asset classes portfolio are stationary. This is underpinned by the results of the augmented Dickey-Fuller test, which rejects the null of a unit root against a trend-stationary alternative for every return series. Assessing the null hypothesis that the squared demeaned series exhibit no autocorrelation for $k = 1, ..., 20$ lags against the alternative that some autocorrelation coefficient $\rho(k)$, $k = 1, ..., 20$, is nonzero, the portmanteau tests of Ljung and Box (1978) reject the null hypothesis for every return series and every lag k. The results of Engle's (1982) Lagrange multiplier test state that there is enough evidence to conclude that the returns on each of the indices in the portfolio are heteroscedastic. Capturing this time-varying volatility is at the core of the univariate models presented in chapter 2, which are hence well equipped to model these characteristics of the time series.

Table 5.3
Summary statistics: Multi asset classes portfolio

	SMI	DAX	UKX	SPX	NAR	EPR	CNE	CEN
Mean	0.001	0.001	0.001	0.001	0.001	0.000	-0.000	0.000
Std	0.026	0.032	0.023	0.023	0.029	0.023	0.018	0.043
Max	0.148	0.172	0.136	0.102	0.202	0.123	0.073	0.197
Min	-0.140	-0.168	-0.127	-0.165	-0.246	-0.161	-0.094	-0.290
Skew	-0.445	-0.709	-0.252	-0.607	-0.924	-0.688	-0.407	-0.433
Kurt	6.946	6.626	6.434	7.429	15.025	10.447	6.360	6.456
JB	833.3^a	771.9^a	613.5^a	1073.8^a	7536.3^a	2920.1^a	608.4^a	646.5^a
Correlations								
SMI		0.776	0.770	0.672	0.448	0.548	0.183	0.002
DAX			0.763	0.718	0.463	0.570	0.235	0.046
UKX				0.730	0.512	0.620	0.244	0.116
SPX					0.683	0.544	0.284	0.108
NAR						0.546	0.283	0.126
EPR							0.250	0.122
CNE								0.320
Stationarity Tests								
LMC(1)	0.099^c	0.083^c	0.061^c	0.094^c	0.210^a	-1.335^c	0.090^c	0.066^c
LMC(5)	0.074^c	0.086^c	0.059^c	0.070^c	0.037^c	0.122^b	0.088^c	0.048^c
LMC(10)	0.082^c	0.103^c	0.083^c	0.101^c	0.046^c	0.108^c	0.058^c	0.043^c
ADF(1)	-37.7^a	-38.5^a	-38.1^a	-37.7^a	-35.8^a	-36.5^a	-35.3^a	-35.2^a
ADF(5)	-15.2^a	-15.2^a	-16.0^a	-15.3^a	-14.3^a	-12.8^a	-14.9^a	-13.8^a
ADF(10)	-10.5^a	-10.3^a	-11.5^a	-11.0^a	-12.2^a	-9.6^a	-9.5^a	-9.6^a
Heteroscedasticity Tests								
LBQ(1)	168.5^a	70.5^a	103.1^a	50.5^a	199.8^a	255.6^a	80.8^a	66.0^a
LBQ(5)	229.6^a	162.3^a	176.7^a	169.6^a	588.4^a	494.0^a	325.5^a	147.7^a
LBQ(10)	293.9^a	220.6^a	266.9^a	247.5^a	1038.4^a	872.5^a	451.6^a	272.0^a
ELM(1)	165.2^a	70.3^a	100.0^a	49.1^a	197.5^a	254.9^a	80.8^a	65.9^a
ELM(5)	179.8^a	109.0^a	122.8^a	107.8^a	321.0^a	304.9^a	169.2^a	98.6^a
ELM(10)	192.7^a	116.6^a	142.0^a	140.4^a	354.1^a	379.6^a	172.9^a	136.2^a

Summary statistics of the weekly returns over the full sample period from January 3, 1990 to June 5, 2013 for the SMI, DAX 30, FTSE 100 (UKX), S&P 500 (SPX), FTSE EPRA/NAREIT North America Index (NAR), FTSE EPRA/NAREIT Europe Index (EPR), S&P GSCI Non Energy Index ER (CNE) and the S&P GSCI Energy Index ER (CEN). Mean, Std, Skew and Kurt denote the mean, standard deviation, skewness and kurtosis. JB is the test statistic of the Jarque-Bera test for normality of the unconditional distribution of the returns. The correlations report Pearson's linear unconditional sample correlations between the weekly returns over the full sample period. LMC(k) is the statistic of Leybourne and McCabe's (1999) test assessing the null hypothesis of a trend stationary AR(k) process against the alternative of a nonstationary ARIMA(k,1,1) process. ADF(k) is the statistic of the augmented Dickey-Fuller (1979) test for a unit root against a trend-stationary alternative augmented with k lagged difference terms. LBQ (k) is the statistic of the Ljung-Box (1978) portmanteau Q-test assessing the null hypothesis of no autocorrelation in the squared (mean-subtracted) residuals at k lags. ELM (k) is Engle's (1982) Lagrange multiplier statistic for heteroscedasticity obtained by regressing the squared returns on k lags. Significance is denoted by superscripts at the 1% (^{a}), 5% (^{b}) and 10% (^{c}) levels.

Chapter 6

In-Sample Analysis

This chapter estimates the models using the complete data sets and analyzes their goodness-of-fit. Firstly, the models for the univariate risk factor evolutions are estimated and their appropriateness is evaluated. In the second part the different dependence models are calibrated, the dependence structures of the different portfolios is assessed and the model fit to the data is discussed.

6.1 Univariate Models

The marginal models play an important role for dependence modeling since they filter the univariate risk factor evolutions from autocorrelation, heteroscedasticity and leverage and yield the input data for the copula estimation. Misspecification of the marginals can thus result in biased copula parameter estimates which in turn yields inadequate portfolio risk forecasts. In order to compare the fit of the models presented in chapter 2 and choose the most appropriate model specification for each of the return series, the likelihood ratio test is not applicable since the different models are not nested. Instead, the information criterion of Akaike (1973) (AIC) and the Bayesian Information Criterion (BIC) of Schwarz (1978) can be used to find a suitable univariate model. Given the m models $M_1, ..., M_m$ for the n-dimensional return vector Y, where model j has k_j parameters denoted $\theta_j = (\theta_{j1}, ..., \theta_{jk})'$ with the likelihood function $\mathcal{L}_j$, the criteria are defined as

$$AIC(M_j) = -2\log \mathcal{L}_j(\hat{\theta}_j; Y) + 2k_j, \tag{6.1}$$

$$BIC(M_j) = -2\log \mathcal{L}_j(\hat{\theta}_j; Y) + k_j \log(n), \tag{6.2}$$

where $\hat{\theta}_j$ denotes the maximum likelihood estimation of θ_j. Each of these criteria has two terms; the first term measures the goodness-of-fit and the second term penalizes model complexity. The model favored is the one for which the respective criterion is minimized. Since the penalty of the BIC is a function of the sample size of Y, it is more severe for the applications at hand. The BIC thus favors more parsimonious models which is why it is the model selection criterion of choice in case the two information criteria favor different model specifications. The univariate model for each index return series was found by selecting the AIC and BIC optimal model considering ARMA(p,q) specifications for the conditional mean up to order (p=3, q=3) and GARCH(P,Q), EGARCH(P,Q), and GJR-GARCH(P,Q) volatility models up to order (P=3, Q=3).

The parameters of the accordingly selected univariate models for the equity index portfolio are listed in table 6.1. Note that the automatic model selection by AIC / BIC yields parsimonious specifications, as no model is of an order higher than one. The results in table 6.1 show that a moving average parameter is not part of any optimal model for the return series. With the DAX 30, the CAC 40 and the S&P 500 three out of the eight series contain an autoregressive part for the conditional mean, while the mean of the other series is simply characterized by a constant. Leverage appears to be a feature in all of the return series as the information criteria favor a leveraged conditional volatility model over the standard GARCH model for all indices. The EGARCH specification is selected by the information criteria to be more suitable than the GJR model for every equity return series. All leverage parameters are negative which is in line with the economic interpre-

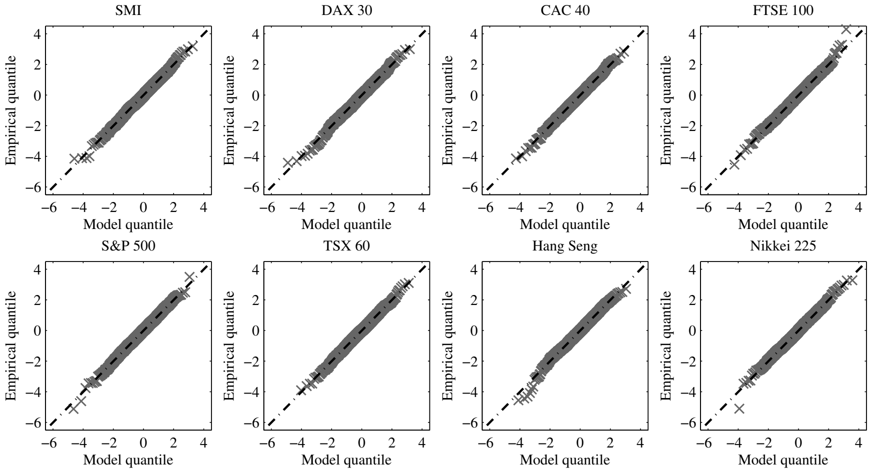

Figure 6.1: Quantile-quantile plots of the empirical versus the fitted skewed-t quantiles of the equity models' standardized innovations. If the standardized univariate model residuals adhere to the skewed-t distribution (with parameters for each series listed in table 6.1) then the data markers will fall on the dashed 45° line.

tation, as in the EGARCH a negative parameter captures the phenomenon of negative innovations to returns having a more significant impact on volatility than positive return innovations. The estimated skewness parameters λ are all negative and the degrees of freedom parameters range from 7.731 to 15.069 both with significantly low standard errors.[1] Recalling that the skewed-t distribution converges to a Student-t distribution when $\lambda = 0$ and to a normal distribution when $\lambda = 0$ and $\nu \to \infty$, one may conclude that the skewed-t is a more suitable model for the residual distribution than either the Gaussian or the standard Student-t. The negative skewness parameters indicate that the residual distributions exhibit longer left tails compared to the right tails emphasizing the importance of accounting for asymmetric return distributions. The DAX 30 reveals the most prominent left tail incorporating the lowest degrees of freedom and the third largest skewness. In case the fat tails are neglected or underestimated, the respective quantiles are inadequate which in

[1] The usual t-statistic $\hat{\alpha}/\hat{\sigma}_{\hat{\alpha}}$ is used to gauge the significance of a parameter α.

turn distorts the risk forecasts.

A first inspection of the quality of the fittings is done graphically through quantile-quantile plots. Figure 6.1 shows the plots of the empirical versus the fitted skewed-t quantiles for the equity data set and points to a fairly good fit since the pairs of quantiles lay almost all very close to the main diagonal. The skewed-t distribution seems to be particularly capable of adequately capturing the lower tails of the residual distributions. To formally test whether the estimated models are appropriate, three goodness-of-fit tests are applied: The Kolmogorov-Smirnov (KS), the Cramer-von Mises (CvM) and the Anderson-Darling (AD) test with the test statistics defined as:

$$KS_i = \max_t \left| \hat{U}_{i,(t)} - \frac{t}{T} \right|, \tag{6.3}$$

$$CvM_i = \sum_{t=1}^{T} \left(\hat{U}_{i,(t)} - \frac{t}{T} \right)^2, \tag{6.4}$$

$$AD = -T - \frac{1}{T} \sum_{t=1}^{T} \Big((2t-1)\log(\hat{U}_{i,(t)}) + (2T+1-2t)\log(1-\hat{U}_{i,(t)}) \Big), \tag{6.5}$$

where $\hat{U}_{i,(t)}$ is the t^{th} order statistic of $\left\{ \hat{U}_{i,j} \right\}_{j=1}^{T}$, i.e. the t^{th} largest value of $\left\{ \hat{U}_{i,j} \right\}_{j=1}^{T}$. The test statistics are based on the estimated probability integral transformations $\hat{U}_{i,(t)} \equiv F_{skewed-t}(\hat{\epsilon}_{it}; \hat{\nu}_i, \hat{\lambda}_i)$ and have asymptotic distributions that are known in the absence of parameter estimation errors. However, these asymptotic distributions cannot be applied here as the univariate models are based on a number of estimated parameters. Since the parametric models for the mean, variance and error distribution completely characterize the conditional distribution, this can be overcome by obtaining critical values using the following simulation-based method (Genest and Rémillard, 2008; Patton, 2013):

1. Simulate T returns from a univariate model using the estimated param-

eters.

2. Estimate the model on the simulated returns.
3. Compute the test statistics on the estimated probability integral transforms of the simulated returns.
4. Repeat steps (1) to (3) $R = 1'000$ times.
5. Use the upper 1-α quantile of $\{(KS_{(r)}, CvM_{(r)}, AD_{(r)})\}_{r=1}^{R}$ as the critical value for the tests.

The implementation of this procedure yields the p-values of the three test statistics listed in the tables 6.1, 6.2 and 6.3. For the equity indices, the lowest p-values for all three tests are found for the model of the DAX 30 which yields 0.377 (KS), 0.382 (CvM) and 0.441 (AD). The three tests therewith cannot reject the null hypothesis that the skewed-t models are well-suited for the equity index return series. The magnitude of the p-values further confirms the conclusion suggested by the quantile-quantile plots, that the skewed-t distribution adapts very well to the empirical data.

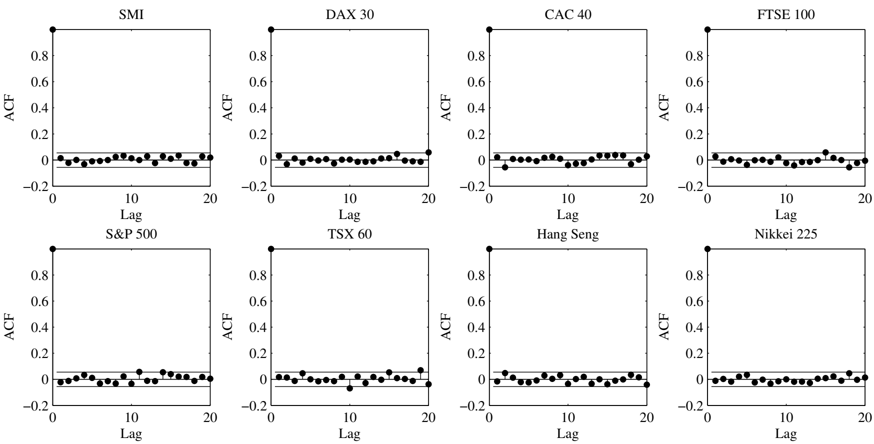

Figure 6.2: Autocorrelation functions of the squared standardized residuals of the univariate models for the equity returns.

The standardized residuals, obtained by filtering each of the returns series with its AIC / BIC-optimal model, should be independent and identically distributed. This means that the series of squared standardized residuals must also be independent and identically distributed. Figure 6.2 shows a graphical display for the estimates of serial correlation at different lags together with 95% confidence bounds. The visual inspection of these correlograms indicates that an independent and identical distribution seems to be given for all series, as the autocorrelation remains within the confidence bounds for all lags $k = 1, ..., 20$. In order to formally test the null hypothesis of zero autocorrelation, the portmanteau test of Ljung and Box (1978) is performed on the squared standardized residuals using lags up to the twentieth order. To account for the estimated parameters, the degrees of freedom of the statistic's limiting χ^2 distribution is adjusted for the number of estimated parameters p (excluding constants): $LBQ(k) \xrightarrow{d} \chi^2_{k-p}$. Table 6.1 lists the p-values of the tests at lags 7, 10, 15 and 20 for the equity indices.[2] The test results for all tested orders indicate that there is not enough evidence to reject the null hypothesis of zero correlation in the squared standardized residuals of any of the univariate models. The Ljung-Box test therewith supports the findings of the goodness-of-fit tests and contributes to the conclusion that the univariate models are well fitted.

Next, the outcome of the estimation of the univariate models of the commodities are analyzed. Table 6.2 lists the parameter estimates for the AIC / BIC optimal models. Neither an autoregressive parameter nor a moving average parameter for the conditional mean is supported by the criteria for any of the series. The information criteria favor the exponential volatility model (E) for every commodity return series. Except for one index, the EGARCH(1,1) is the optimal model. For the return series of the wheat futures index, the op-

[2] Note that with the adjustment for the number of estimated parameters, the test is valid only for $k > p$. Tsay (2005) finds evidence of increased test power by setting $k \approx \log(T)$; for the equity and the commodity return series $k = \log(1300) \approx 7$ and for the multi asset classes portfolio returns $k = \log(1222) \approx 7$.

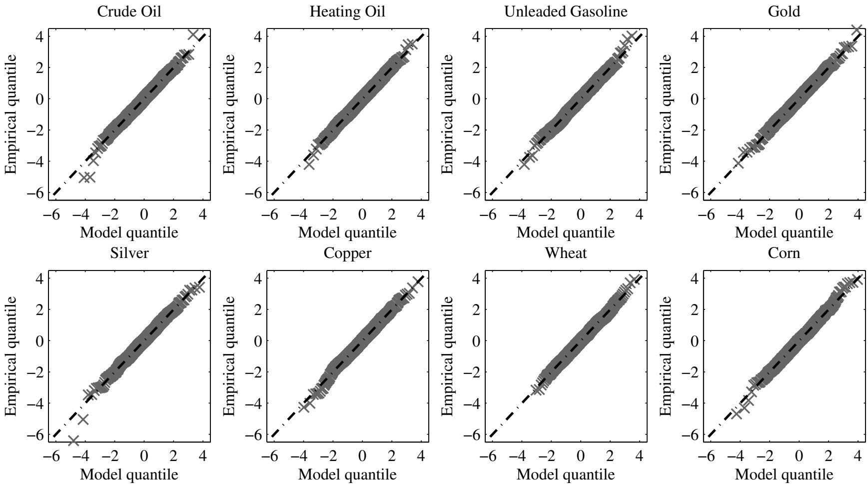

Figure 6.3: Commodity portfolio: Quantile-quantile plots of the empirical versus the fitted skewed-t quantiles of the standardized innovations. If the standardized univariate model residuals adhere to the skewed-t distribution (with parameters for each series listed in table 6.2) then the data markers will fall on the dashed 45° line.

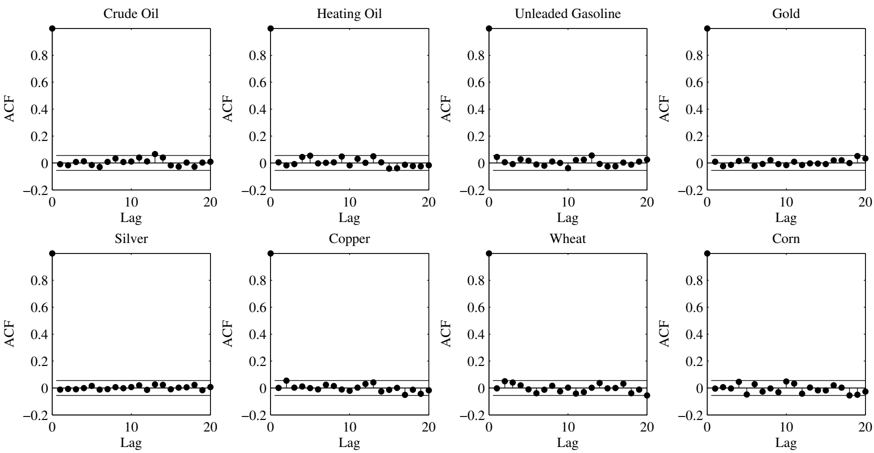

Figure 6.4: Autocorrelation functions of the squared standardized residuals of the univariate models for the commodity returns.

timal specification is the EGARCH(1,0), i.e. an exponential ARCH. The low degrees of freedom parameters for the standardized innovation distribution for all commodities and particularly for the metals and the grains indicate that the normal distribution is not a suitable model for these assets. Note that in comparison to the equity data set, the average innovation distribution for the commodity models is not as skewed. While the skewness parameter is negative for the energy and the metal futures indices, it is positive for the two grains futures indices which indicates that their right tails are longer than their left tails.

All three goodness-of-fit tests indicate with high p-values that the selected models adapt very well to the commodity data. This is confirmed by the quantile-quantile plots in figure 6.3 which shows that with the exception of two or three outliers in the lower tails of the models for the crude oil futures index and the silver futures index, the residuals adhere very closely to the diagonal. Given a well fitting model, the standardized residuals should not display any autocorrelation. The according plots of the autocorrelation functions are depicted in figure 6.4 and show that for all lags $k = 1, ..., 20$ and for all indices in the commodity portfolio, the autocorrelation is inside the 95% confidence bounds. Additionally, the statistical tests of the null hypothesis of no autocorrelation in the squared standardized residuals of the models cannot be rejected for any index series on any lag $k = 1, ..., 20$. One may thus conclude that the univariate models are well capable of capturing the dynamics in the commodity series.

Table 6.3 presents the estimated univariate model parameters for the multi asset classes portfolio. The information criteria select the EGARCH volatility model for all portfolio constituents. The results for the four equity indices do not differ much in comparison to those for the one and a half years longer equity data set (see table 6.1). While for the European REITs index an AR(1)-EGARCH model is AIC / BIC-optimal, no autoregressive component is supported for the North American REITs index and the two commodity indices. The ν parameters are rather low also for the indices of the non-traditional asset classes which shows that all distributions of the standardized residuals are fat-tailed. Furthermore, all latter distributions are negatively skewed, even though significantly less than the ones of the equity index series.

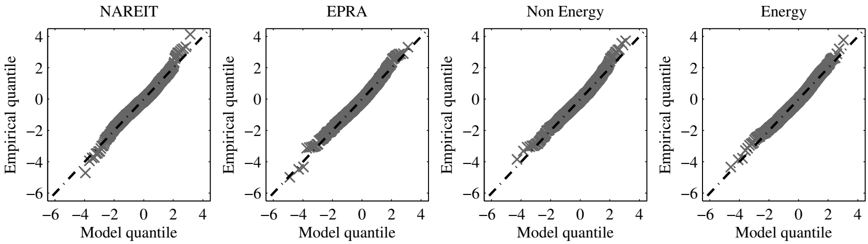

Figure 6.5: Multi asset classes portfolio: Quantile-quantile plots of the empirical versus the fitted skewed-t quantiles of the standardized innovations. If the standardized univariate model residuals adhere to the skewed-t distribution (with parameters for each series listed in table 6.3) then the data markers will fall on the dashed 45° line.

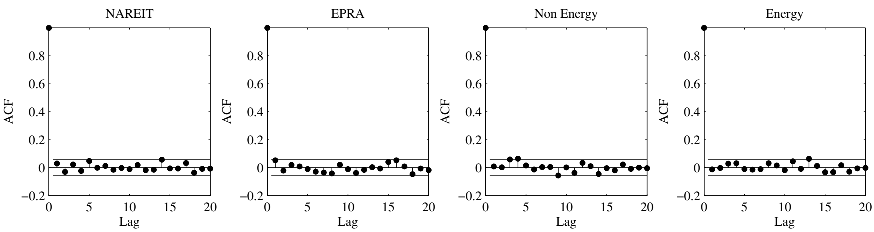

Figure 6.6: Autocorrelation functions of the squared standardized residuals of the univariate models for the additional multi asset classes portfolio return series.

Figure 6.5 depicts the quantile-quantile plots for the REITs and commodity index models which point to a fairly good fit, as the pairs of quantiles lay almost all very close to the main diagonal. The high p-values of the three goodness-of-fit tests underpin the conclusion of well-fitted marginal models. The autocorrelation functions of the squared standardized residuals depicted in figure 6.6 show no $\rho(k)$ outside the confidence bounds for any lag $k = 1, ..., 20$. The results for the tests of Ljung and Box further confirm that there is not enough statistical evidence to reject the null hypothesis of no autocorrelation in the squared standardized residuals of the univariate multi asset classes portfolio models. One may thus conclude that the EGARCH models are properly calibrated and capable of accurately capturing the features of the univariate time series.

Having found well-fitting models for each individual time series, one may now turn to the dependence structure which links the univariate series to form a joint distribution.

Table 6.1
Univariate model parameters: Equity portfolio

Index	SMI	DAX	CAC	UKX	SPX	TSX	HS	Nik
Model	E	E	E	E	E	E	E	E
μ	0.002	0.002	0.001	0.001	0.001	0.001	0.002	0.000
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
ϕ		-0.073	-0.088		-0.104			
		(0.034)	(0.034)		(0.037)			
α_0	-0.873	-0.576	-0.583	-0.511	-0.519	-0.311	-0.306	-0.583
	(0.382)	(0.332)	(0.316)	(0.451)	(0.328)	(0.326)	(0.234)	(0.379)
α_1	0.257	0.243	0.224	0.176	0.174	0.194	0.266	0.212
	(0.085)	(0.078)	(0.079)	(0.080)	(0.075)	(0.082)	(0.065)	(0.085)
γ_1	-0.186	-0.133	-0.144	-0.174	-0.162	-0.067	-0.041	-0.109
	(0.053)	(0.051)	(0.049)	(0.052)	(0.054)	(0.046)	(0.047)	(0.048)
β_1	0.884	0.919	0.919	0.934	0.933	0.960	0.955	0.917
	(0.125)	(0.122)	(0.138)	(0.123)	(0.102)	(0.129)	(0.081)	(0.137)
ν	8.840	7.731	13.784	11.707	9.394	15.069	14.837	8.464
	(0.901)	(0.912)	(1.463)	(1.263)	(0.696)	(1.194)	(1.081)	(1.257)
λ	-0.218	-0.269	-0.272	-0.204	-0.275	-0.186	-0.215	-0.137
	(0.040)	(0.036)	(0.037)	(0.038)	(0.037)	(0.037)	(0.043)	(0.043)
KS	0.418	0.377	0.996	0.935	0.938	0.782	0.991	0.657
CvM	0.506	0.382	0.962	0.985	0.974	0.706	0.918	0.671
AD	0.589	0.441	0.968	0.973	0.968	0.752	0.843	0.710
LBQ(7)	0.668	0.332	0.152	0.557	0.247	0.412	0.187	0.473
LBQ(10)	0.678	0.629	0.216	0.725	0.264	0.138	0.242	0.631
LBQ(15)	0.793	0.910	0.275	0.459	0.074	0.154	0.368	0.841
LBQ(20)	0.762	0.648	0.247	0.458	0.218	0.085	0.436	0.846

Parameter estimates for the models of the weekly returns over the full sample period from June 30, 1988 to June 5, 2013 for the SMI, DAX 30, CAC 40, FTSE 100 (UKX), S&P 500 (SPX), S&P/TSX 60, Hang Seng (HSI) and Nikkei 225 index. The *Model* line indicates the volatility model selected by the AIC / BIC, where E stands for the EGARCH specification. KS, CvM and AD report the p-values of the parameter estimation error adjusted results (based on 1,000 simulations) of the Kolmogorov-Smirnov (KS), Cramer-von Mises (CvM) and Anderson-Darling (AD) goodness-of-fit tests for the models of the conditional marginal distributions. LBQ(k) reports the p-value of the Ljung-Box Q-statistics (adjusted for the number of estimated parameters) assessing the null hypothesis of no autocorrelation of the squared standardized residuals for k lags.

Table 6.2
Univariate model parameters: Commodity portfolio

Index	OIL	HOL	GAS	GLD	SLV	CPP	WHT	CRN
Model	*E*	*E*	*E*	*E*	*E*	*E*	*E*	*E*
μ	0.001	0.001	0.002	-0.000	-0.001	0.001	-0.002	-0.001
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
α_0	-0.193	-0.248	-0.167	-0.227	-0.155	-0.249	-6.579	-0.262
	(0.843)	(0.627)	(0.546)	(0.367)	(0.478)	(0.556)	(0.094)	(0.623)
α_1	0.195	0.225	0.197	0.208	0.143	0.222	0.282	0.238
	(0.093)	(0.062)	(0.074)	(0.075)	(0.062)	(0.077)	(0.076)	(0.057)
γ_1	-0.013	0.022	0.014	0.063	0.055	-0.003	0.050	0.034
	(0.047)	(0.040)	(0.048)	(0.043)	(0.038)	(0.045)	(0.057)	(0.041)
β_1	0.969	0.961	0.973	0.970	0.976	0.963		0.960
	(0.134)	(0.099)	(0.088)	(0.048)	(0.073)	(0.082)		(0.092)
ν	11.976	18.413	13.272	5.434	5.709	9.664	8.853	6.290
	(2.836)	(5.586)	(2.466)	(0.482)	(0.544)	(1.715)	(3.862)	(0.677)
λ	-0.141	-0.050	-0.069	-0.033	-0.059	-0.034	0.102	0.040
	(0.039)	(0.038)	(0.040)	(0.040)	(0.040)	(0.042)	(0.041)	(0.035)
KS	0.914	0.971	0.786	0.833	0.421	0.750	0.592	0.204
CvM	0.932	0.977	0.743	0.708	0.455	0.564	0.816	0.413
AD	0.937	0.968	0.731	0.732	0.437	0.475	0.835	0.451
LBQ(7)	0.676	0.145	0.326	0.583	0.912	0.295	0.085	0.096
LBQ(10)	0.788	0.178	0.457	0.792	0.993	0.547	0.229	0.094
LBQ(15)	0.277	0.155	0.369	0.974	0.990	0.572	0.260	0.155
LBQ(20)	0.483	0.219	0.598	0.887	0.998	0.448	0.172	0.078

Parameter estimates for the models of the weekly returns over the full sample period from June 30, 1988 to June 5, 2013 for the crude oil (OIL), heating oil (HOL), unleaded gasoline (GAS), gold (GLD), silver (SLV), copper (CPP), wheat (WHT) and corn (CRN) futures index. The *Model* line indicates the volatility model selected by the AIC / BIC, where E stands for the EGARCH specification. KS, CvM and AD report the p-values of the parameter estimation error adjusted results (based on 1,000 simulations) of the Kolmogorov-Smirnov (KS), Cramer-von Mises (CvM) and Anderson-Darling (AD) goodness-of-fit tests for the models of the conditional marginal distributions. LBQ(k) reports the p-value of the Ljung-Box Q-statistics (adjusted for the number of estimated parameters) assessing the null hypothesis of no autocorrelation of the squared standardized residuals for k lags.

Table 6.3
Univariate model parameters: Multi asset classes portfolio

Index	SMI	DAX	UKX	SPX	NAR	EPR	CNE	CEN
Model	E	E	E	E	E	E	E	E
μ	0.002	0.002	0.001	0.001	0.002	0.000	-0.000	0.000
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.000)	(0.001)
ϕ		-0.077		-0.108		0.913		
		(0.036)		(0.038)		(0.557)		
α_0	-0.782	-0.575	-0.563	-0.525	-0.268	-0.192	-0.058	-0.250
	(0.352)	(0.356)	(0.433)	(0.468)	(0.847)	(0.865)	(0.825)	(1.146)
α_1	0.248	0.249	0.179	0.184	0.268	0.266	0.139	0.217
	(0.094)	(0.078)	(0.083)	(0.073)	(0.109)	(0.087)	(0.082)	(0.083)
γ_1	-0.183	-0.136	-0.196	-0.166	-0.054	-0.056	0.021	0.012
	(0.056)	(0.053)	(0.063)	(0.054)	(0.076)	(0.054)	(0.049)	(0.055)
β_1	0.896	0.919	0.927	0.932	0.963	0.975	0.993	0.961
	(0.143)	(0.137)	(0.140)	(0.120)	(0.116)	(0.114)	(0.102)	(0.179)
ν	9.946	7.764	11.540	9.414	5.227	8.581	8.854	14.056
	(0.993)	(0.951)	(0.793)	(0.675)	(0.291)	(0.517)	(0.415)	(1.848)
λ	-0.227	-0.282	-0.229	-0.274	-0.071	-0.099	-0.043	-0.091
	(0.041)	(0.034)	(0.034)	(0.033)	(0.039)	(0.047)	(0.039)	(0.038)
KS	0.301	0.347	0.811	0.984	0.882	0.991	0.575	0.990
CvM	0.413	0.381	0.970	0.981	0.779	0.987	0.640	0.993
AD	0.512	0.416	0.968	0.976	0.797	0.983	0.694	0.998
LBQ(7)	0.624	0.312	0.628	0.197	0.160	0.029	0.039	0.559
LBQ(10)	0.657	0.662	0.765	0.308	0.433	0.083	0.053	0.665
LBQ(15)	0.860	0.886	0.566	0.167	0.433	0.176	0.075	0.286
LBQ(20)	0.848	0.604	0.642	0.369	0.571	0.150	0.235	0.460

Parameter estimates for the models of the weekly returns over the full sample period from January 3, 1990 to June 5, 2013 for the SMI, DAX 30, FTSE 100 (UKX), S&P 500 (SPX), FTSE EPRA/NAREIT North America Index (NAR), FTSE EPRA/NAREIT Europe Index (EPR), S&P GSCI Non Energy Index ER (CNE) and the S&P GSCI Energy Index ER (CEN). The *Model* line indicates the volatility model selected by the AIC / BIC, where E stands for the EGARCH specification where E stands for the EGARCH specification. KS, CvM and AD report the p-values of the parameter estimation error adjusted results (based on 1,000 simulations) of the Kolmogorov-Smirnov (KS), Cramer-von Mises (CvM) and Anderson-Darling (AD) goodness-of-fit tests for the models of the conditional marginal distributions. LBQ(k) reports the p-value of the Ljung-Box Q-statistics (adjusted for the number of estimated parameters) assessing the null hypothesis of no autocorrelation of the squared standardized residuals for k lags.

6.2 Static Dependence

A central characteristic often observed in financial time series data is asymmetric dependence. This asymmetry refers to the observation that in times of crisis returns have a tendency to be more dependent than in normal times. There are multiple approaches to quantify this characteristic. For financial applications it is interesting to measure the ordinary sort of dependence between returns in the center of the distribution, as well as dependence amongst extreme events. While the normal distribution is well capable of capturing the former, it fails with the latter. Risk management, however, mostly deals with the latter, as it identifies the negative extremes in the return distributions as critical (Chollete et al., 2009).

6.2.1 Exceedance Correlation and Tail Dependence

Among the recent literature that focuses on extremal dependence, a widely used measure introduced by Longin and Solnik (2001) is *exceedance correlation*. The left exceedance correlation between two random variables $(z_{1,t}, z_{2,t})$ for a threshold q is defined as the correlation between the realizations that are below the q-quantiles, denoted by $F_1^{-1}(q)$ and $F_2^{-1}(q)$, respectively, with $q \in (0, 0.5)$ and F_i the cumulative distribution function of $z_{i,t}$, for $i = 1, 2$. The right exceedance correlation therewith is the correlation between the realizations that are above the $(1-q)$-quantile, $q \in (0, 0.5)$. Formally, exceedance correlation is

$$\begin{aligned} \rho^-(q) &= \mathrm{Corr}\left(z_{1,t}, z_{2,t} | F_1(z_{1,t}) < q, F_2(z_{2,t}) < q\right), \\ \rho^+(q) &= \mathrm{Corr}\left(z_{1,t}, z_{2,t} | F_1(z_{1,t}) \geq 1-q, F_2(z_{2,t}) \geq 1-q\right). \end{aligned} \tag{6.6}$$

In the case of symmetric distributions, one expects $\rho^-(q) = \rho^+(q)$, $\forall\, q \in (0, 0.5)$. A main finding of studies employing this measure is that in contrast to bull markets financial return series have a tendency to display excess

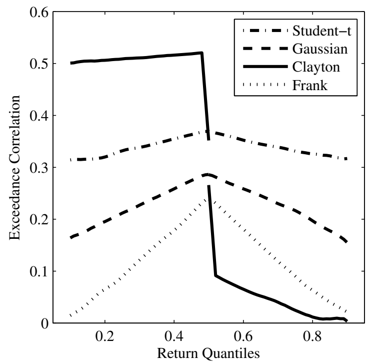

Figure 6.7: Exceedance correlations of 100,000 bivariate random numbers simulated from each of the four copula models. The transformation of all copula parameters to Kendall's tau equals 0.4, with Student-t $\nu = 5$.

correlation in bear markets (see e.g. Jondeau, 2010; Longin and Solnik, 2001; Patton, 2006a). The normal distribution, however, is not capable of capturing this characteristic. This is shown in figure 6.7, which visualizes the different exceedance correlation patterns of the four standalone copula models under consideration. The plot is based on 100,000 bivariate random numbers drawn from the different copula models, which all imply the identical level of dependence, as measured by a Kendall's tau of 0.4. For the Student-t, the Frank and the Gaussian copula, the simulated data confirms that $\rho^{-}(q) = \rho^{+}(q)$. The data simulated from the Clayton copula, however, implies a very different pattern of exceedance correlation which reflects the asymmetric dependence structure of the Clayton copula.

To investigate whether the portfolios under consideration exhibit asymmetries in their dependence structures, the concept of the exceedance correlation is applied to the three data sets. Since this statistic is bivariate in nature, the average pairwise empirical exceedance correlation is computed across all return series in one portfolio. Figure 6.8 depicts the average exceedance correlation for the equity index portfolio, while figures 6.9 and 6.10 visualize the measures for the commodity futures and multi asset classes index portfolio,

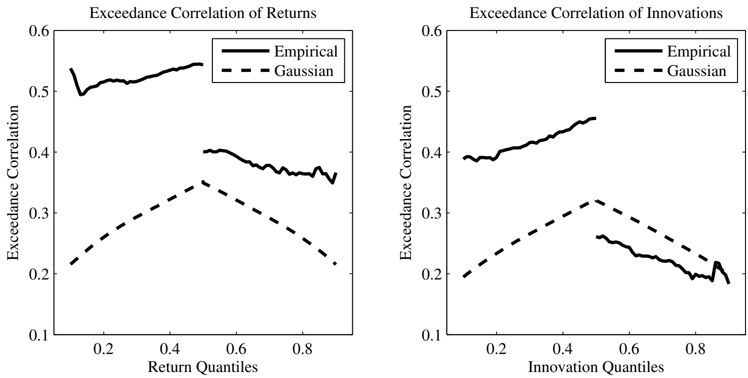

Figure 6.8: Average pairwise exceedance correlations of the international equity index portfolio. The left side presents the exceedance correlation of the returns, standardized by their unconditional means and standard deviation. The plot on the right depicts the exceedance correlations of the standardized innovations obtained by filtering with the according univariate GARCH models.

respectively. The left hand side of the plots reports the statistics computed using the portfolio's returns, which were standardized by their unconditional means and standard deviation. The right hand side shows the exceedance correlations of the standardized innovations of the corresponding AIC / BIC optimal univariate ARMA-GARCH filter. The statistics are computed for the thresholds from 0.1 to 0.9 by increments of 0.01. The dashed line shows exceedance correlations implied by a Gaussian distribution using the average linear correlation estimated from the data. The graphs give a visualization of the fact that asymmetric exceedance correlations cannot be captured by a normal distribution: the threshold correlations in the normal distribution are symmetric, and also decrease quite rapidly in the tails. The plots however show that the empirical exceedance correlation patterns are asymmetric: the downside exceedance correlations are substantially larger than the upside exceedance correlations. The comparison of the left-hand plots based on returns with the right-hand plots based on innovations illustrates one advantage of using copulas. If the univariate models for the marginals were capable of sufficiently capturing all the asymmetries and nonnormalities in the data, no

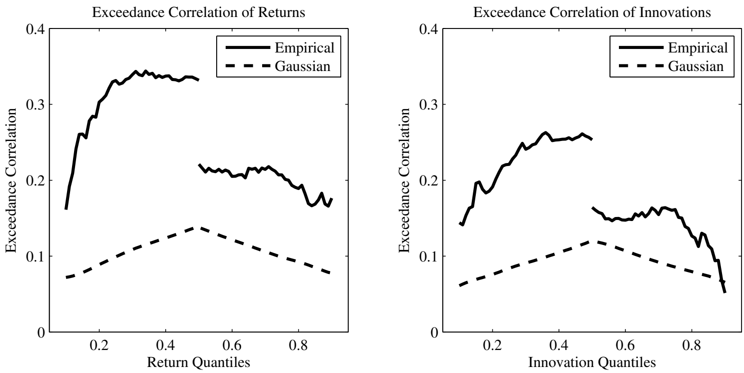

Figure 6.9: Average pairwise exceedance correlations of the commodity futures index portfolio. The left side presents the exceedance correlation of the returns, standardized by their unconditional means and standard deviation. The plot on the right depicts the exceedance correlations of the standardized innovations obtained by filtering with the according univariate GARCH models.

asymmetries would be observable in the plots on the right. This is evidently not the case. Even though the level of the empirical exceedance correlations is shifted downwards when comparing the right-hand side plot with the left, the asymmetry pattern seems to be rather similar. This points to a limited role of the marginal models in capturing multivariate asymmetries, and indicates the need for copula models which incorporate asymmetries.

Exceedance correlation, however, is not without issues. Firstly, as it is a bivariate concept, the averaging of empirical exceedance correlations may induce biases. Secondly, the statistic is calculated only from those observations that are below respectively above the threshold. This means that exceedance correlation becomes more and more imprecise, the further out into the tails it is computed. Thirdly, like Pearson's linear correlation, exceedance correlation is not independent of the marginal distributions.

While the exceedance correlation is a function of the dependence structure and of the marginal distributions, the tail dependence is uniquely a function of the dependence structure, regardless of the marginal distributions (Garcia and Tsafack, 2011). The concept of tail dependence was first introduced by

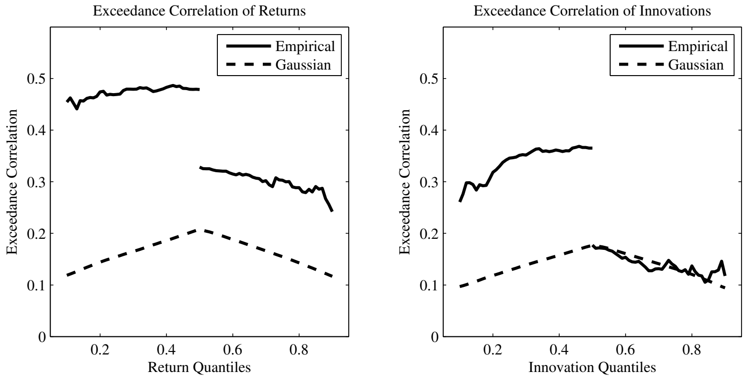

Figure 6.10: Average pairwise exceedance correlations of the multi asset classes index portfolio. The left side presents the exceedance correlation of the returns, standardized by their unconditional means and standard deviation. The plot on the right depicts the exceedance correlations of the standardized innovations obtained by filtering with the according univariate GARCH models.

Sibuya (1960) and advanced by Ledford and Tawn (1996). The coefficient of tail dependence between two assets is the probability that one of the two assets undergoes a large loss (or gain) given that the other asset also undergoes a large loss (or gain). To forecast the risk of a portfolio, lower tail dependence is of particular interest. For two random variables $(z_{1,t}, z_{2,t})$ with cumulative distribution functions F_1 and F_2, the coefficient of lower tail dependence is given by

$$\lambda_l := \lim_{q \to 0^+} P\left(z_{2,t} \leq F_2^{-1}(q) | z_{1,t} \leq F_1^{-1}(q)\right), \tag{6.7}$$

provided that a limit $\lambda_l \in [0, 1]$ exists. If $\lambda_l > 0$, $(z_{1,t}, z_{2,t})$ are lower tail dependent and in case $\lambda_l = 0$, the variables are asymptotically independent in the lower tail. For continuous distributions F_1 and F_2, Bayes' rule may be applied to obtain a simple expression for the population lower tail dependence:

$$\lambda_l = \lim_{q \to 0^+} \frac{P\left(z_{2,t} \leq F_2^{-1}(q) | z_{1,t} \leq F_1^{-1}(q)\right)}{P(z_{1,t} \leq F_1^{-1}(q))}. \tag{6.8}$$

Note that sample tail dependence cannot be taken simply as the limit in (6.8), since by setting q close enough to zero one is assured that the estimate will be zero. The estimation of the lower tail dependence coefficient of a finite sample thus requires an alternative approach. McNeil et al. (2005) show that the coefficients of tail dependence are measures that depend only on the copula of $(z_{1,t}, z_{2,t})$, such that (6.8) becomes

$$\lambda_l = \lim_{q \to 0^+} \frac{C(q,q)}{q}. \tag{6.9}$$

The calculation of these lower tail dependence coefficients is straightforward if the copula C has a simple closed form, like for example the Clayton copula. While the asymmetric Clayton copula is asymptotically independent in the upper tail, its lower tail dependence coefficient is

$$\frac{\left(2q^{-\theta}-1\right)^{-1/\theta}}{q} = \left(2-q^{\theta}\right)^{-\frac{1}{\theta}},$$

$$\longrightarrow 2^{-1/\theta} = \lambda_l. \tag{6.10}$$

For the Student-t copula, McNeil et al. (2005) show that the lower tail dependence coefficient is (equal to the upper one) given by

$$\lambda = 2t_{\nu+1}\left(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}}\right). \tag{6.11}$$

Provided that $\rho > -1$, the copula of the bivariate Student-t distribution thus exhibits both upper and lower tail dependence. The t-copula is tail dependent even for negative or zero correlation. Among the copulas considered in this thesis, the Frank and the Gaussian copula do not exhibit neither lower nor upper tail dependence being asymptotically independent in both tails (McNeil et al., 2005). The illustration of the mixture copula concept in figure 3.1 visualizes the differences in tail behavior between the Gaussian and the Clayton copula. The pronounced density of the Clayton copula at the lower left corner depicts its lower tail dependence. To determine the tail dependence coefficients given by (6.10) or (6.11) of the data sets under scrutiny, the according copula parameters first have to be estimated.

6.2.2 Static Copulas

To calibrate the static copulas, the filtered standardized residuals from the univariate models are transformed to uniform variates by inversion using the corresponding cumulative skewed-t distribution function. According to the multi-stage estimation procedure, the same uniform variates are used for the estimation of the different copulas. In this chapter, the copulas are estimated using the data of the entire sample period. Besides the four standalone static copulas presented in chapter 3, three static mixture copulas are constructed by combining the asymmetric Clayton copula with the other three (symmetric) copulas. The combination of the two Archimedean copulas into a mixture construct yields a parsimonious model which is able of capturing lower tail dependence. Mixing the Clayton with the Gaussian copula combines the parameter plurality of the elliptical copula with the lower tail dependence feature of the Clayton copula and creates a flexible model which is capable of capturing asymmetries in the dependence structure. Adding the Clayton to the Student-t copula finally results in an adaptive model capable of modeling different degrees of upper and lower tail dependence.

Table 6.4 lists the parameter estimates for all static copulas for the equity indices. The standard errors, which are displayed in parentheses are computed as outlined in section 4.2 with 1,000 bootstrap replications. The Archimedean copulas capture the dependence structure each with a single parameter θ_F (Frank copula) and θ_C (Clayton copula). The θ parameters of the two Archimedean copulas seem very different at first sight, but as their impact depends on each generator function, a direct comparison is not very meaningful. Using the functional relationship between the two copula parameters and Kendall's tau outlined in section 4.1, the according rank correlation measures are $\tau_F = 0.3455$ and $\tau_C = 0.2631$, which can be seen as the average bivariate rank correlation. With the calibrated Clayton copula, lower tail dependence can be quantified by inserting θ_C into equation (6.10). The resulting tail dependence coefficient of 0.379 indicates that there is substantial dependence among the extreme negative returns of the different equity index series. Since the Frank copula is not capable of capturing tail dependence, it has to account for this relation between the extreme losses by an overall higher dependence reflected in the larger average rank correlation τ_F.

To gauge how well the different copula models fit the data, standard goodness-of-fit tests based on the comparison with the empirical copula employing the test statistics in equations (6.3), (6.4) and (6.5) are not applicable. Patton (2013) emphasizes that such tests rely on the empirical copula to serve as a non-parametric estimate of the true conditional copula, but when the true conditional copula is not constant, the empirical copula cannot be used for such tests. Therefore, following Dias and Embrechts (2010), Guégan and Zhang (2010) and Chollete et al. (2011), the information criteria outlined in section 2 are used to rank the fit of the different models.

The higher log likelihood value and therewith - as both copulas have a single parameter - lower AIC and BIC indicate that the Clayton copula provides a better fit to the data among the two standalone Archimedean copulas. The Clayton's superiority is due to its ability to capture the returns' lower tail dependence. The mixture of the two Archimedean copulas proves to be better capable to adapt to the returns, based on the analysis of both information criteria. Intuitively, the Frank-Clayton mixture model improves the adaptability of the Frank dependence structure by creating a possibly asymmetric structure allowing for lower tail dependence. With 58.7% and a low standard error of 6.5%, the Clayton copula has the larger weight in the mixture structure. Clayton's θ_C is further substantially higher in the mixture than in the standalone Clayton copula, which reflects the increased flexibility of the mixture copula to adapt to the data. The Clayton proportion of the mixture copula captures the asymmetric dependence, while the Frank proportion captures symmetric dependence. The higher standard errors of the Clayton and particularly of the Frank copula parameter shows that the increase in flexibility comes at a price of less reliable parameters.

With their correlation matrices, the elliptical copulas contain a multiple of the number of parameters in the Archimedean copulas. Their increased likelihood value thus reflects the additional number of parameters. However, both information criteria account for the number of parameters, and both indicate a better fit of the standalone elliptical copulas compared to both the standalone and the mixture Archimedean copulas. The degrees of freedom parameter $\nu = 10.655$ of the Student-t copula is to be considered as rather low which confirms tail dependence in the return data. However, the Student-t copula implies the same degree of lower and upper tail dependence, even if only the former is present, as insinuated by figure 6.8. This drawback is addressed by mixing the asymmetric Clayton with one of the symmetric elliptical copulas. The weight of the Clayton in the Gaussian-Clayton mixture amounts to 19.9%, however θ_C is rather low. Furthermore, the uncertainty in the Clayton copula parameter indicated by the according standard error is high. As one might expect, the Student-t-Clayton mixture displays a higher ν, since the Clayton copula fraction of the mixture accounts for lower tail dependence. However, for both the Gaussian-Clayton and the Student-t-Clayton mixture copula, the standard errors do not allow to conclude with certainty that θ_C is different from zero. Using the information criteria to rank the fit of the static models, the Student-t-Clayton mixture fits best according to both AIC and BIC, followed by the Student-t copula and the Gauss-Clayton mixture.

Next, the parameters of the copulas for the commodity indices which are listed in table 6.5 are analyzed. Comparing the fit of the two Archimedean copulas the Clayton is more suitable to describe the dependence compared to the Frank copula which indicates that the commodity index returns are lower tail dependent. The lower tail dependence coefficient amounts to 0.115, which points to a substantially lower probability of joint negative extreme returns of the commodity indices compared to the equity indices. There is a large difference in likelihood value and in both information criteria values between the purely Archimedean copulas and the other dependence models which contain an elliptical copula. The Archimedean copulas' likelihood values range from 661 for the Frank copula to 788 for the Frank-Clayton mixture while the other copulas' likelihood are more than three times as large, all exceeding 2746. This substantial difference can be attributed to the fact that the standalone Archimedean copulas have to capture the dependence structure with only one parameter and the pure Archimedean mixture only contains three parameters. While for the equity indices this also results in a lower likelihood compared to the elliptical models, for the commodity indices the difference is more pronounced. The reason lies in the dependence structure which is much more diverse for the commodities compared to the equities. This diversity can be seen for example in the range of the Gaussian copula correlation matrix which for the equities includes values from 0.381 (SMI:NIK) to 0.793 (DAX:CAC), while for the commodities the values span from 0.082 (OIL:WHT) to 0.875 (OIL:HOL). The low likelihood values of the Archimedean copulas express the difficulty of these dependence models to capture such a diverse structure with only one respectively three parameters. Containing a correlation matrix, the rest of the models have at least 28 parameters to characterize the dependence structure among the eight indices. This allows a more precise fit which materializes in higher likelihood values and in lower values of both information criteria.

The ranking of the model fit based on both the AIC and BIC is as follows: the Student-t-Clayton mixture is best capable to characterize the commodity indices' dependence followed by the Student-t copula and in third place the Gaussian-Clayton mixture. This ranking is in analogy to the one for the equity index data. The degrees of freedom of the Student-t copula of 16.804 indicate tail dependence which is substantiated by the improved fit (i.e. lower AIC and BIC values) of the Student-t copula compared to the Gaussian one. In the Student-t-Clayton mixture ν amounts to 18.579 which shows that the Clayton copula, even though it only accounts for 2.7% of the overall mixture, covers a part of the lower tail dependence allowing for the increased degrees of freedom in the Student-t fraction. Even though θ_C is lower compared to the Student-t-Clayton mixture, the Gaussian-Clayton mixture substantiates the conclusion of lower tail dependence in the data with a higher mixture weight w_C.

The analysis of the static copula parameters for the multi asset classes portfolio, listed in table 6.6, is done next. Clayton's θ_C implies a lower tail dependence coefficient of 0.191. This insinuates that there is a higher probability of observing a joint negative return in the multi asset classes portfolio than in the commodity portfolio. Compared to the equity index portfolio, which has the largest tail dependence coefficient of the three data sets, this probability is about half as large. Both the larger likelihood value as well as the lower values for both information criteria corroborate the conclusion that the returns of the multi asset classes portfolio are lower tail dependent.

Similar to the commodity index portfolio, the dependence structure is quite heterogeneous with values of the Gaussian copula correlation matrix ranging from close to independence with 0.030 (SMI:CEN) to rather strong dependence between the SMI and the DAX: 0.704. The low likelihood values and the comparably large AIC and BIC values of the standalone Archimedean copulas reflect their difficulty of capturing this heterogeneity in the dependence with only one parameter. The best fit among the purely Archimedean copulas is shown by the Frank-Clayton mixture, which is not surprising, as it comes with two additional parameters and nests both Archimedean copulas. About three quarters of the Archimedean mixture are formed by the Clayton copula with a rather high θ_C of 0.708 while the rest is accounted for by a Frank copula with a very low θ_F of 0.106.

The dependence models containing an elliptical copula are much more capable to characterize the dependence structure as their likelihood values are about twice as large compared to the purely Archimedean models and their AIC / BIC values are about twice as low. The Student-t copula accounts for the tail dependence in the returns with a rather low degrees of freedom parameter of 13.905. The joint negative extreme returns are captured in the Gaussian-Clayton mixture copula by the Clayton fraction of 16.1% with a θ_C of 0.237 while in the Student-t-Clayton mixture both the ν of 16.876 and the Clayton part of the mixture amounting to 12.8% capture the lower tail dependence in the series. According to both information criteria, the best fitting model is the Student-t-Clayton mixture. The AIC ranks the Gaussian-Clayton in the second place and the Student-t copula in the third. According to the BIC, which imposes a larger penalty for additional parameters, the Student-t is ranked second and the Gaussian-Clayton is third. However, the differences in AIC respectively BIC values for the copulas on ranks two and three are negligibly small.

Overall, the in-sample analysis shows that the best fitting static dependence model for each of the three data sets is the Student-t-Clayton mixture copula. The two models with the second-best fit are the Gaussian-Clayton mixture copula and the Student-t copula. All Archimedean copulas rank far behind with the standalone Archimedean copulas having the largest AIC / BIC values and the lowest likelihood value, respectively. With a lower tail dependence coefficient of 0.379, the equity index portfolio has the highest probability of a joint extreme negative return, followed by the multi asset class portfolio with $\lambda_l = 0.191$. With a lower tail dependence coefficient of 0.115, the commodity futures index portfolio displays the lowest probability of a joint crash of the portfolio constituents.

Table 6.4 Equity portfolio: Static copula parameters with standard errors in parentheses. The mixture copulas are abbreviated: Frank-Clayton mixture (FC Mix), Gaussian-Clayton mixture (GC Mix) and Student-t-Clayton mixture (TC Mix). w_C denotes the weight of the Clayton copula in the mixture.

Copula	Clayton	Frank	Gaussian	Student-t	FC Mix	GC Mix	TC Mix
SMI:DAX			0.697 (0.018)	0.720 (0.016)		0.807 (0.023)	0.781 (0.019)
SMI:CAC			0.683 (0.019)	0.716 (0.018)		0.824 (0.028)	0.796 (0.026)
SMI:FTS			0.660 (0.019)	0.690 (0.017)		0.789 (0.030)	0.757 (0.025)
SMI:SPX			0.591 (0.021)	0.607 (0.019)		0.690 (0.026)	0.659 (0.023)
SMI:TSX			0.499 (0.025)	0.505 (0.024)		0.575 (0.030)	0.544 (0.027)
SMI:HSE			0.418 (0.027)	0.447 (0.025)		0.523 (0.041)	0.493 (0.033)
SMI:NIK			0.381 (0.026)	0.407 (0.026)		0.456 (0.040)	0.448 (0.035)
DAX:CAC			0.793 (0.015)	0.818 (0.014)		0.895 (0.018)	0.880 (0.015)
DAX:FTS			0.670 (0.022)	0.698 (0.020)		0.794 (0.027)	0.770 (0.024)
DAX:SPX			0.637 (0.021)	0.658 (0.019)		0.740 (0.025)	0.715 (0.022)
DAX:TSX			0.547 (0.021)	0.551 (0.020)		0.618 (0.028)	0.591 (0.024)
DAX:HSE			0.489 (0.025)	0.505 (0.024)		0.561 (0.036)	0.544 (0.029)
DAX:NIK			0.439 (0.025)	0.463 (0.024)		0.509 (0.037)	0.506 (0.033)
CAC:FTS			0.725 (0.017)	0.754 (0.015)		0.843 (0.023)	0.822 (0.021)
CAC:SPX			0.641 (0.021)	0.661 (0.020)		0.750 (0.027)	0.724 (0.024)
CAC:TSX			0.560 (0.023)	0.566 (0.021)		0.640 (0.030)	0.612 (0.025)
CAC:HSE			0.490 (0.023)	0.501 (0.024)		0.559 (0.033)	0.539 (0.027)
CAC:NIK			0.459 (0.026)	0.482 (0.025)		0.530 (0.037)	0.523 (0.032)
FTS:SPX			0.662 (0.020)	0.667 (0.019)		0.741 (0.023)	0.720 (0.022)
FTS:TSX			0.566 (0.022)	0.573 (0.020)		0.645 (0.027)	0.620 (0.024)
FTS:HSE			0.532 (0.026)	0.535 (0.025)		0.600 (0.032)	0.576 (0.027)
FTS:NIK			0.432 (0.024)	0.448 (0.023)		0.509 (0.037)	0.493 (0.032)
SPX:TSX			0.721 (0.015)	0.714 (0.014)		0.757 (0.016)	0.740 (0.015)
SPX:HSE			0.483 (0.027)	0.490 (0.026)		0.561 (0.035)	0.530 (0.028)
SPX:NIK			0.431 (0.024)	0.438 (0.026)		0.485 (0.034)	0.471 (0.031)
TSX:HSE			0.455 (0.025)	0.443 (0.025)		0.488 (0.033)	0.465 (0.027)
TSX:NIK			0.392 (0.027)	0.389 (0.027)		0.440 (0.032)	0.419 (0.030)
HSE:NIK			0.416 (0.030)	0.430 (0.029)		0.469 (0.039)	0.461 (0.032)
ν				10.655 (0.803)			13.355 (1.324)
θ_F		3.453 (0.141)			2.311 (1.282)		
θ_C	0.714 (0.038)				1.166 (0.194)	0.280 (0.306)	0.330 (0.411)
w_C					0.587 (0.065)	0.199 (0.029)	0.149 (0.029)
$\log \mathcal{L}$	2269	2234	3172	3345	2522	3345	3399
AIC	-4536	-4467	-6289	-6632	-5039	-6630	-6735
BIC	-4531	-4461	-6143	-6482	-5023	-6475	-6576

Table 6.5 Commodity portfolio: Static copula parameters with standard errors in parentheses. The mixture copulas are abbreviated: Frank-Clayton mixture (FC Mix), Gaussian-Clayton mixture (GC Mix) and Student-t-Clayton mixture (TC Mix). w_C denotes the weight of the Clayton copula in the mixture.

Copula	Clayton	Frank	Gaussian	Student-t	FC Mix	GC Mix	TC Mix
OIL:HOL			0.875 (0.010)	0.879 (0.010)		0.892 (0.010)	0.887 (0.011)
OIL:GAS			0.846 (0.014)	0.852 (0.013)		0.866 (0.013)	0.864 (0.014)
OIL:GLD			0.222 (0.038)	0.231 (0.037)		0.236 (0.038)	0.231 (0.039)
OIL:SLV			0.214 (0.036)	0.224 (0.038)		0.234 (0.038)	0.231 (0.039)
OIL:CPP			0.207 (0.043)	0.208 (0.044)		0.210 (0.045)	0.208 (0.045)
OIL:WHT			0.082 (0.031)	0.088 (0.032)		0.080 (0.034)	0.086 (0.033)
OIL:CRN			0.120 (0.031)	0.122 (0.032)		0.115 (0.034)	0.119 (0.033)
HOL:GAS			0.817 (0.015)	0.822 (0.015)		0.837 (0.015)	0.831 (0.016)
HOL:GLD			0.209 (0.036)	0.213 (0.035)		0.215 (0.038)	0.213 (0.037)
HOL:SLV			0.200 (0.037)	0.206 (0.037)		0.212 (0.038)	0.212 (0.038)
HOL:CPP			0.196 (0.042)	0.191 (0.044)		0.197 (0.044)	0.191 (0.044)
HOL:WHT			0.101 (0.030)	0.111 (0.030)		0.107 (0.032)	0.111 (0.031)
HOL:CRN			0.126 (0.033)	0.127 (0.033)		0.121 (0.035)	0.124 (0.034)
GAS:GLD			0.167 (0.037)	0.175 (0.036)		0.186 (0.038)	0.178 (0.038)
GAS:SLV			0.172 (0.036)	0.183 (0.037)		0.194 (0.038)	0.190 (0.038)
GAS:CPP			0.190 (0.039)	0.191 (0.040)		0.199 (0.041)	0.193 (0.041)
GAS:WHT			0.085 (0.029)	0.091 (0.031)		0.087 (0.032)	0.091 (0.031)
GAS:CRN			0.109 (0.032)	0.111 (0.032)		0.108 (0.034)	0.110 (0.033)
GLD:SLV			0.728 (0.022)	0.739 (0.020)		0.763 (0.020)	0.755 (0.021)
GLD:CPP			0.251 (0.035)	0.266 (0.035)		0.267 (0.039)	0.271 (0.037)
GLD:WHT			0.127 (0.032)	0.133 (0.033)		0.136 (0.033)	0.134 (0.033)
GLD:CRN			0.164 (0.033)	0.169 (0.034)		0.169 (0.036)	0.171 (0.034)
SLV:CPP			0.287 (0.033)	0.297 (0.033)		0.305 (0.037)	0.305 (0.035)
SLV:WHT			0.141 (0.032)	0.143 (0.033)		0.145 (0.034)	0.144 (0.034)
SLV:CRN			0.171 (0.035)	0.172 (0.035)		0.173 (0.037)	0.175 (0.036)
CPP:WHT			0.133 (0.035)	0.136 (0.037)		0.135 (0.038)	0.134 (0.037)
CPP:CRN			0.127 (0.041)	0.132 (0.041)		0.131 (0.043)	0.132 (0.042)
WHT:CRN			0.598 (0.026)	0.599 (0.025)		0.618 (0.027)	0.606 (0.026)
ν				16.804 (1.804)			18.579 (2.556)
θ_F		1.565 (0.143)			0.632 (1.745)		
θ_C	0.321 (0.029)				0.548 (0.133)	0.163 (0.091)	0.303 (0.215)
w_C					0.598 (0.069)	0.043 (0.013)	0.027 (0.012)
$\log \mathcal{L}$	717	661	2747	2815	788	2785	2824
AIC	-1432	-1319	-5439	-5574	-1570	-5510	-5586
BIC	-1427	-1314	-5294	-5423	-1555	-5355	-5426

Table 6.6 Multi asset classes portfolio: Static copula parameters with standard errors in parentheses. The mixture copulas are abbreviated: Frank-Clayton mixture (FC Mix), Gaussian-Clayton mixture (GC Mix) and Student-t-Clayton mixture (TC Mix). w_C denotes the weight of the Clayton copula in the mixture.

Copula	Clayton	Frank	Gaussian	Student-t	FC Mix	GC Mix	TC Mix
SMI:DAX			0.704 (0.018)	0.721 (0.018)		0.804 (0.018)	0.785 (0.019)
SMI:FTS			0.682 (0.021)	0.700 (0.020)		0.789 (0.023)	0.764 (0.021)
SMI:SPX			0.600 (0.022)	0.608 (0.022)		0.687 (0.023)	0.661 (0.023)
SMI:NAR			0.422 (0.030)	0.427 (0.028)		0.471 (0.038)	0.457 (0.032)
SMI:EPR			0.514 (0.024)	0.524 (0.024)		0.603 (0.029)	0.578 (0.027)
SMI:CNE			0.141 (0.030)	0.144 (0.030)		0.154 (0.043)	0.147 (0.036)
SMI:CEN			0.030 (0.040)	0.040 (0.041)		0.025 (0.056)	0.030 (0.047)
DAX:FTS			0.693 (0.019)	0.713 (0.017)		0.794 (0.021)	0.775 (0.020)
DAX:SPX			0.653 (0.022)	0.664 (0.022)		0.741 (0.023)	0.722 (0.023)
DAX:NAR			0.429 (0.029)	0.440 (0.027)		0.487 (0.038)	0.477 (0.030)
DAX:EPR			0.551 (0.022)	0.563 (0.022)		0.628 (0.028)	0.610 (0.023)
DAX:CNE			0.168 (0.032)	0.174 (0.032)		0.185 (0.044)	0.180 (0.038)
DAX:CEN			0.056 (0.038)	0.065 (0.039)		0.060 (0.056)	0.064 (0.046)
FTS:SPX			0.667 (0.023)	0.674 (0.022)		0.731 (0.024)	0.719 (0.023)
FTS:NAR			0.437 (0.029)	0.447 (0.029)		0.487 (0.039)	0.480 (0.032)
FTS:EPR			0.610 (0.022)	0.610 (0.022)		0.661 (0.027)	0.645 (0.024)
FTS:CNE			0.164 (0.033)	0.175 (0.034)		0.199 (0.044)	0.189 (0.039)
FTS:CEN			0.129 (0.035)	0.138 (0.036)		0.117 (0.051)	0.126 (0.042)
SPX:NAR			0.620 (0.022)	0.628 (0.022)		0.673 (0.027)	0.665 (0.023)
SPX:EPR			0.500 (0.022)	0.501 (0.022)		0.558 (0.032)	0.539 (0.025)
SPX:CNE			0.195 (0.032)	0.203 (0.032)		0.225 (0.047)	0.213 (0.039)
SPX:CEN			0.088 (0.038)	0.098 (0.038)		0.071 (0.055)	0.082 (0.046)
NAR:EPR			0.412 (0.030)	0.407 (0.030)		0.453 (0.041)	0.435 (0.034)
NAR:CNE			0.164 (0.032)	0.164 (0.032)		0.180 (0.047)	0.169 (0.039)
NAR:CEN			0.048 (0.043)	0.054 (0.042)		0.052 (0.059)	0.052 (0.048)
EPR:CNE			0.151 (0.032)	0.146 (0.033)		0.159 (0.045)	0.147 (0.040)
EPR:CEN			0.068 (0.036)	0.068 (0.037)		0.051 (0.052)	0.056 (0.044)
CNE:CEN			0.253 (0.030)	0.249 (0.029)		0.277 (0.036)	0.264 (0.031)
ν				13.905 (1.252)			16.876 (1.725)
θ_F		2.034 (0.118)			0.106 (0.138)		
θ_C	0.419 (0.031)				0.708 (0.061)	0.237 (0.232)	0.295 (0.169)
w_C					0.752 (0.039)	0.161 (0.028)	0.128 (0.024)
$\log \mathcal{L}$	1052	904	2082	2166	1190	2169	2201
AIC	-2102	-1807	-4108	-4274	-2373	-4278	-4340
BIC	-2097	-1801	-3965	-4126	-2359	-4125	-4182

6.3 Time-Varying Dependence

A plethora of evidence suggests that correlations between asset returns vary over time.[3] To visualize the evolution of the dependence level between the asset returns over time, time series plots of rolling six months rank correlation are presented. For the international equity index portfolio, figure 6.11 shows a plot of the six months rolling Kendall's tau computed via the one-to-one mapping of Frank's multivariate copula parameter θ_F and Kendall's tau stated in (3.38). The plot further includes 90% bootstrap confidence intervals obtained from 500 bootstrap replications of the data. The visualization confirms that the level of dependence between the equity indices measured by the multivariate Frank copula is not constant but displays some rather abrupt changes over time. Furthermore, dependence seems to have increased substantially over the time horizon under scrutiny: starting at 0.2 the measure hovered around 0.2 to 0.3 in the first third of the sample period and rose to about 0.5 in the last third of the sample. The structure of the dependence also does not appear to be constant. This is visualized by the lower tail dependence of the international equity index portfolio in figure 6.11. The plot shows the evolution of the lower tail dependence coefficient computed with equation (6.10). In the first half of the observation period, lower tail dependence fluctuated around 0.1 while in the second half it increased to about 0.3. One of the most prominent surges in tail dependence is in the year 2001 coinciding with the market turmoil around the events of 09/11 and a second one goes along with the outbreak of the financial crisis post 2007.

Figure 6.12 depicts the level of dependence of the commodity data set over time. The evolution can be fragmented into two distinct periods: from 1988 until about 2005, the level of dependence remained very low hovering around 0.1 with little variation. From the year 2005 onwards, variations in the level of dependence surged, along with a significant increase of the rank correlation. The two periods are also clearly shown by the lower tail depen-

[3] See references in section 3.3.

dence coefficient over time, depicted in figure 6.12. Until about 2005, there was virtually no lower tail dependence among the returns of the commodity futures indices. Starting in 2005, substantial spikes in tail dependence can be observed, indicating that along with the level of dependence, the structure of dependence was as well subject to change.

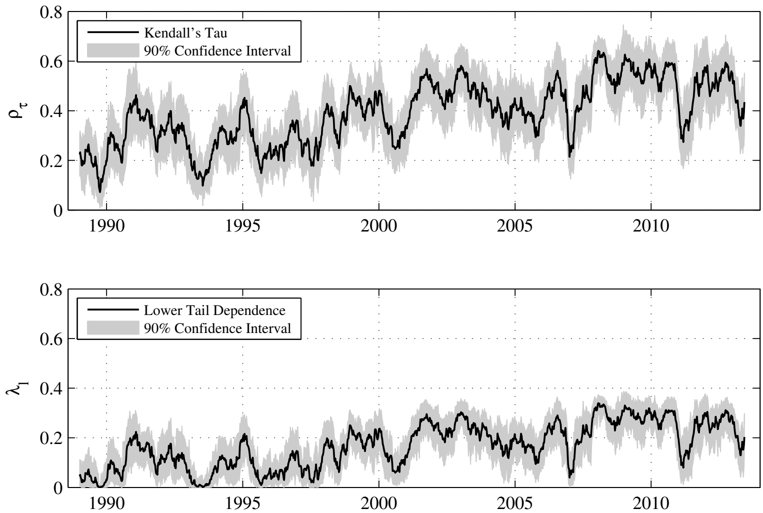

Figure 6.11: Kendall's tau implied by the multivariate Frank copula (upper graph) and lower tail dependence implied by the multivariate Clayton copula (lower graph) of the international equity index portfolio over a six months rolling window along with 90% bootstrap confidence intervals obtained from 500 bootstrap replications of the data.

The recent literature on the financialization of commodities ascribes this change to the emergence of commodities as an asset class, which has become increasingly held by institutional investors in search for diversification benefits (see e.g. Basak and Pavlova, 2013; Büyükşahin and Robe, 2014; Singleton, 2014). Indeed, beginning in the year 2004, institutional investors have been building up substantial positions in commodity futures. The U.S. Commodity Futures Trading Commission (CFTC) estimates in its staff report (2008) that institutional holdings have increased from USD 15 billion in 2003 to over USD

200 billion in 2008. Many of the institutional investors hold commodities through commodity futures indices, such as the Standard & Poors Goldman Sachs Commodity Indices (SPGSCI) (Basak and Pavlova, 2013).

The level of dependence of the multi asset classes index portfolio over time, visualized by the multivariate Frank copula implied Kendall's tau in figure 6.12, also varies substantially over the entire sample period. There seem to be two distinct periods of dependence: before the year 2001, Kendall's tau remained around 0.2 with little variation. After 2001, the level of dependence is increasing, reaching peaks of more than 0.5. Tail dependence over time is depicted in the bottom graph of figure 6.12 and underpins the segmentation into two different periods separated around the year 2001. During the first period, one cannot say with certainty that there was any tail dependence at all, as the confidence bands almost always included a coefficient of zero. After 2001, the returns of the equity, commodity and REIT indices display some tail dependence which then increases significantly with the propagation of the financial crisis in 2008. The crisis seems to emphasize the link between the equity, commodity and real estate market, which is expressed by surges in tail dependence and overall level of dependence.

The analysis of the dependence over time confirms that there are substantial changes in both structure and level of dependence during the observation period for all the portfolios under scrutiny. Neglecting this time variation might result in inaccurate risk forecasts. To capture the dynamics of the variations in level and structure of the dependence, two approaches are followed in this thesis: a regime switching model and fully dynamic copulas.

6.3.1 Regime Switching Copulas

This section presents the results of the estimation of the two-regime and three-regime switching copula models calibrated to the entire sample data. All copula models are estimated using the same residuals which result from

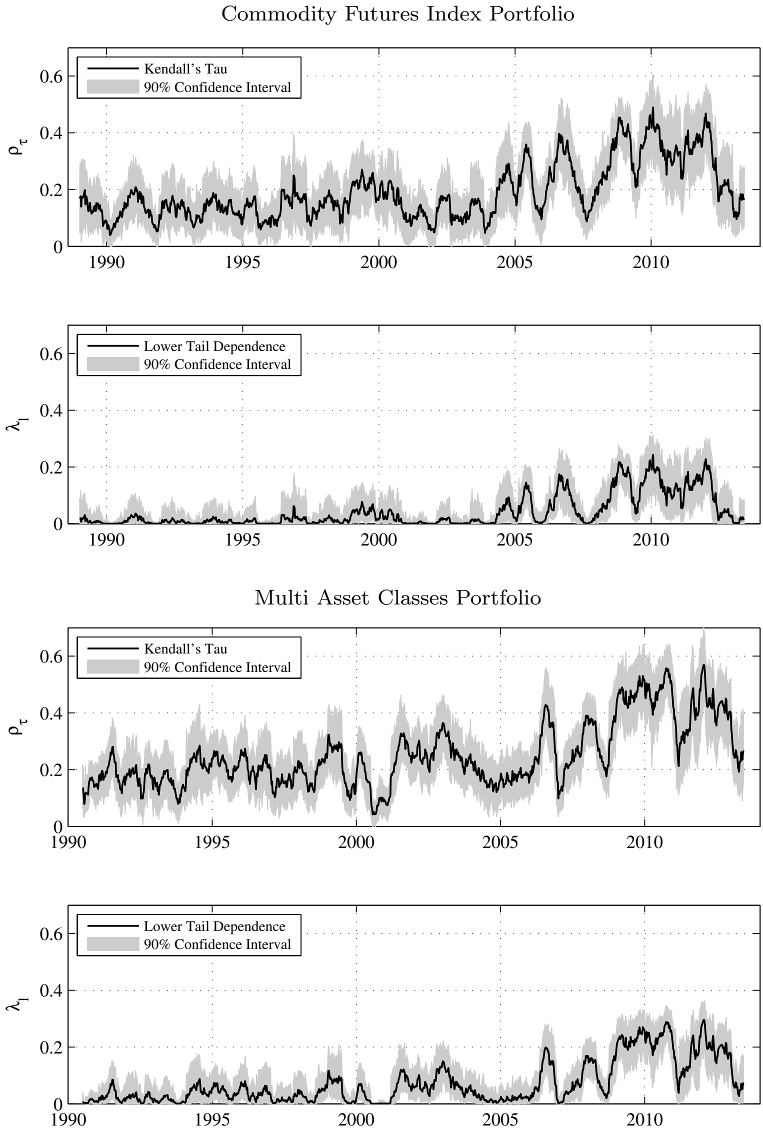

Figure 6.12: Kendall's tau implied by the multivariate Frank copula and lower tail dependence implied by the multivariate Clayton copula of the commodity futures index portfolio (upper part) and the multi asset classes portfolio (lower part) over a six months rolling window along with 90% bootstrap confidence intervals obtained from 500 bootstrap replications of the data.

the filtering with the univariate models listed in tables 6.1 to 6.3. The first three regime switching models combine the elliptical copulas into a two-regime setup. While the Gaussian/Gaussian (G/G) model allows for two regimes with different levels of dependence, it does not capture tail dependence in any of them whereas the Gaussian/Student-t (G/T) copula allows for tail dependence in one regime. The underlying idea for the latter is that returns may have (presumably tranquil) periods without tail dependence, which may be well described with a Gaussian copula, while the other (presumably turbulent) periods with tail dependence are described by the Student-t copula. The Student-t/Student-t (T/T) regime switching model then allows for tail dependence in both regimes.

As the purely elliptical models are symmetric, the Gaussian/Student-t model is enhanced by mixing either one with the asymmetric Clayton which firstly results in the Gaussian/Student-t-Clayton mixture (G/TC). This setup is thus capable of capturing different levels of lower and upper tail dependence in one regime. Secondly, the Student-t/Gaussian-Clayton mixture copula (T/GC) allows for equal (lower and upper) tail dependence in one regime and an asymmetric dependence with a probability of joint negative extreme returns in the other regime.

In addition to the two-state models, regime switching copulas with three separate regimes are constructed. Firstly, the Gaussian/Clayton/Frank copula (G/C/F), which has an elliptical and an Archimedean regime modeling the interrelation between the returns without tail dependence and one asymmetric regime with lower tail dependence. Secondly, the Gaussian/Student-t/Clayton copula (G/T/C), which has one regime with asymptotic independence in the tails (Gaussian), a second regime with equal lower and upper tail dependence (Student-t) and a third regime with only lower tail dependence (Clayton). Note that the number of possible combinations of the standalone copulas presented in chapter 3.2 into regime switch structures exceeds the amount of models presented in this thesis. In particular, models with three regimes consisting of all elliptical copulas are not considered, as with the increase in the number of parameters these models are far from parsimonious and would require an even larger calibration period.

The regime switching copula estimation results for the equity index portfolio is discussed next. Table 6.8 presents the parameter estimates for the equity indices. The first five models are two-regime models of which three contain only elliptical copulas. The regimes' parameters are listed in table 6.8 in the order indicated by the abbreviated name, i.e. for the G/T regime switch copula, the Gaussian regime parameters are listed under *Regime 1* and the Student-t copula parameters under *Regime 2*. The results for the two-regime models indicate that the maximum likelihood estimation procedure identifies a high and a low dependence regime, which is consistent with the findings of Chollete et al. (2009) and Braun (2011). Note that the labeling of high respectively low dependence regimes is done ex post, i.e. by analyzing the results of the maximum likelihood estimation. The copula correlation coefficients in the more dependent regime are larger for all index pairs which means that all returns in the portfolio are more dependent when the economy is in that regime. The high dependence regime features some very large correlations such as for example between the German DAX and the French CAC, where it exceeds 0.9 in every model. In contrast, the DAX:CAC copula correlation in the low dependence regime is as low as 0.5. Generally, the highest copula correlations are between the returns of the European equity indices.

The Student-t copula appears to be particularly well capable of describing periods of high dependence, as it is responsible for the high dependence regime in every structure it is a part of. This also holds for the three-regime structures, where the comparison of the level of dependence can be made using the transformations in (3.38). For the G/T/C copula, the according Kendall's tau for the average copula correlation amounts to 0.456 for the Student-t, 0.232 for the Gaussian and Kendall's tau corresponding to the Clayton copula with $\theta_C = 1.179$ equals 0.371. For the G/C/F regime switch copula, the average Gaussian copula correlation translates to a Kendall's tau of 0.458, while the parameters of the Clayton and the Frank copula correspond to Kendall's tau of 0.197 and 0, respectively. The fact that the Student-t copula consistently models the high dependence regimes reflects its versatility due to the parameter plurality (in comparison to the Archimedean copulas) and its capability to model tail dependence (in contrast to the Gaussian copula). Indeed, the degrees of freedom of the Student-t copulas in the mixture structures is between 8.884 and 10.434, which clearly indicates that the returns are tail dependent in the high dependence regimes.

In the G/T/C model, besides a high and a low dependence state, the third regime seems to model some middle or "normal" state of the economy with a dependence level midway between the ones of the extreme states. In contrast to what one might expect, this "normal" state is not modeled by the Gaussian copula but by the Clayton copula. The low probability of staying in the third regime conditional on being in regime three suggests that this regime is more of a pass-through between the other regimes which both display much higher probabilities $p_{1|1}, p_{2|2}$. Following Hamilton (1989), the transition probabilities can be used to compute the probability to stay k weeks in regime j as $P(D_j = k) = p_{j|j}^{k-1}(1-p_{j|j})$, where D_j denotes the number of periods the Markov chain is in state j. This implies that the expected duration of this state is

$$E(D_j) = \sum_{k=1}^{\infty} k p_{j|j}^{k-1}(1 - p_{j|j}) = \frac{1}{1 - p_{j|j}}. \tag{6.12}$$

Table 6.7 lists the expected durations of the regimes in the different models. The top part refers to the equity data set and shows that for the two-regime models, the high dependence regime (marked in bold) has a much longer expected duration compared to the other regime(s) in every copula model. The table illustrates that although the transition probabilities $p_{1|1}$ and $p_{2|2}$ are high probabilities (exceeding 0.82 for every two-state model), they reflect a very different regime persistence. The additional length in expected duration in the two-regime setups range from one third in the T/T model to more than two thirds in the T/GC structure. The differences in the three-regime

Table 6.7
Expected regime durations $E(D_{R_j})$ in weeks

	G/G	G/T	T/T	T/GC	G/TC	G/T/C	G/C/F
Equities							
$E(D_{R_1})$	**13**	8	**8**	**22**	11	5	**17**
$E(D_{R_2})$	9	**13**	6	13	**13**	**11**	5
$E(D_{R_3})$						1	3
Commodities							
$E(D_{R_1})$	4	6	**8**	9	**10**	**7**	**24**
$E(D_{R_2})$	**4**	**5**	8	**13**	8	5	1
$E(D_{R_3})$						1	1
Multi Asset Classes							
$E(D_{R_1})$	8	10	**7**	**16**	8	5	**8**
$E(D_{R_2})$	**9**	**14**	6	14	**11**	**8**	1
$E(D_{R_3})$						2	6

This table presents the expected regime durations under the regime switching models for all three data sets under consideration. The durations are computed by applying (6.12) to the transition probabilities listed in the tables 6.8, 6.9 and 6.10. The expected duration of the high dependence regime is marked in bold. The copulas are abbreviated: Gaussian (G), Student-t (T), Clayton (C), Frank (F), Gaussian-Clayton mixture (GC) and Student-t-Clayton mixture (TC). The forward slash indicates the separation of the regimes.

models are even larger. However, the minimal expected duration of the third regime in the G/T/C model suggests that this regime may be irrelevant. In the G/C/F model, the regime modeled by the Frank copula has a parameter θ_F of virtually zero, which means that in this setup there are times when the returns are independent. Even though the transition probability $p_{3|3}$ is not as low as in the other three regime model, it has the second lowest expected duration (three weeks).

Comparing the fit of the models based on the information criteria, it is evident that the differences between the AIC / BIC values of the regime switching models are much smaller compared to the static copulas. The setup that all regime switching models contain at least one correlation matrix and therewith have a higher average number of parameters explains the higher average likelihood of the regime switch models compared to the static copulas. According to both AIC and BIC, the best fitting model is the T/GC followed by the G/G and the G/T. Both information criteria further clearly indicate that despite having more than twice the number of parameters, the Gaussian/Gaussian regime switch copula fits the data much better than the static Gaussian copula. However, adding a third regime as done with the G/T/C and G/C/F model does not improve the fit. To the contrary, amalgamating the Clayton copula into a mixture with an elliptical copula to model one of two regimes (as done in the G/TC and T/GC models) yields a better fit than letting each of the involved copula separately model one regime (as done in the G/T/C). Based on the lower values of both information criteria and on the small expected duration of a third regime, one may conclude that two-regime models are more suitable to capture the dynamics of the equity index portfolio.

Figure 6.13 depicts the Kim filtered evolution of the state probabilities over the entire sample period (see (3.28)) for the equities. All seven models depict a shift from low to high dependence over the observation period. After the year 2001, all models remain in the high dependence regime (marked with the solid black line) for the rest of the time. Comparing the Kim filtered regime paths with the rolling Kendall's tau in figure 6.11, one can identify the intermediate heights of the average rank correlation around 1991, 1995 and before the turn of the millennium in the Kim filtered regime probabilities of each model, since these heights are captured by a high dependence regime probability of close to one in every model. Furthermore, for the second half of the period under scrutiny, the high dependence regime clearly takes over in every regime switching model, having a regime probability of close to one for the vast majority of the time.

The regime probability path of the G/T/C model confirms the finding of the expected regime duration that the Clayton regime is not relevant, since the according regime probability remains close to zero at all times. The regime probabilities of the G/C/F copula over time show that while the third regime has a few short spikes in probability in the first half of the observation time, it remains at zero for the rest which indicates that this third regime is overall of subordinate importance. Taken as a whole, the Kim filtered regime probabilities show that most usually it is quite clear which regime the system is in at every point in time as the Kim filtered probabilities are close to either zero or one most of the time.

Next, the results for the commodity futures portfolio are analyzed. The according parameter estimates are listed in table 6.9. In all commodity regime switching models with two states, the parameter estimates indicate one high and one low dependence regime. Some copula correlations which are low already in the static Gaussian copula (see table 6.5) are now close to zero in the low dependence regime. This is the case for example for the copula correlation between crude oil (OIL) and wheat (WHT) in the Gaussian regime of the G/T model, where it is as low as 0.025. The degrees of freedom of 9.557 in the high dependence Student-t regime of the G/T copula suggests that tail dependence is a feature of this state of the economy. This is consistent with the impression one obtains by analyzing figure 6.12. However, in the T/T model both ν parameters are equally low suggesting tail dependence in both regimes. Then again, the weights of the Clayton copula in the mixture structures of the T/GC and G/TC models and the according values of θ_c are lower compared to the models for the equity indices, indicating that asymmetries in the dependence structure may be a less prominent feature for commodity futures index returns.

To compare the levels of dependence in the different regimes of the models with three regimes, the mapping in equation 3.38 is used. For the G/C/F model, the θ_C of 0.304 transformed into a rank correlation measure yields a Kendall's tau of 0.1319, the Frank copula parameter of $\theta_F = 0.506$ corresponds to 0.056 while the average Gaussian copula correlation translates to a Kendall's tau amounting to 0.184. The second regime (Clayton) thus forms the midpoint dependence regime. However, the importance of the Clayton regime is negligible which can be seen in both the probability $p_{2|2}$ of virtually zero and the according minimal expected regime duration listed in table 6.7.

This suggests that the Clayton regime in the G/C/F structure is irrelevant and that the other two regimes would be sufficient to capture the dependence structure of the commodity futures indices. This finding is substantiated by both information criteria, which display the highest values for the G/C/F model. The ranking of the model fit according to both BIC and AIC is: Student-t/Gaussian-Clayton mixture in the first, the Gaussian/Student-t-Clayton mixture in the second and the Gaussian/Student-t/Clayton in the third place.

The expected regime durations in the mid-section of table 6.7 show that the persistence of the regimes in the purely elliptical models is identical for the G/G and the T/T while being only one week apart in the G/T copula. As soon as asymmetric dependence is introduced to the model, the differences in expected duration become larger. Analogous to the models for the equity data, the high dependence regime in these models has a higher persistence. Except for the G/T model, the high dependence regime for the commodities is modeled by the Gaussian copula, which in contrast to the equity returns points to insignificant tail dependence in the high dependence regime of the commodity returns.

The Kim filtered regime probabilities over time are depicted in figure 6.14. At first sight it becomes apparent that the plot for the G/C/F model is different to the other six, as the probability of the Gaussian regime is close to one almost all the time. The Clayton and the Frank copula probabilities in this G/C/F model are virtually zero, which is also reflected by their minimal expected regime duration in table 6.7. With two of three regimes being insignificant, the G/C/F regime switch copula is evidently an inadequate model for the data at hand. The G/T regime probability paths in figure 6.14 differentiates itself as it gives the least clear idea about which regime the system was in at any point in time until about 2007. However, all regime probability paths capture the spikes in dependence identified in figure 6.12 by the high dependence regime. In contrast to the corresponding plot 6.13 for the equities, the regime switching models do not show a consensus shift from the low to the high dependence regime over the observation period even if such a shift could be expected from analyzing figure 6.12.

Table 6.10 shows the regime switching models' parameter estimates for the multi asset classes portfolio, which are now put into focus. In accordance with the results for the commodity and the equity data set, the models with a two-state Markov chain contain a high dependence and a low dependence regime. The copula correlations in the latter regimes attain negative values for the pairs containing the energy futures index (CEN), which suggests that this index provides interesting diversification potential in this state of the economy. Also in the high dependence regimes, the SPGSCI energy futures index displays the lowest correlation with the other indices. Congruent with the results for the equity data set, the Student-t copula models the high dependence regime in every dependence structure it is included in. The rather low ν parameters of these Student-t copulas indicate the tail dependence in this state of the economy. The Gaussian copula is incapable of capturing this tail dependence. As a result, the Gaussian copula modeling the high dependence regime in the G/G structure (regime 2) displays higher correlations than the Student-t copula, which models the high dependence regime in the G/T setup and captures the dependencies of the extreme movements with its low degrees of freedom parameter. However, the T/T model suggests that there is tail dependence also in the low dependence regime, since the ν parameters of both regimes are equally low. The results for the T/GC model corroborate this interpretation, as the weight of the Clayton copula which captures lower tail dependence with $\theta_C = 0.443$ in the Gaussian-Clayton mixture construct amounts to 15.9%. In the G/TC model both the Student-t and the Clayton copula capture lower tail dependence in the same regime. The according mixture weight of the Clayton copula w_C is smaller with a larger θ_C, which shows that the Clayton fraction models the dependence of the rare but highly dependent extreme negative returns.

To compare the level of dependence in the regimes of the three-regime models, the transformation of the average G/T/C copula correlation yields a Kendall's tau of 0.174 for the Gaussian, 0.324 for the Student-t copula, while the mapping of Clayton's θ_C to Kendall's tau yields 0.141. Interestingly, the regime with the lowest level of dependence is modeled by the Clayton copula. This could mean that despite a low level of dependence in that state of the economy, there is lower tail dependence. It could however also be that with the larger number of parameters, the Student-t and the Gaussian are simply better equipped to adapt to the periods of higher dependence. Comparing the Kendall's tau transformations of the G/C/F model shows that the Gaussian copula models the higher ($\rho_\tau = 0.256$), the Clayton is responsible for the average ($\rho_\tau = 0.132$) and the Frank captures the low dependence periods ($\rho_\tau = 0.056$).

As for the commodity data set, the second regime in the G/C/F copula governed by the Clayton copula can be classified as irrelevant based on its transition probability $p_{2|2}$ of zero. This evidently translates into the minimal expected regime duration of 1, as depicted in the bottom part of table 6.7. Table 6.7 further shows that the high dependence regimes are more persistent in every regime switching model for the multi asset classes portfolio, as the according expected regime durations are larger than the ones for the lower, respectively average dependence regimes. Based on the values of the AIC / BIC, the dependence model that best fits the returns of the multi asset classes portfolio is the Student-t / Gaussian-Clayton mixture copula, followed by the G/T regime switch copula. The third best fit, according to the AIC is the G/T/C. However, the BIC - which imposes a larger penalty for additional parameters - ranks the T/T model in third place as it attains the identical likelihood value as the G/T/C with less parameters.

The Kim filtered regime probability paths over time for the multi asset classes portfolio are depicted in figure 6.15. All probability evolutions share two distinguishable periods which are segmented around the year 2001. While in the first part of the observation period the low dependence regime (marked in gray) prevailed, the post 2001 era appears to be dominated by the high dependence regime. The high dependence probability of the G/C/F copula remains close to one for the post millennium period but has the noisiest evolution of all before that. However, the G/C/F has the lowest likelihood, the highest information criteria values and a transition probability $p_{2|2} = 0$ for the Clayton regime all indicating an inferior fit compared to the other six regime switching models.

The five models containing two regimes ascribe the same time frames to the high respectively low dependence regimes. The clearest idea about which regime the system was in at any point in time is given by the T/GC copula, which shows clear-cut regime switches with regime probabilities of close to either zero or one most of the time.

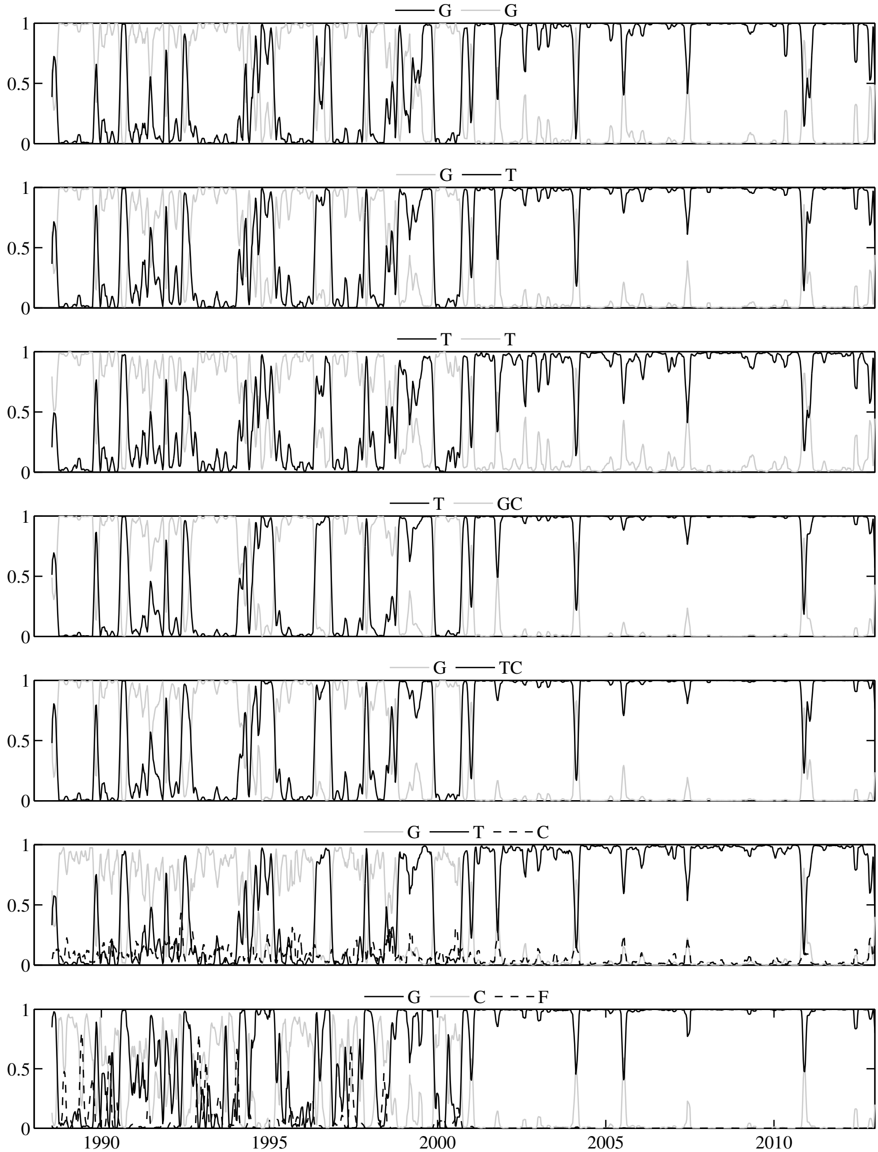

Figure 6.13: Equity portfolio: Kim filtered regime probabilities of the various regime switching copula models over the entire sample period. The copulas are abbreviated: Gaussian (G), Student-t (T), Frank (F), Clayton (C), Gaussian-Clayton mixture (GC) and Student-t-Clayton mixture (TC). The solid black line marks the high dependence regime.

Table 6.8

Equity portfolio: Regime switching copula parameters with standard errors in parentheses. The copulas are abbreviated: Frank (F), Clayton (C), Gaussian (G), Student-t (T), Gaussian-Clayton mixture (GC) and Student-t-Clayton mixture (TC). w_C denotes the weight of the Clayton copula in the mixture. The forward slash indicates the separate regimes i.e. G/T/C and G/C/F are three-state models. $p_{i|i}$ denotes the probability of staying in regime i.

Copula	G/G	G/T	T/T	T/GC	G/TC	G/T/C	G/C/F
			Regime 1				
SMI:DAX	0.838 (0.154)	0.496 (0.108)	0.822 (0.125)	0.826 (0.090)	0.452 (0.104)	0.518 (0.119)	0.827 (0.016)
SMI:CAC	0.858 (0.188)	0.409 (0.141)	0.847 (0.166)	0.853 (0.130)	0.340 (0.139)	0.452 (0.169)	0.848 (0.016)
SMI:FTS	0.838 (0.179)	0.414 (0.129)	0.820 (0.157)	0.830 (0.139)	0.327 (0.139)	0.386 (0.179)	0.818 (0.017)
SMI:SPX	0.717 (0.160)	0.408 (0.094)	0.712 (0.102)	0.690 (0.105)	0.381 (0.100)	0.441 (0.125)	0.700 (0.023)
SMI:TSX	0.614 (0.141)	0.364 (0.081)	0.564 (0.063)	0.583 (0.087)	0.377 (0.084)	0.315 (0.105)	0.577 (0.029)
SMI:HSE	0.612 (0.220)	0.183 (0.136)	0.561 (0.152)	0.605 (0.160)	0.133 (0.156)	0.160 (0.192)	0.564 (0.033)
SMI:NIK	0.573 (0.249)	0.086 (0.142)	0.546 (0.170)	0.518 (0.195)	0.093 (0.185)	0.072 (0.228)	0.487 (0.038)
DAX:CAC	0.932 (0.138)	0.570 (0.100)	0.922 (0.127)	0.919 (0.095)	0.501 (0.100)	0.600 (0.118)	0.909 (0.012)
DAX:FTS	0.849 (0.185)	0.429 (0.129)	0.821 (0.168)	0.823 (0.128)	0.335 (0.136)	0.457 (0.166)	0.815 (0.020)
DAX:SPX	0.798 (0.173)	0.406 (0.108)	0.791 (0.140)	0.768 (0.121)	0.386 (0.120)	0.436 (0.146)	0.766 (0.020)
DAX:TSX	0.657 (0.150)	0.392 (0.081)	0.621 (0.076)	0.644 (0.088)	0.394 (0.089)	0.356 (0.108)	0.633 (0.025)
DAX:HSE	0.641 (0.193)	0.290 (0.107)	0.602 (0.123)	0.657 (0.135)	0.225 (0.126)	0.326 (0.156)	0.596 (0.029)
DAX:NIK	0.619 (0.234)	0.170 (0.135)	0.597 (0.154)	0.563 (0.195)	0.193 (0.172)	0.139 (0.217)	0.543 (0.033)
CAC:FTS	0.883 (0.158)	0.495 (0.115)	0.861 (0.144)	0.873 (0.114)	0.418 (0.116)	0.536 (0.133)	0.858 (0.013)
CAC:SPX	0.801 (0.185)	0.404 (0.121)	0.804 (0.152)	0.791 (0.138)	0.383 (0.133)	0.432 (0.165)	0.774 (0.022)
CAC:TSX	0.683 (0.171)	0.380 (0.103)	0.649 (0.097)	0.673 (0.108)	0.338 (0.105)	0.366 (0.130)	0.653 (0.026)
CAC:HSE	0.646 (0.192)	0.284 (0.105)	0.604 (0.126)	0.650 (0.139)	0.207 (0.132)	0.333 (0.157)	0.602 (0.031)
CAC:NIK	0.632 (0.238)	0.220 (0.133)	0.613 (0.152)	0.581 (0.188)	0.256 (0.169)	0.168 (0.216)	0.556 (0.033)
FTS:SPX	0.775 (0.145)	0.517 (0.094)	0.763 (0.113)	0.776 (0.102)	0.460 (0.095)	0.543 (0.115)	0.760 (0.021)
FTS:TSX	0.708 (0.174)	0.370 (0.103)	0.664 (0.102)	0.687 (0.107)	0.355 (0.104)	0.361 (0.131)	0.665 (0.026)
FTS:HSE	0.664 (0.191)	0.353 (0.112)	0.623 (0.139)	0.673 (0.138)	0.251 (0.127)	0.429 (0.161)	0.631 (0.032)
FTS:NIK	0.594 (0.230)	0.209 (0.126)	0.559 (0.147)	0.540 (0.178)	0.244 (0.160)	0.188 (0.198)	0.525 (0.032)
SPX:TSX	0.751 (0.054)	0.669 (0.032)	0.732 (0.034)	0.744 (0.028)	0.640 (0.026)	0.691 (0.037)	0.758 (0.016)
SPX:HSE	0.638 (0.186)	0.281 (0.116)	0.593 (0.130)	0.643 (0.132)	0.214 (0.127)	0.325 (0.156)	0.600 (0.029)
SPX:NIK	0.561 (0.194)	0.249 (0.116)	0.550 (0.131)	0.521 (0.162)	0.296 (0.152)	0.229 (0.176)	0.504 (0.035)
TSX:HSE	0.548 (0.129)	0.303 (0.082)	0.496 (0.067)	0.554 (0.091)	0.239 (0.090)	0.347 (0.109)	0.513 (0.032)
TSX:NIK	0.511 (0.166)	0.220 (0.097)	0.472 (0.097)	0.452 (0.123)	0.246 (0.118)	0.219 (0.139)	0.459 (0.033)
HSE:NIK	0.590 (0.209)	0.186 (0.123)	0.558 (0.135)	0.560 (0.177)	0.196 (0.172)	0.157 (0.207)	0.505 (0.041)
ν_1			10.218 (0.865)	10.362 (0.989)			

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Table 6.8 continued

				Regime 2			
SMI:DAX	0.513 (0.154)	0.813 (0.066)	0.511 (0.126)	0.477 (0.134)	0.826 (0.090)	0.811 (0.095)	
SMI:CAC	0.450 (0.185)	0.846 (0.088)	0.433 (0.168)	0.415 (0.190)	0.853 (0.130)	0.838 (0.149)	
SMI:FTS	0.425 (0.184)	0.813 (0.085)	0.425 (0.164)	0.368 (0.192)	0.830 (0.139)	0.804 (0.158)	
SMI:SPX	0.405 (0.158)	0.691 (0.068)	0.424 (0.106)	0.383 (0.132)	0.690 (0.105)	0.674 (0.116)	
SMI:TSX	0.360 (0.142)	0.563 (0.050)	0.385 (0.068)	0.340 (0.106)	0.583 (0.087)	0.552 (0.089)	
SMI:HSE	0.190 (0.218)	0.575 (0.092)	0.211 (0.152)	0.110 (0.207)	0.605 (0.160)	0.537 (0.183)	
SMI:NIK	0.117 (0.250)	0.543 (0.113)	0.144 (0.169)	0.053 (0.222)	0.518 (0.195)	0.540 (0.209)	
DAX:CAC	0.608 (0.139)	0.917 (0.072)	0.593 (0.131)	0.573 (0.133)	0.919 (0.095)	0.918 (0.102)	
DAX:FTS	0.438 (0.193)	0.815 (0.092)	0.431 (0.172)	0.422 (0.176)	0.823 (0.128)	0.812 (0.141)	
DAX:SPX	0.408 (0.169)	0.772 (0.080)	0.413 (0.138)	0.387 (0.156)	0.768 (0.121)	0.781 (0.134)	
DAX:TSX	0.394 (0.152)	0.611 (0.060)	0.407 (0.077)	0.366 (0.113)	0.644 (0.088)	0.591 (0.096)	
DAX:HSE	0.291 (0.186)	0.618 (0.085)	0.296 (0.125)	0.231 (0.168)	0.657 (0.135)	0.576 (0.155)	
DAX:NIK	0.174 (0.233)	0.592 (0.105)	0.203 (0.147)	0.116 (0.214)	0.563 (0.195)	0.601 (0.206)	
CAC:FTS	0.508 (0.164)	0.861 (0.087)	0.521 (0.149)	0.500 (0.146)	0.873 (0.114)	0.849 (0.127)	
CAC:SPX	0.411 (0.185)	0.783 (0.087)	0.402 (0.151)	0.386 (0.183)	0.791 (0.138)	0.785 (0.154)	
CAC:TSX	0.400 (0.176)	0.638 (0.069)	0.392 (0.099)	0.376 (0.142)	0.673 (0.108)	0.630 (0.121)	
CAC:HSE	0.294 (0.189)	0.625 (0.089)	0.286 (0.131)	0.247 (0.161)	0.650 (0.139)	0.585 (0.160)	
CAC:NIK	0.210 (0.234)	0.606 (0.107)	0.242 (0.146)	0.140 (0.214)	0.580 (0.188)	0.617 (0.201)	
FTS:SPX	0.494 (0.149)	0.747 (0.063)	0.485 (0.123)	0.518 (0.126)	0.776 (0.102)	0.735 (0.106)	
FTS:TSX	0.381 (0.183)	0.663 (0.067)	0.380 (0.111)	0.377 (0.137)	0.687 (0.107)	0.645 (0.117)	
FTS:HSE	0.368 (0.193)	0.640 (0.084)	0.334 (0.140)	0.341 (0.172)	0.673 (0.138)	0.611 (0.161)	
FTS:NIK	0.206 (0.226)	0.555 (0.100)	0.223 (0.146)	0.165 (0.194)	0.540 (0.178)	0.556 (0.188)	
SPX:TSX	0.670 (0.055)	0.742 (0.020)	0.677 (0.037)	0.679 (0.037)	0.744 (0.028)	0.728 (0.030)	
SPX:HSE	0.283 (0.186)	0.598 (0.075)	0.273 (0.133)	0.249 (0.161)	0.643 (0.132)	0.571 (0.143)	
SPX:NIK	0.246 (0.191)	0.535 (0.086)	0.243 (0.128)	0.208 (0.169)	0.521 (0.162)	0.543 (0.168)	
TSX:HSE	0.326 (0.131)	0.513 (0.051)	0.308 (0.065)	0.300 (0.106)	0.554 (0.091)	0.481 (0.092)	
TSX:NIK	0.224 (0.168)	0.469 (0.071)	0.236 (0.085)	0.200 (0.133)	0.452 (0.123)	0.456 (0.125)	
HSE:NIK	0.181 (0.199)	0.566 (0.084)	0.181 (0.135)	0.121 (0.187)	0.560 (0.179)	0.561 (0.182)	
ν_2		9.913 (0.688)	10.434 (0.895)		9.527 (0.681)	8.884 (0.789)	
θ_C				0.866 (0.400)	2.268 (0.312)	1.179 (0.414)	0.489 (0.399)
θ_F							0.000 (0.356)
w_C				0.156 (0.068)	0.071 (0.049)		
$p_{1\|1}$	0.923 (0.052)	0.868 (0.091)	0.880 (0.052)	0.954 (0.091)	0.913 (0.125)	0.815 (0.141)	0.941 (0.031)
$p_{2\|2}$	0.891 (0.047)	0.924 (0.050)	0.828 (0.045)	0.924 (0.105)	0.948 (0.071)	0.907 (0.109)	0.786 (0.161)
$p_{3\|3}$						0.190 (0.110)	0.614 (0.166)
$\log \mathcal{L}$	3641	3631	3590	3655	3617	3618	3479
AIC	-7166	-7144	-7060	-7188	-7112	-7114	-6892
BIC	-6866	-6839	-6750	-6873	-6787	-6799	-6721

Table 6.9
Commodity portfolio: Regime switching copula parameters with standard errors in parentheses. The copulas are abbreviated: Frank (F), Clayton (C), Gaussian (G), Student-t (T), Gaussian-Clayton mixture (GC) and Student-t-Clayton mixture (TC). w_C denotes the weight of the Clayton copula in the mixture. The forward slash indicates the separate regimes i.e. G/T/C and G/C/F are three-state models. $p_{i|i}$ denotes the probability of staying in regime i.

Copula	G/G	G/T	T/T	T/GC	G/TC	G/T/C	G/C/F
			Regime 1				
OIL:HOL	0.797 (0.071)	0.819 (0.059)	0.946 (0.070)	0.782 (0.054)	0.936 (0.030)	0.942 (0.034)	0.894 (0.010)
OIL:GAS	0.751 (0.095)	0.771 (0.082)	0.924 (0.077)	0.763 (0.062)	0.910 (0.040)	0.923 (0.039)	0.867 (0.014)
OIL:GLD	0.102 (0.153)	0.093 (0.113)	0.301 (0.114)	0.163 (0.126)	0.284 (0.083)	0.272 (0.093)	0.234 (0.038)
OIL:SLV	0.090 (0.138)	0.030 (0.100)	0.332 (0.112)	0.187 (0.117)	0.247 (0.081)	0.252 (0.086)	0.231 (0.038)
OIL:CPP	0.143 (0.114)	0.093 (0.103)	0.217 (0.112)	0.242 (0.129)	0.156 (0.090)	0.153 (0.092)	0.208 (0.045)
OIL:WHT	0.060 (0.069)	0.025 (0.064)	0.083 (0.072)	0.153 (0.085)	0.055 (0.060)	0.039 (0.057)	0.097 (0.034)
OIL:CRN	0.104 (0.062)	0.050 (0.058)	0.095 (0.065)	0.160 (0.073)	0.047 (0.056)	0.034 (0.054)	0.111 (0.034)
HOL:GAS	0.725 (0.092)	0.746 (0.076)	0.886 (0.076)	0.729 (0.065)	0.878 (0.039)	0.887 (0.042)	0.842 (0.015)
HOL:GLD	0.108 (0.139)	0.093 (0.101)	0.269 (0.107)	0.162 (0.108)	0.260 (0.076)	0.263 (0.082)	0.210 (0.039)
HOL:SLV	0.079 (0.131)	0.028 (0.092)	0.310 (0.109)	0.169 (0.106)	0.227 (0.077)	0.226 (0.081)	0.207 (0.039)
HOL:CPP	0.149 (0.107)	0.090 (0.102)	0.203 (0.120)	0.215 (0.122)	0.143 (0.095)	0.146 (0.096)	0.196 (0.045)
HOL:WHT	0.082 (0.070)	0.043 (0.067)	0.083 (0.070)	0.216 (0.083)	0.064 (0.059)	0.045 (0.059)	0.125 (0.033)
HOL:CRN	0.107 (0.060)	0.053 (0.058)	0.086 (0.063)	0.168 (0.072)	0.051 (0.056)	0.038 (0.055)	0.117 (0.036)
GAS:GLD	0.056 (0.140)	0.065 (0.113)	0.239 (0.111)	0.084 (0.112)	0.245 (0.075)	0.238 (0.080)	0.186 (0.038)
GAS:SLV	0.051 (0.138)	0.020 (0.106)	0.294 (0.120)	0.099 (0.120)	0.241 (0.072)	0.245 (0.077)	0.192 (0.039)
GAS:CPP	0.124 (0.114)	0.091 (0.104)	0.197 (0.119)	0.192 (0.118)	0.169 (0.088)	0.156 (0.087)	0.196 (0.042)
GAS:WHT	0.081 (0.072)	0.029 (0.070)	0.056 (0.073)	0.184 (0.087)	0.045 (0.061)	0.027 (0.057)	0.101 (0.033)
GAS:CRN	0.094 (0.064)	0.040 (0.063)	0.082 (0.066)	0.150 (0.079)	0.034 (0.052)	0.014 (0.052)	0.103 (0.035)
GLD:SLV	0.612 (0.103)	0.663 (0.082)	0.688 (0.068)	0.820 (0.061)	0.692 (0.046)	0.696 (0.042)	0.765 (0.020)
GLD:CPP	0.091 (0.182)	0.145 (0.111)	0.210 (0.112)	0.301 (0.125)	0.237 (0.087)	0.243 (0.092)	0.270 (0.039)
GLD:WHT	0.093 (0.086)	0.081 (0.097)	0.097 (0.092)	0.178 (0.103)	0.085 (0.077)	0.069 (0.075)	0.136 (0.034)
GLD:CRN	0.089 (0.098)	0.081 (0.084)	0.143 (0.068)	0.254 (0.083)	0.102 (0.073)	0.123 (0.070)	0.170 (0.037)
SLV:CPP	0.125 (0.168)	0.161 (0.107)	0.227 (0.100)	0.344 (0.116)	0.244 (0.082)	0.279 (0.086)	0.306 (0.036)
SLV:WHT	0.127 (0.083)	0.110 (0.101)	0.104 (0.091)	0.187 (0.096)	0.097 (0.082)	0.081 (0.077)	0.143 (0.036)
SLV:CRN	0.119 (0.095)	0.096 (0.096)	0.152 (0.075)	0.261 (0.086)	0.105 (0.071)	0.136 (0.071)	0.175 (0.038)
CPP:WHT	0.153 (0.081)	0.092 (0.086)	0.161 (0.082)	0.210 (0.097)	0.095 (0.075)	0.082 (0.074)	0.151 (0.039)
CPP:CRN	0.057 (0.110)	0.030 (0.094)	0.162 (0.091)	0.181 (0.114)	0.069 (0.085)	0.091 (0.087)	0.129 (0.043)
WHT:CRN	0.649 (0.123)	0.675 (0.111)	0.480 (0.130)	0.728 (0.089)	0.515 (0.071)	0.483 (0.070)	0.620 (0.026)
ν_1			10.304 (1.135)	9.729 (1.060)			

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Table 6.9 continued

				Regime 2			
OIL:HOL	0.946 (0.073)	0.945 (0.062)	0.788 (0.070)	0.942 (0.049)	0.778 (0.065)	0.768 (0.061)	
OIL:GAS	0.935 (0.098)	0.939 (0.069)	0.761 (0.078)	0.918 (0.050)	0.765 (0.074)	0.741 (0.067)	
OIL:GLD	0.365 (0.149)	0.380 (0.110)	0.091 (0.100)	0.243 (0.117)	0.120 (0.135)	0.165 (0.140)	
OIL:SLV	0.359 (0.134)	0.416 (0.100)	0.057 (0.103)	0.226 (0.105)	0.173 (0.131)	0.209 (0.128)	
OIL:CPP	0.269 (0.124)	0.324 (0.115)	0.190 (0.096)	0.147 (0.104)	0.266 (0.148)	0.252 (0.148)	
OIL:WHT	0.140 (0.074)	0.145 (0.079)	0.023 (0.061)	0.046 (0.072)	0.151 (0.110)	0.148 (0.102)	
OIL:CRN	0.138 (0.063)	0.177 (0.071)	0.064 (0.054)	0.049 (0.064)	0.192 (0.090)	0.145 (0.083)	
HOL:GAS	0.904 (0.096)	0.907 (0.071)	0.718 (0.079)	0.890 (0.056)	0.733 (0.075)	0.732 (0.073)	
HOL:GLD	0.335 (0.131)	0.349 (0.096)	0.106 (0.092)	0.224 (0.104)	0.119 (0.110)	0.174 (0.118)	
HOL:SLV	0.338 (0.122)	0.388 (0.091)	0.066 (0.100)	0.216 (0.099)	0.182 (0.115)	0.231 (0.113)	
HOL:CPP	0.242 (0.114)	0.297 (0.112)	0.151 (0.103)	0.131 (0.106)	0.236 (0.144)	0.234 (0.140)	
HOL:WHT	0.159 (0.075)	0.162 (0.076)	0.068 (0.060)	0.051 (0.072)	0.204 (0.106)	0.224 (0.100)	
HOL:CRN	0.143 (0.060)	0.178 (0.070)	0.088 (0.051)	0.052 (0.062)	0.200 (0.087)	0.167 (0.080)	
GAS:GLD	0.297 (0.134)	0.299 (0.107)	0.044 (0.097)	0.211 (0.105)	0.050 (0.113)	0.081 (0.115)	
GAS:SLV	0.306 (0.135)	0.349 (0.112)	0.012 (0.114)	0.219 (0.096)	0.093 (0.129)	0.111 (0.128)	
GAS:CPP	0.253 (0.121)	0.294 (0.115)	0.152 (0.101)	0.153 (0.100)	0.191 (0.141)	0.180 (0.137)	
GAS:WHT	0.116 (0.079)	0.135 (0.082)	0.070 (0.060)	0.036 (0.073)	0.180 (0.106)	0.181 (0.098)	
GAS:CRN	0.120 (0.061)	0.159 (0.076)	0.074 (0.052)	0.037 (0.062)	0.192 (0.090)	0.160 (0.086)	
GLD:SLV	0.840 (0.104)	0.823 (0.070)	0.796 (0.066)	0.700 (0.058)	0.826 (0.062)	0.832 (0.055)	
GLD:CPP	0.403 (0.184)	0.382 (0.121)	0.247 (0.094)	0.221 (0.115)	0.314 (0.144)	0.311 (0.146)	
GLD:WHT	0.185 (0.094)	0.168 (0.097)	0.110 (0.090)	0.079 (0.100)	0.220 (0.119)	0.219 (0.119)	
GLD:CRN	0.257 (0.097)	0.247 (0.066)	0.146 (0.068)	0.094 (0.085)	0.245 (0.100)	0.212 (0.091)	
SLV:CPP	0.436 (0.167)	0.434 (0.108)	0.279 (0.090)	0.254 (0.110)	0.400 (0.129)	0.324 (0.130)	
SLV:WHT	0.179 (0.084)	0.158 (0.095)	0.138 (0.086)	0.090 (0.100)	0.242 (0.125)	0.222 (0.118)	
SLV:CRN	0.241 (0.088)	0.243 (0.075)	0.157 (0.070)	0.096 (0.090)	0.265 (0.106)	0.229 (0.098)	
CPP:WHT	0.167 (0.080)	0.166 (0.094)	0.061 (0.073)	0.082 (0.092)	0.228 (0.118)	0.186 (0.117)	
CPP:CRN	0.205 (0.104)	0.216 (0.106)	0.032 (0.077)	0.072 (0.104)	0.198 (0.139)	0.155 (0.135)	
WHT:CRN	0.564 (0.123)	0.539 (0.089)	0.667 (0.123)	0.506 (0.106)	0.735 (0.083)	0.763 (0.085)	
ν_2		9.557 (0.954)	9.132 (1.153)		8.132 (1.063)	9.527 (0.848)	
θ_C				0.572 (0.295)	0.317 (0.295)	0.309 (0.239)	0.304 (0.579)
θ_F							0.506 (0.647)
w_C				0.027 (0.020)	0.046 (0.038)		
$p_{1\|1}$	0.738 (0.087)	0.825 (0.110)	0.876 (0.082)	0.893 (0.142)	0.900 (0.074)	0.860 (0.074)	0.959 (0.012)
$p_{2\|2}$	0.764 (0.088)	0.785 (0.132)	0.878 (0.072)	0.922 (0.105)	0.867 (0.149)	0.794 (0.159)	0.000 (0.200)
$p_{3\|3}$						0.205 (0.073)	0.176 (0.158)
$\log \mathcal{L}$	2880	2883	2879	2904	2899	2892	2789
AIC	-5644	-5648	-5638	-5686	-5676	-5662	-5512
BIC	-5344	-5343	-5328	-5371	-5361	-5347	-5341

Table 6.10
Multi asset classes portfolio: Regime switching copula parameters with standard errors in parentheses. The copulas are abbreviated: Frank (F), Clayton (C), Gaussian (G), Student-t (T), Gaussian-Clayton mixture (GC) and Student-t-Clayton mixture (TC). w_C denotes the weight of the Clayton copula in the mixture. The forward slash indicates the separate regimes i.e. G/T/C and G/C/F are three-state models. $p_{i|i}$ denotes the probability of staying in regime i.

Copula	G/G	G/T	T/T	T/GC	G/TC	G/T/C	G/C/F
			Regime 1				
SMI:DAX	0.532 (0.146)	0.507 (0.091)	0.835 (0.130)	0.844 (0.050)	0.494 (0.092)	0.699 (0.088)	0.828 (0.015)
SMI:FTS	0.485 (0.156)	0.466 (0.096)	0.831 (0.140)	0.839 (0.056)	0.461 (0.099)	0.615 (0.096)	0.805 (0.018)
SMI:SPX	0.449 (0.141)	0.426 (0.068)	0.697 (0.093)	0.712 (0.046)	0.430 (0.068)	0.547 (0.074)	0.697 (0.020)
SMI:NAR	0.267 (0.187)	0.307 (0.098)	0.523 (0.080)	0.507 (0.075)	0.291 (0.111)	0.307 (0.119)	0.448 (0.035)
SMI:EPR	0.294 (0.166)	0.292 (0.089)	0.612 (0.111)	0.673 (0.066)	0.271 (0.094)	0.397 (0.100)	0.603 (0.026)
SMI:CNE	0.015 (0.104)	0.043 (0.100)	0.240 (0.090)	0.211 (0.086)	0.043 (0.104)	0.040 (0.124)	0.131 (0.037)
SMI:CEN	0.002 (0.117)	-0.010 (0.113)	0.055 (0.080)	0.086 (0.093)	0.000 (0.136)	-0.001 (0.137)	-0.017 (0.050)
DAX:FTS	0.511 (0.164)	0.479 (0.103)	0.843 (0.144)	0.843 (0.055)	0.470 (0.112)	0.668 (0.118)	0.808 (0.019)
DAX:SPX	0.471 (0.155)	0.450 (0.088)	0.792 (0.125)	0.808 (0.055)	0.441 (0.103)	0.576 (0.105)	0.756 (0.022)
DAX:NAR	0.230 (0.216)	0.269 (0.113)	0.576 (0.095)	0.564 (0.079)	0.250 (0.121)	0.283 (0.123)	0.475 (0.033)
DAX:EPR	0.387 (0.147)	0.396 (0.074)	0.658 (0.072)	0.672 (0.051)	0.375 (0.067)	0.511 (0.069)	0.617 (0.026)
DAX:CNE	0.016 (0.120)	0.037 (0.099)	0.315 (0.098)	0.282 (0.092)	0.017 (0.112)	0.037 (0.128)	0.167 (0.038)
DAX:CEN	0.005 (0.114)	-0.024 (0.112)	0.085 (0.090)	0.121 (0.102)	-0.001 (0.138)	0.006 (0.135)	0.025 (0.048)
FTS:SPX	0.510 (0.144)	0.510 (0.086)	0.774 (0.112)	0.799 (0.054)	0.506 (0.084)	0.580 (0.089)	0.738 (0.024)
FTS:NAR	0.263 (0.218)	0.292 (0.109)	0.586 (0.094)	0.568 (0.082)	0.292 (0.120)	0.300 (0.130)	0.468 (0.036)
FTS:EPR	0.488 (0.144)	0.500 (0.064)	0.661 (0.071)	0.705 (0.054)	0.494 (0.073)	0.552 (0.080)	0.650 (0.025)
FTS:CNE	0.016 (0.131)	0.014 (0.099)	0.332 (0.103)	0.303 (0.096)	0.028 (0.119)	0.037 (0.132)	0.179 (0.040)
FTS:CEN	0.084 (0.092)	0.063 (0.091)	0.131 (0.092)	0.192 (0.100)	0.073 (0.116)	0.036 (0.117)	0.086 (0.043)
SPX:NAR	0.449 (0.147)	0.477 (0.100)	0.727 (0.086)	0.742 (0.069)	0.474 (0.098)	0.515 (0.111)	0.672 (0.024)
SPX:EPR	0.348 (0.155)	0.365 (0.077)	0.598 (0.072)	0.605 (0.056)	0.341 (0.082)	0.427 (0.088)	0.546 (0.027)
SPX:CNE	0.022 (0.135)	0.045 (0.105)	0.368 (0.100)	0.332 (0.091)	0.032 (0.106)	0.050 (0.119)	0.200 (0.041)
SPX:CEN	0.022 (0.117)	-0.000 (0.138)	0.108 (0.114)	0.174 (0.111)	-0.022 (0.156)	-0.059 (0.151)	0.034 (0.048)
NAR:EPR	0.287 (0.201)	0.305 (0.181)	0.499 (0.125)	0.481 (0.114)	0.292 (0.192)	0.337 (0.210)	0.435 (0.037)
NAR:CNE	0.012 (0.110)	0.059 (0.115)	0.287 (0.123)	0.248 (0.109)	0.066 (0.121)	0.050 (0.133)	0.156 (0.042)
NAR:CEN	-0.003 (0.115)	-0.040 (0.106)	0.077 (0.097)	0.115 (0.086)	-0.061 (0.104)	-0.064 (0.109)	0.028 (0.052)
EPR:CNE	0.012 (0.116)	0.100 (0.149)	0.235 (0.171)	0.200 (0.137)	0.060 (0.167)	0.044 (0.189)	0.133 (0.041)
EPR:CEN	0.039 (0.098)	0.036 (0.128)	0.064 (0.112)	0.094 (0.101)	-0.001 (0.128)	-0.037 (0.126)	0.012 (0.047)
CNE:CEN	0.107 (0.116)	0.087 (0.098)	0.253 (0.111)	0.354 (0.083)	0.075 (0.112)	0.092 (0.119)	0.263 (0.033)
ν_1			10.481 (1.033)	11.935 (0.859)			

Continued on next page

Table 6.10 continued

				Regime 2			
SMI:DAX	0.857 (0.148)	0.833 (0.050)	0.505 (0.133)	0.607 (0.105)	0.841 (0.079)	0.836 (0.062)	
SMI:FTS	0.857 (0.162)	0.831 (0.052)	0.460 (0.140)	0.527 (0.105)	0.843 (0.087)	0.852 (0.066)	
SMI:SPX	0.740 (0.146)	0.710 (0.047)	0.437 (0.097)	0.493 (0.070)	0.710 (0.070)	0.743 (0.061)	
SMI:NAR	0.561 (0.188)	0.494 (0.059)	0.298 (0.079)	0.276 (0.127)	0.486 (0.107)	0.553 (0.103)	
SMI:EPR	0.703 (0.168)	0.660 (0.055)	0.297 (0.117)	0.328 (0.106)	0.637 (0.097)	0.694 (0.084)	
SMI:CNE	0.265 (0.111)	0.205 (0.068)	0.057 (0.085)	0.018 (0.123)	0.185 (0.128)	0.221 (0.109)	
SMI:CEN	0.086 (0.129)	0.080 (0.078)	-0.002 (0.088)	-0.002 (0.161)	0.054 (0.124)	0.098 (0.107)	
DAX:FTS	0.858 (0.172)	0.836 (0.052)	0.496 (0.150)	0.579 (0.124)	0.846 (0.083)	0.855 (0.067)	
DAX:SPX	0.813 (0.158)	0.789 (0.052)	0.458 (0.133)	0.521 (0.112)	0.788 (0.078)	0.821 (0.062)	
DAX:NAR	0.606 (0.217)	0.534 (0.063)	0.258 (0.098)	0.242 (0.134)	0.534 (0.113)	0.597 (0.105)	
DAX:EPR	0.698 (0.155)	0.654 (0.048)	0.394 (0.081)	0.463 (0.080)	0.639 (0.067)	0.692 (0.063)	
DAX:CNE	0.326 (0.127)	0.269 (0.068)	0.054 (0.090)	0.016 (0.134)	0.263 (0.132)	0.312 (0.120)	
DAX:CEN	0.118 (0.133)	0.116 (0.081)	-0.003 (0.097)	-0.012 (0.155)	0.091 (0.129)	0.141 (0.112)	
FTS:SPX	0.801 (0.156)	0.778 (0.049)	0.534 (0.111)	0.560 (0.090)	0.775 (0.073)	0.812 (0.068)	
FTS:NAR	0.595 (0.221)	0.531 (0.067)	0.302 (0.101)	0.262 (0.144)	0.527 (0.118)	0.593 (0.113)	
FTS:EPR	0.720 (0.145)	0.682 (0.044)	0.512 (0.074)	0.509 (0.086)	0.679 (0.075)	0.717 (0.071)	
FTS:CNE	0.344 (0.130)	0.285 (0.072)	0.046 (0.095)	0.016 (0.128)	0.275 (0.143)	0.326 (0.127)	
FTS:CEN	0.178 (0.110)	0.181 (0.075)	0.106 (0.093)	0.052 (0.130)	0.151 (0.129)	0.216 (0.116)	
SPX:NAR	0.771 (0.152)	0.718 (0.057)	0.494 (0.088)	0.488 (0.120)	0.720 (0.094)	0.767 (0.086)	
SPX:EPR	0.625 (0.163)	0.600 (0.050)	0.379 (0.077)	0.409 (0.096)	0.551 (0.074)	0.623 (0.074)	
SPX:CNE	0.373 (0.135)	0.310 (0.065)	0.060 (0.094)	0.026 (0.124)	0.310 (0.132)	0.349 (0.118)	
SPX:CEN	0.165 (0.133)	0.155 (0.089)	0.056 (0.111)	-0.043 (0.178)	0.131 (0.159)	0.197 (0.126)	
NAR:EPR	0.508 (0.204)	0.478 (0.080)	0.340 (0.121)	0.269 (0.234)	0.423 (0.164)	0.508 (0.160)	
NAR:CNE	0.291 (0.112)	0.229 (0.068)	0.066 (0.123)	0.026 (0.141)	0.218 (0.159)	0.260 (0.140)	
NAR:CEN	0.109 (0.129)	0.101 (0.065)	-0.011 (0.092)	-0.060 (0.135)	0.079 (0.107)	0.132 (0.104)	
EPR:CNE	0.245 (0.122)	0.184 (0.085)	0.067 (0.159)	0.030 (0.196)	0.167 (0.193)	0.227 (0.176)	
EPR:CEN	0.090 (0.106)	0.085 (0.074)	0.053 (0.106)	0.009 (0.152)	0.064 (0.116)	0.118 (0.110)	
CNE:CEN	0.383 (0.114)	0.346 (0.061)	0.131 (0.102)	0.080 (0.145)	0.342 (0.113)	0.365 (0.101)	
ν_2		10.387 (0.856)	10.773 (0.991)		9.232 (0.850)	10.320 (0.986)	
θ_C				0.443 (0.207)	1.516 (0.207)	0.328 (0.247)	1.348 (0.210)
θ_F							0.822 (0.247)
w_C				0.159 (0.080)	0.074 (0.063)		
$p_{1\|1}$	0.871 (0.063)	0.902 (0.106)	0.866 (0.083)	0.939 (0.120)	0.869 (0.111)	0.793 (0.127)	0.873 (0.034)
$p_{2\|2}$	0.890 (0.083)	0.928 (0.069)	0.833 (0.079)	0.928 (0.115)	0.913 (0.117)	0.878 (0.138)	0.000 (0.136)
$p_{3\|3}$						0.515 (0.139)	0.819 (0.204)
$\log \mathcal{L}$	2316	2329	2277	2339	2320	2329	2228
AIC	-4516	-4540	-4434	-4556	-4518	-4536	-4390
BIC	-4220	-4237	-4128	-4244	-4206	-4224	-4221

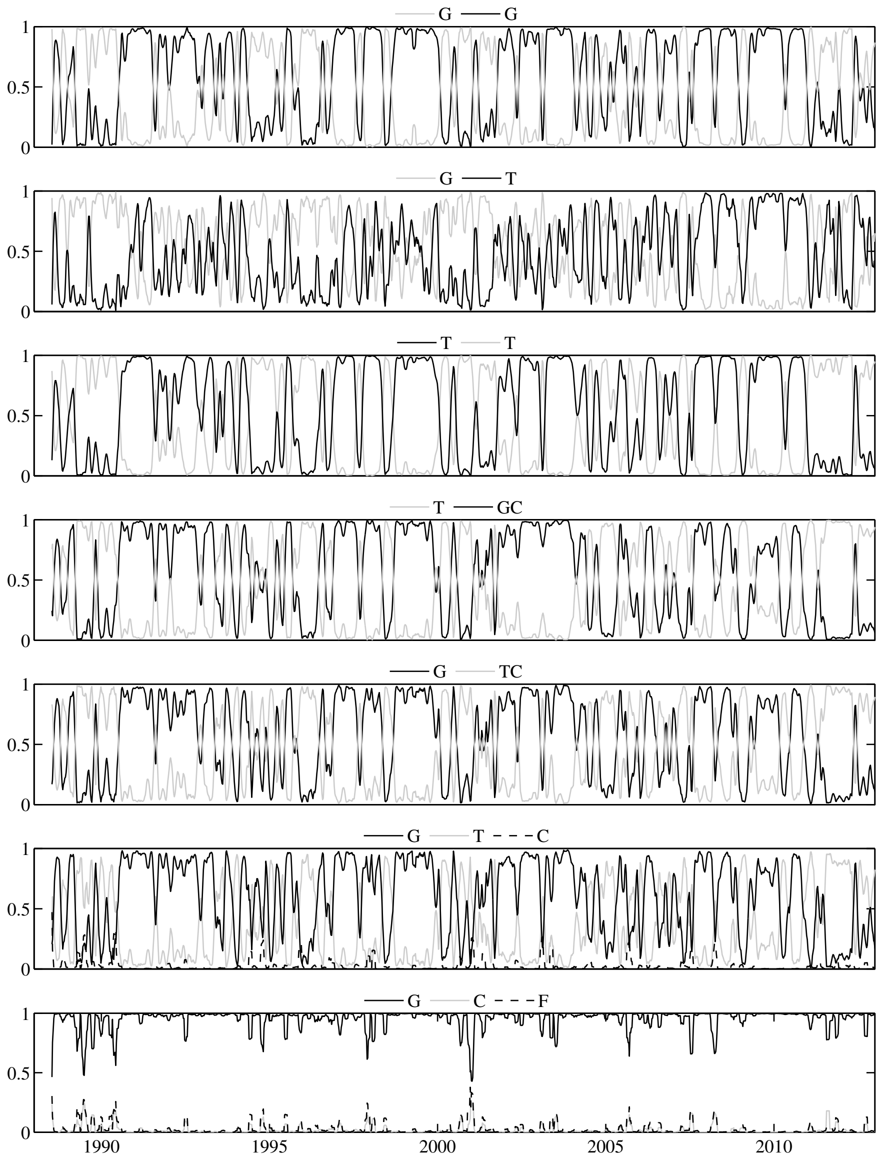

Figure 6.14: Commodity portfolio: Kim filtered regime probabilities of the various regime switching copula models over the entire sample period. The copulas are abbreviated: Gaussian (G), Student-t (T), Frank (F), Clayton (C), Gaussian-Clayton mixture (GC) and Student-t-Clayton mixture (TC). The solid black line marks the high dependence regime.

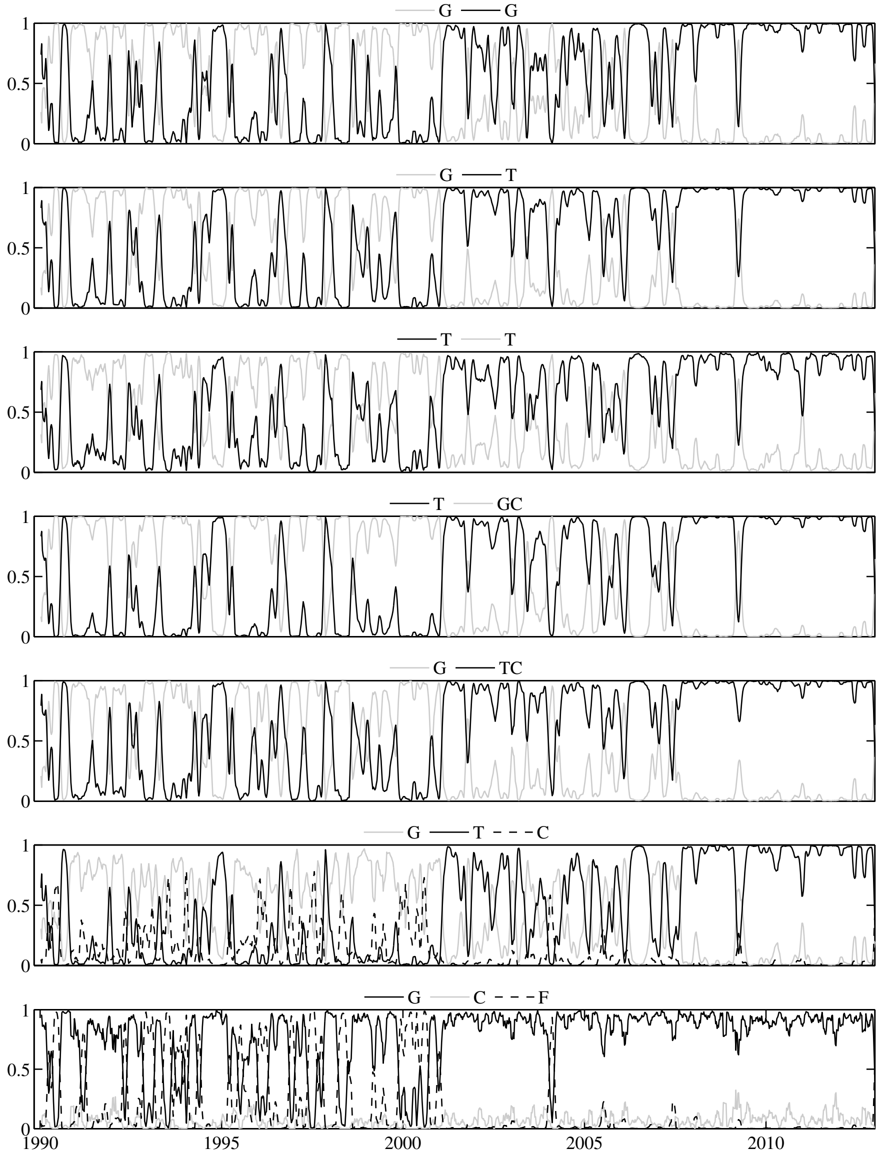

Figure 6.15: Multi Asset Classes Portfolio: Kim filtered regime probabilities of the various regime switching copula models over the entire sample period. The copulas are abbreviated: Gaussian (G), Student-t (T), Frank (F), Clayton (C), Gaussian-Clayton mixture (GC) and Student-t-Clayton mixture (TC). The solid black line marks the high dependence regime.

6.3.2 Dynamic Copulas

The fully dynamic copula models presented in section 3.4 are calibrated to each of the three portfolios based on the same residuals obtained by filtering the entire sample period with the univariate models in 6.1. In contrast to the regime switching models in the previous section, where the copula parameters remain static and time variation in the dependence structure is induced by the latent Markov chain, the parameters of the dynamic copulas are allowed to change in every time step. The considered time-varying models are the dynamic versions of the static copulas applied in section 6.2.2, i.e. dynamic versions of both Archimedean and elliptical copulas and dynamic mixtures of these dynamic copulas. This allows for a direct comparison of a copula versus its dynamic counterpart with regards to in-sample fit and predictive power (see section 7). In this section, the results of the in-sample parameter estimation of the dynamic models are discussed. Table 6.12 lists the results for the equity index data set. Figure 6.16 visualizes the evolution of the dynamic Clayton copula parameter θ_c governed by its parameter estimates listed in table 6.12.

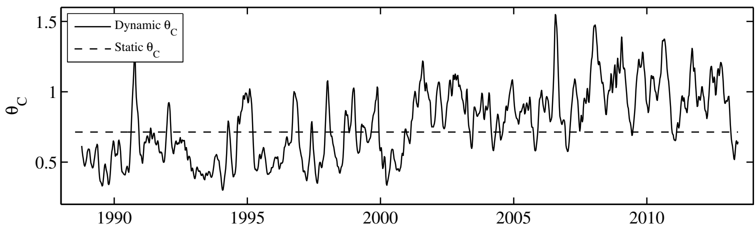

Figure 6.16: Equities: Smoothed evolution of the dynamic Clayton copula parameter (solid line) versus the static Clayton copula parameter (dashed line) over the entire sample period.

The plot illustrates the adaption of the dynamic Clayton copula to the changes of the equity indices' dependence over time, as found in figure 6.11. As a comparison, figure 6.16 further shows the parameter estimate of the static

Clayton (see table 6.4) as an overlaid dashed line. The parameter of the static Clayton can be seen as the average of the dynamic θ_C. The comparison with the static Clayton parameter indicates that without allowing for time variation in the parameter, the static copula overestimates the magnitude of the parameter in most of the first half of the sample period while underestimating it most of the time in the second half. Specifically, this means that during the unfolding of the financial crisis, the parameter of the dynamic Clayton reaches peaks in excess of 1.5 which is more than twice the magnitude of the static Clayton parameter of 0.714. The values of both the AIC and the BIC for the dynamic Clayton copula are substantially lower than for the static version which confirms that allowing for time variation in the Clayton parameter results in an improved fit to the data. The same conclusion holds for all dynamic copulas in table 6.12, whose AIC / BIC values indicate a remarkably better fit, compared to their static versions in table 6.4.

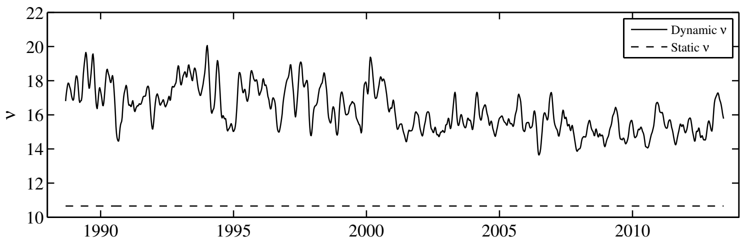

Figure 6.17: Equities: Smoothed evolution of the dynamic Student-t copula's degrees of freedom parameter (solid line) versus the ν parameter of the static Student-t copula (dashed line) over the entire sample period.

Figure 6.17 allows a comparison of the degrees of freedom of the dynamic Student-t copula with the ν parameter of its static counterpart. The evolution of the dynamic degrees of freedom follows a downward trend over the sample period, indicating that tail dependence has increased over time. In contrast to the findings for the Clayton copula parameter θ_C, the static degrees of freedom parameter (dashed line) is nowhere near the average of the dynamic

ν as it remains below the dynamic parameter (solid line) at all times. This is consistent with Dias and Embrechts (2010), who find on the bivariate level that the degrees of freedom of the dynamic Student-t copulas is always larger than for the static Student-t copula. One may conclude that ignoring time variation in the copula parameters might induce spurious heavier conditional tails.

Within the dynamic mixture copulas, time variation in the dependence structure can be captured in the evolution of each of the two copula models as well as with the process for the dynamic mixture weight. Figure 6.18 depicts the evolution of the three dependence parameters of the dynamic Frank-Clayton mixture copula, induced by the parameter estimates in 6.12, over the entire sample period. While the Frank copula parameter shows a rather steady evolution capturing the overall increase in the level of dependence over time, θ_C also captures the variations in lower tail dependence. Clayton's mixture weight largely remains at the level of the static mixture weight of 0.587 (see table 6.4) showing some short-lived spikes, whose magnitude is larger for the second half of the observation period. Based on the observation that the dynamic mixture weight mostly remains close to the static one and further considering the large standard errors for the parameters of the dynamic mixture weight process, one may conclude that combining a dynamic Frank and a dynamic Clayton copula with a static weight would even be sufficient to capture the equity index returns' dependence structure over time. According to the information criteria, the most suitable dynamic dependence model for the equity data is the dynamic Student-t-Clayton mixture copula, followed by the dynamic Student-t copula and the dynamic Gaussian-Clayton mixture.

The dynamic copula models' estimation results for the commodity futures index returns are listed in table 6.13. The comparison of the Akaike and Bayesian information criteria values with the ones of their counterparts in table 6.5 reveals that the dynamic version of each of the copulas has a better fit to the commodity future portfolio than the version with constant param-

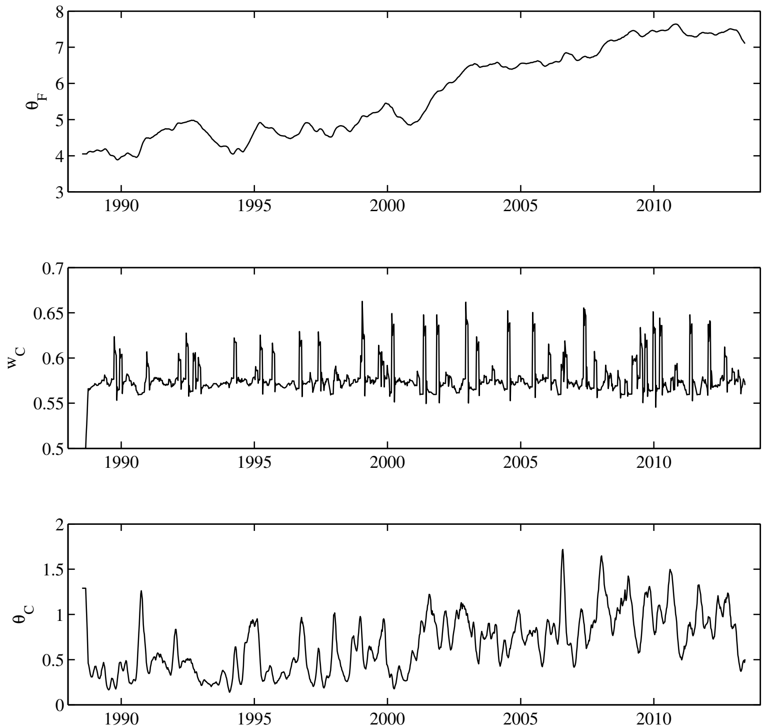

Figure 6.18: Equities: Smoothed evolution of the dynamic Frank-Clayton mixture copula's (DFC) parameters over the entire sample period. The top figure depicts the evolution of the mixture's dynamic Frank parameter θ_F. In the center, the variations in the mixture weight of the dynamic Clayton copula over time are depicted and the bottom graph shows the evolution of the mixture's dynamic Clayton copula parameter θ_C.

eters. As with the static versions, the purely Archimedean copulas attain significantly lower likelihood values compared to the models with elliptical copulas. This results in higher AIC and BIC values indicating an inferior fit. Furthermore, the standard errors of the Archimedean copulas in the dynamic dependence models are larger for the commodity futures data than for the equity portfolio pointing towards difficulties of the Archimedean copulas to capture the dependence evolution of the commodity futures index portfolio.

The lower standard errors of the elliptical copulas and their lower AIC / BIC values indicate that the elliptical copulas are better models for the commodities futures index portfolio. The size of the standard errors of the dynamic mixture weight parameters shows that one cannot say with certainty that the parameters are different from zero. Mixing the dynamic copulas with a static weight may hence be sufficient to capture the dependence dynamics.

The dynamic Student-t copula is identified as the best fitting dynamic model for the commodity data by both information criteria, which further rank the dynamic Student-t-Clayton mixture copula in second and the dynamic version of the Gaussian copula in third place. The parameters for the dynamic Student-t copula's degrees of freedom induce an evolution ranging between $\nu = 16.493$ and $\nu = 21.870$ with a mean of 18.703. The static Student-t copula's ν is with 16.804 (see table 6.5) below the average dynamic ν providing support for the conclusion that ignoring time variation in the copula parameters might induce spuriously increased conditional tail dependence.

Table 6.14 lists the parameter estimates of the dynamic copulas for the multi asset classes portfolio. As with the equity and the commodity data set, all dynamic models attain higher likelihood and lower information criteria values than the static versions (see table 6.6), which indicates the dynamic models' superior fit. While the dynamic models including an elliptical copula attain information criteria values below -4500, the according values of the dynamic Archimedean copulas are on average less than half that size. This shows that also for the multi asset classes data set, dependence models comprising elliptical copulas yield a better fit to the data. The model with the best fit is the dynamic Student-t-Clayton mixture. In addition to the adaptability of the dynamic Student-t copula, this mixture can further adjust to the structure of dependence and its changes over time with the mixture weight process and the parameters of the dynamic Clayton copula. The versatility of this model is depicted in figure 6.19, which shows the evolution of the dynamic Student-t-Clayton mixture copula's parameters induced by the estimates in table 6.14 over the entire sample period. The top plot visualizes the progression of the average Student-t copulas correlation with a clearly recognizable increase after 2008. The second graph shows the Student-t copula's degrees of freedom, which reach the lowest values during the propagation of the financial crisis after the year 2008. In addition to the increase in the level of dependence this copula hence also identifies an augmentation of tail dependence after the outbreak of the crisis. While the lower ν parameter value signifies an increase in both lower and upper tail dependence, the enormous spikes and overall rise in the θ_C parameter of the Clayton fraction of the mixture (depicted in the bottom graph of figure 6.19) starting with the outbreak of the financial crisis specifically imply a heavier lower tail. Note that the peaks of Clayton's θ_C after 2005 coincide with the plunges of the Student-t's ν. The increase in lower tail dependence is thus captured by both copulas in the mixture. The weight of the Clayton copula inside the mixture structure remains almost constant with only a few and short-term departures from the long-run average, which is depicted in the third graph (from the top) of figure 6.19. Combined with the large standard errors of the dynamic weight process, one may conclude that a static mixture weight would be sufficient. The second and third lowest AIC and BIC values and therewith best fitting models after the dynamic Student-t-Clayton mixture copula are the Dynamic Student-t that ranks second and the dynamic Gaussian-Clayton mixture copula that ranks third.

Table 6.11 shows the overall ranking of the in-sample model fit among the static, regime-switching and dynamic copulas according to the two information criteria. For the equity index portfolio, the AIC favors the regime switching models while the BIC indicates that the fully dynamic copulas are more adequate. Compared to the dynamic copulas, the regime switching models contain on average more parameters which entails a larger penalty for model complexity in the BIC than in the AIC. For the commodity and the multi asset classes data sets, both criteria agree on the first four ranks. The rankings indicate the superiority of the time-varying copulas' in-sample fit compared to the static versions for all three portfolios. The only static copula to attain a top five ranking is the static Student-t-Clayton mixture, which ranks fifth according to the BIC for the commodity portfolio. The dynamic Student-t and the dynamic Student-t-Clayton mixture stand out as they dominate the top two ranks for the equity portfolio (according to the BIC) and for the commodity and the multi asset classes portfolio (according to both AIC and BIC). The best-ranked copulas indicate the importance of accounting for time variation, highlight that positive tail dependence is a crucial feature of a well-fitting model for each of the three portfolios and that the capability of capturing asymmetries in the dependence structure yields the winning edge for the equity and the multi asset classes portfolio.

While this chapter concentrated on the in-sample fit of the models, the focus is now shifted to their out-of-sample forecast performance, which is the subject of the following chapter.

Table 6.11
In-sample model fit ranking

	1.	2.	3.	4.	5.
AIC					
Equity Indices	T/GC	G/G	G/T	DTC	G/T/C
Commodity Futures Indices	DT	DTC	DG	DGC	T/GC
Multi Asset Classes	DTC	DT	DGC	T/GC	G/T
BIC					
Equity Indices	DTC	DT	DGC	DG	T/GC
Commodity Futures Indices	DT	DTC	DG	DGC	TC
Multi Asset Classes	DTC	DT	DGC	T/GC	G/T

This table presents the top in-sample model fit rankings for the static, regime-switching and fully dynamic copulas for the three portfolios under consideration according to the Akaike (AIC) and the Bayesian (BIC) information criteria. The copulas are abbreviated: Gaussian (G), Student-t (T), Clayton (C), Gaussian-Clayton mixture (GC) and Student-t-Clayton mixture (TC). The forward slash indicates the separation of the regimes in the Markov switching models while the prefixed D denotes fully dynamic copula models.

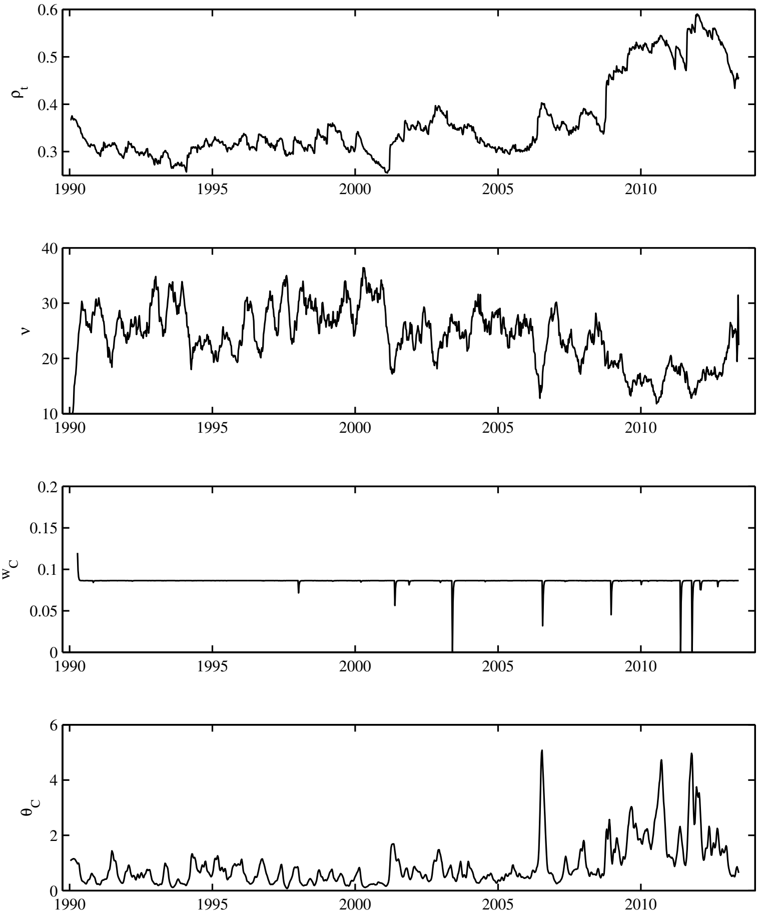

Figure 6.19: This figure visualizes the smoothed evolution of the dynamic Student-t-Clayton mixture copula (DTC) parameters for the Multi Asset Classes data: The first graph (on top) depicts the average Student-t copula correlation over time. The second plot shows the evolution of the Student-t copula's degrees of freedom. The weight of the dynamic Clayton copula in the mixture structure over time is shown in the third graph and the bottom plot depicts the θ_C of the dynamic Clayton copula over time.

Table 6.12
Equity portfolio: Dynamic copula parameters.

Copula	DC	DF	DG	DT	DFC	DGC	DTC
α			0.012	0.011		0.013	0.013
			(0.006)	(0.006)		(0.011)	(0.010)
β			0.986	0.986		0.985	0.985
			(0.129)	(0.146)		(0.195)	(0.175)
α_C	-1.199				-2.650	-4.418	-3.608
	(0.351)				(0.682)	(3.131)	(1.587)
β_C	0.917				-1.035	-0.745	-0.969
	(0.436)				(2.468)	(4.170)	(1.007)
ω_C	0.031				1.867	4.492	3.896
	(0.014)				(1.223)	(4.121)	(1.935)
α_F		-0.472			-0.074		
		(0.138)			(0.552)		
β_F		0.908			0.990		
		(0.069)			(0.284)		
ω_F		0.837			0.140		
		(0.328)			(0.513)		
α_W					-0.932	1.680	0.012
					(21.275)	(0.924)	(5.588)
β_W					-1.098	-0.249	-6.460
					(2.377)	(7.370)	(7.217)
ω_W					0.243	2.008	7.780
					(1.038)	(6.942)	(6.696)
ς				-1.435			-2.270
				(0.241)			(0.273)
φ				0.526			0.568
				(0.219)			(0.258)
$\log \mathcal{L}$	2427	2369	3476	3546	2705	3540	3581
AIC	-4848	-4732	-6948	-7084	-5392	-7064	-7142
BIC	-4833	-4717	-6938	-7063	-5346	-7023	-7090

Standard errors are listed in parentheses. The prefixed D stands for 'dynamic' and the copula models are abbreviated as follows: F (Frank), C (Clayton), G (Gaussian), T (Student-t), FC (Frank & Clayton mixture), GC (Gaussian & Clayton mixture) and TC (Student-t & Clayton mixture). The subscript W indicates the parameters of the dynamic mixture weight process. ς and φ are the parameters of the dynamic Student-t copula's degrees of freedom process.

Table 6.13
Commodity portfolio: Dynamic copula parameters.

Copula	DC	DF	DG	DT	DFC	DGC	DTC
α			0.019	0.019		0.019	0.020
			(0.005)	(0.005)		(0.006)	(0.005)
β			0.948	0.951		0.947	0.951
			(0.051)	(0.045)		(0.056)	(0.048)
α_C	-0.542				-0.659	-2.221	-2.032
	(0.295)				(0.661)	(3.089)	(3.364)
β_C	-0.385				-8.581	2.691	1.737
	(1.073)				(4.964)	(3.830)	(3.859)
ω_C	-0.987				-0.411	2.212	0.519
	(0.516)				(1.103)	(4.643)	(4.618)
α_F		-0.078			-0.054		
		(0.862)			(1.987)		
β_F		0.981			0.850		
		(0.848)			(0.530)		
ω_F		0.143			0.965		
		(0.273)			(0.227)		
α_W					-2.746	6.081	0.321
					(22.250)	(7.048)	(30.436)
β_W					-1.024	0.415	-3.036
					(1.678)	(14.788)	(31.267)
ω_W					-0.891	4.464	7.926
					(1.497)	(14.443)	(28.447)
ς				-0.862			-1.748
				(0.432)			(0.412)
φ				0.162			0.153
				(0.278)			(0.264)
$\log \mathcal{L}$	737	686	2856	2910	889	2853	2907
AIC	-1468	-1365	-5708	-5811	-1759	-5689	-5795
BIC	-1452	-1350	-5698	-5791	-1713	-5648	-5743

Standard errors are listed in parentheses. The prefixed D stands for 'dynamic' and the copula models are abbreviated as follows: F (Frank), C (Clayton), G (Gaussian), T (Student-t), FC (Frank& Clayton mixture), GC (Gaussian & Clayton mixture) and TC (Student-t & Clayton mixture). The subscript W indicates the parameters of the dynamic mixture weight process. ς and φ are the parameters of the dynamic Student-t copula's degrees of freedom process.

Table 6.14
Multi asset classes portfolio: Dynamic copula parameters.

Copula	DC	DF	DG	DT	DFC	DGC	DTC
α			0.011	0.012		0.015	0.014
			(0.006)	(0.006)		(0.007)	(0.007)
β			0.983	0.980		0.979	0.977
			(0.154)	(0.158)		(0.160)	(0.169)
α_C	-1.031				-1.377	-5.296	-4.704
	(0.383)				(0.737)	(2.234)	(2.001)
β_C	2.195				2.427	-2.095	-1.787
	(2.241)				(3.949)	(3.683)	(3.043)
ω_C	-0.623				-0.413	6.200	5.496
	(0.498)				(1.205)	(3.327)	(2.888)
α_F		-0.138			-0.078		
		(0.402)			(1.883)		
β_F		0.971			0.982		
		(0.320)			(0.431)		
ω_F		0.241			0.193		
		(0.357)			(0.358)		
α_W					-40.007	0.014	0.004
					(13.882)	(5.620)	(7.538)
β_W					-1.129	4.208	4.472
					(1.372)	(5.966)	(6.961)
ω_W					1.242	-1.575	-1.727
					(0.778)	(5.463)	(6.456)
ς				-2.673			-3.585
				(0.419)			(0.436)
φ				1.549			1.691
				(0.335)			(0.351)
$\log \mathcal{L}$	1113	970	2254	2311	1244	2300	2337
AIC	-2220	-1935	-4504	-4615	-2471	-4584	-4653
BIC	-2205	-1919	-4494	-4594	-2425	-4543	-4602

Standard errors are listed in parentheses. The prefixed D stands for 'dynamic' and the copula models are abbreviated as follows: F (Frank), C (Clayton), G (Gaussian), T (Student-t), FC (Frank & Clayton mixture), GC (Gaussian & Clayton mixture) and TC (Student-t & Clayton mixture). The subscript W indicates the parameters of the dynamic mixture weight process. ς and φ are the parameters of the dynamic Student-t copula's degrees of freedom process.

Chapter 7

Forecast Evaluation

To test the predictive power of the different copula models, this chapter presents the methodology and results of out-of-sample backtests. To this end, the univariate models are linked by means of the different copula dependence structures in order to perform Monte Carlo simulations to obtain forecasted profit and loss distributions. Since the same univariate models are coupled with different copulas, differences in the return distribution forecasts are attributable to the copula functions only. The most accurate forecasts are thus produced by the model whose copula function is best capable of describing the multivariate dependence structure. The models' performance is evaluated in a comprehensive backtesting scheme by comparing their forecasts with the observed historical portfolio returns.

The remainder of this chapter is organized as follows: firstly, the backtesting procedure and the different approaches to evaluate the predictive performance of the copula models are presented. Secondly, the results of the backtest evaluations over the entire out-of-sample period are discussed before, finally, the focus is narrowed to the performance of the models during the last financial crisis.

7.1 Backtesting Procedure

The backtesting procedure is based on a rolling window scheme with 520 returns.[1] Specifically, to compute a one-week forecast of the portfolios' profit and loss distribution in time t, the univariate models and the copula functions are calibrated to the information from $t-520$ until $t-1$ with the multi-stage maximum likelihood estimation outlined in section 4.1. Thereafter, dependent uniform variates with the specified copula are simulated with the algorithms described in chapter 3 and subsequently transformed by inversion of the according marginal cumulative distribution function to obtain standardized residuals. These standardized residuals are then employed as the independent and identically distributed noise processes of the respective GARCH models outlined in chapter 2, which reestablish the heteroscedasticity and the autocorrelation of the original returns. Using 10,000 accordingly simulated weekly returns for each portfolio constituent, the forecasted profit and loss distribution of the equally weighted portfolio in time t is computed and compared with the historical portfolio return in time t. The portfolio with equal weights ensures that the impact of each univariate model is limited to the same extent. Taking advantage of the entire forecasted portfolio return distributions, both risk measure forecasts and density forecasts are evaluated.

The univariate models employed in the backtesting procedure differ from the ones outlined in chapter 6. While the latter were found using the entire data set, the univariate models for the backtesting procedure are determined by choosing those specifications which reveal the lowest AIC / BIC value for the first 520 returns in the respective data set. Table 7.1 shows the resulting AIC / BIC optimal models employed in the backtesting process. For six of the eight equity indices, EGARCH is still the optimal model. While for five of them, the EGARCH (1,1) is most suitable, for the Hang Seng index, the

[1] Note that the rolling window scheme (as opposed to the expanding window method) provides some "shield" for the static copula models against changing market conditions.

Table 7.1
Univariate model specifications in the backtesting procedure

Equities Model	SMI E	DAX E	CAC E	UKX G	SPX E	TSX GJR	HS E	Nik E
AR	0	0	0	0	1	0	0	0
MA	0	0	0	0	0	0	0	0
P	1	1	1	0	1	1	1	1
Q	1	1	1	1	1	1	3	1
Commodities Model	OIL E	HOL GJR	GAS GJR	GLD E	SLV GJR	CPP GJR	WHT GJR	CRN E
AR	0	0	0	0	0	0	0	0
MA	0	0	0	0	0	0	0	0
P	1	1	1	1	1	1	1	1
Q	1	1	1	1	1	1	1	1
Multi AC Model	SMI E	DAX E	UKX G	SPX E	NAR E	EPR E	CNE E	CEN E
AR	0	0	0	1	0	1	0	0
MA	0	0	0	0	0	1	0	0
P	1	1	1	1	1	1	1	1
Q	1	1	0	1	2	1	1	1

Univariate model specifications used in the backtesting procedure. The models are the AIC / BIC optimal models for the training period consisting of the first 520 returns. AR and MA are the number of autoregressive respectively moving average parameters in the mean equation. P and Q the number of parameters of the conditional volatility models, which are abbreviated with G for GARCH, GJR for the GJR-GARCH and E for the EGARCH model. Multi AC is the multi asset class portfolio.

EGARCH (1,3) has the lowest BIC value. For the UKX, the optimum turns out to be an ARCH process (i.e. GARCH (1,0)). For the TSX, the Bayesian information criterion selects the GJR specification to be optimal.

For the commodity data set, the GJR specification replaces the EGARCH as optimal specification for the heating oil, unleaded gasoline, silver, copper and wheat series. The information criteria select a lag of one to be optimal for all univariate commodity series. The multi asset classes data reaches back to January 3, 1990, which is 1.5 years less than the pure equity portfolio. Still, the AIC / BIC optimal univariate models for the equity indices used in the backtests of the multi asset classes portfolio are identical to the ones used for backtesting the pure equity portfolio. For the non energy and the energy index, the backtesting models turn out to be equal to the ones in the in-sample analysis (see section 6.1). While the AIC / BIC optimal volatility model for the REIT indices based on the first training period still is the EGARCH, the information criteria indicate a superiority of the EGARCH(1,2) specification for the NAREIT. For the EPRA index, the ARMA(1,1)-EGARCH(1,1) is the optimal model based on the first 520 returns.

7.1.1 Backtesting Risk Measures

The most common reported risk measure is the Value-at-Risk (VaR) as it represents the industry and regulatory standard for the calculation of risk capital in banking and insurance (Embrechts et al., 2013). The VaR for a given confidence level α is defined as (Christoffersen, 2012)

$$Pr(-y_t > VaR_t(\alpha)) = 1 - \alpha \quad \Leftrightarrow \quad Pr(y_t < -VaR_t(\alpha)) = 1 - \alpha. \tag{7.1}$$

Hence, VaR(α) is the negative return that will not be exceeded with probability α. Formally, the VaR(α) corresponds to the $(1-\alpha)$ percentile of the portfolio return distribution. The most widely used alternative to VaR is expected shortfall (ES), also known as Conditional VaR (CVaR), which is defined as

$$ES(\alpha) = E[y_t \mid y_t < -VaR_t(\alpha)] \tag{7.2}$$

and possesses some advantages over VaR as it is a coherent risk measure representing the expected return conditional on VaR being violated.[2] Despite theoretical advantages, the vast majority of financial institutions uses VaR and not ES. This may be due to problems with the backtesting procedure of ES, as it requires estimates of the tail expectation to be compared with the forecast of ES. Daníelsson (2011) points out that in a backtest, forecasted ES

[2] See Artzner et al. (1999) or McNeil et al. (2005) for a detailed discussion on the property of coherence.

can only be compared to a model output while VaR can be compared with real observations. The resulting complete profit and loss functions allow to calculate VaR forecasts at arbitrary significance levels α. The most common confidence levels 90%, 95% and 99% are deployed in the thesis at hand. If the actual return of the portfolio over the forecast period falls short of the forecast, then the VaR limit is said to have been violated. In a backtesting procedure, violations over time are a sequence of zeros and ones, also called "hit sequence", denoted η_t. Formally, a violation of the VaR(α) forecast is an event such that

$$\eta_{t,\alpha} = \begin{cases} 1, & \text{if } y_t \leq -VaR_t(\alpha) \\ 0, & \text{if } y_t > -VaR_t(\alpha), \end{cases} \tag{7.3}$$

where y_t is the portfolio return in time t. A judgment on the quality of the models forecast can then be made by calculating the hit ratio, which reflects the percentage of times when the portfolio return exceeds the forecasted VaR(α) in a sample with size T:

$$\text{Hit Ratio}(\alpha) = \frac{\sum_{t=1}^{T} \eta_{t,\alpha}}{T}. \tag{7.4}$$

Backtesting for example the VaR($\alpha = 99\%$) means that one expects to observe a VaR violation in 1% of the time. Naturally, the closer the hit ratio is to the expected value $(1 - \alpha)$, the better the forecasts of the risk model. If the hit ratio is greater than the expectation, then the model underforecasts the portfolio risk; with a hit ratio smaller than $(1 - \alpha)$, the model overforecasts risk.

To formally test whether the difference between observed and expected hit ratio is significant, statistical tests are called for. According to Christoffersen (1998), the measurement of the accuracy of a VaR forecast model can be reduced to determining whether the hit sequence η_α satisfies two properties: the unconditional coverage property and the independence property.

Unconditional Coverage Test

The unconditional coverage test was introduced by Kupiec (1995). The property of unconditional coverage ensures that the empirical hit ratio corresponds with the theoretical significance level $p = (1 - \alpha)$. If a violation on day t occurred, then variable η_t takes the value 1 and 0 otherwise (see (7.3)). η_t is thus a sequence of Bernoulli-distributed random variables. The unconditional coverage test is used to determine the violations' proportion. For VaR violations, the null hypothesis is

$$H_0 : \eta \overset{i.i.d.}{\sim} B(p), \tag{7.5}$$

where B stands for the Bernoulli distribution with density

$$f(\eta_t; p) = (1 - p)^{1-\eta_t}(p)^{\eta_t}, \quad \eta_t = 0, 1. \tag{7.6}$$

To test whether a hit ratio π obtained by a risk model is equal to the expected fraction p, the likelihood of an i.i.d. Bernoulli(π) hit sequence is used:

$$\mathcal{L}(\pi) = \prod_{t=1}^{T}(1 - \pi)^{1-\eta_{t+1}}\pi^{\eta_{t+1}} = (1 - \pi)^{T_0}\pi^{T_1}, \tag{7.7}$$

where $T_1 = \sum_{t=1}^{T} \eta_t$ and $T_0 = T - T_1$ are the number of zeros and ones in the hit sequence η and $\pi = \frac{T_1}{T}$. The test statistic for the null hypothesis $H_0 : \pi = p$ is given by the likelihood ratio

$$LR_{uc} = -2 \log \frac{(1 - p)^{T_0} p^{T_1}}{(1 - \pi)^{T_0} \pi^{T1}}, \tag{7.8}$$

which is asymptotically χ^2 distributed with one degree of freedom. Since the unconditional coverage test does not assume a distribution for the returns, it is an nonparametric test which commonly provides good benchmarks for the assessment of the accuracy of VaR models (Daníelsson, 2011).

Independence Test

Theoretically, violations should spread out over time such that an adequate risk model would not yield VaR violation clusters. Based on this idea, Christoffersen (1998) generalized the approach of Kupiec (1995) to include a test of independence. To fulfill the property of independence, any two observations in the hit sequence must be independent of one another. The fact that a violation has been observed in time t should thus not yield any information about the likelihood of observing a violation in $t+1$. In case previous VaR violations foretell a future VaR violation, this indicates a general inadequacy of the reported risk measure. In order to establish a test of the independence of the VaR violations, Christoffersen (1998, 2012) assumes that the hit sequence is dependent over time and that it can be described by a first-order Markov chain with the transition probability matrix analogous to the one defined in (3.23) with 2 states. The probability of a violation in $t+1$, given a violation in t is defined by $p_{1|1} = Pr(\eta_{t+1} = 1|\eta_t = 1)$, while the probability of a violation in $t+1$, given there is no violation in t is $p_{0|1} = Pr(\eta_{t+1} = 1|\eta_t = 0)$. The probability of no violation in $t+1$ following no violation in t is $1 - p_{0|1}$. With a sample of T observations, the likelihood function of the first-order Markov process is given by

$$\mathcal{L} = (1 - \pi_{01})^{T_{00}} \pi_{01}^{T_{01}} (1 - \pi_{11})^{T_{10}} \pi_{11}^{T_{11}}, \tag{7.9}$$

where $T_{ij},\ i, j = 0, 1$ equals the number of observations with a j following an i. By taking the first derivatives with respect to π_{01} and π_{11} and setting them equal to zero, one may obtain the maximum likelihood estimates $\hat{\pi}_{01} = T_{01}/(T_{00} + T_{01})$ and $\hat{\pi}_{11} = T_{11}/(T_{10} + T_{11})$. The matrix of the estimated transition probabilities is therewith:

$$\hat{\Pi}_1 = \begin{bmatrix} 1 - \hat{\pi}_{01} & \hat{\pi}_{01} \\ 1 - \hat{\pi}_{11} & \hat{\pi}_{11} \end{bmatrix} = \begin{bmatrix} \frac{T_{00}}{(T_{00}+T_{01})} & \frac{T_{01}}{(T_{00}+T_{01})} \\ \frac{T_{10}}{(T_{10}+T_{11})} & \frac{T_{11}}{(T_{10}+T_{11})} \end{bmatrix}. \tag{7.10}$$

If the violations are independent over time, then $\hat{\pi}_{01} = \hat{\pi}_{11} = \hat{\pi}$ such that the transition probability matrix is

$$\hat{\Pi}_0 = \begin{bmatrix} 1-\hat{\pi} & \hat{\pi} \\ 1-\hat{\pi} & \hat{\pi} \end{bmatrix}. \tag{7.11}$$

To test the null hypothesis of independence of the violations $H_0 : \hat{\pi}_{01} = \hat{\pi}_{11}$ the likelihood ratio test is used:

$$LR_{ind} = -2 \log \frac{\mathcal{L}(\hat{\Pi}_0)}{\mathcal{L}(\hat{\Pi}_1)}. \tag{7.12}$$

The statistic is asymptotically χ^2 distributed with one degree of freedom (Christoffersen, 2012).

Joint Test

An accurate VaR forecast has to feature both the independence and unconditional coverage property. A test that jointly examines the unconditional coverage and independence properties thus gives an opportunity to identify VaR measures which are defective in one way or the other. One may conclude, that the joint test should generally be preferred to the individual tests of coverage or independence. However, this is usually not the case since the joint test entails less power to reject a model which only satisfies either one of the properties (Daníelsson, 2011). To jointly test if the observed violations are significantly different from the expected ones and if the violations are independent over time, the joint test also known as conditional coverage test is applied. The test statistic of the joint test is the sum of the test statistics for the individual properties

$$LR_{joint} = LR_{unc} + LR_{ind}, \tag{7.13}$$

which is asymptotically χ^2 distributed with two degrees of freedom.

Basel Three-Zone Approach

The Basel regulatory framework provides a categorization of risk models based on their performance in the backtesting procedure. Accordingly, a model is categorized into one of the three traffic light zones "green", "yellow" and "red" based on the probability of obtaining up to the observed number of x exceedances of the 1% VaR when the true coverage level is 99%. Formally, this corresponds to the binomial cumulative distribution

$$Pr(X \leq x; T, p) = \sum_{i=0}^{x} \binom{T}{i} p^i (1-p)^{T-i}, \tag{7.14}$$

where x is the number of observed exceedances of the model's 1% VaR in the backtesting period, $p = 1\%$ and T equals the size of the backtesting sample. Models, whose backtesting performance is such that $Pr(X \leq x) < 95\%$ are categorized as green, which means that the backtesting results do not suggest a problem with the quality or accuracy of the model. Models with x exceedances such that $95\% \leq Pr(X \leq x) \leq 99.99\%$ are categorized as yellow, which is to be interpreted as a model that raises questions but no definitive conclusion is possible. Finally, a model with a backtesting performance such that the probability of obtaining up to the backtested number of x exceptions equals or exceeds 99.99%, i.e. $Pr(X \leq x) > 99.99\%$ is classified as red, which indicates that there is almost certainly a problem with the risk model (Basel Committee on Banking Supervision, 2013).

Expected Shortfall Evaluation

A backtest of the expected shortfall is a rather intricate task, because the aim is to test an expectation rather than a single quantile. As the expected shortfall is the expected loss, given a violation of VaR, Daníelsson (2011) proposes to compare for all points in time t when the VaR is violated, to which extent on average the ES corresponds to the realized return. This indicates how much of the realized return, given a VaR violation, is on average forecasted by the model:

$$\text{ES Ratio}(\alpha) = T_1^{-1} \left(\sum_{t=1}^{T} \frac{\eta_{t,\alpha} \times ES_{t,\alpha}}{y_t} \right), \tag{7.15}$$

where T_1 is the number of ones in the hit sequence η. Naturally, the closer the ES ratio is to one, the more accurate the forecasts. In case the ES ratio is larger than one, the model overforecasts the expected shortfall and if the ES ratio is smaller than one, the model underestimates the loss given a violation of the VaR. Since ES ratio takes the average over all those times when the VaR forecast of a model is violated, this measure hinges on the model's hit ratio. While it gives a further insight into the predictive power of a single model, it only allows for a direct comparison of competing forecast models if their hit ratios are identical.

Ziegel (2014) points out that other procedures for evaluating ES forecasts such as the one proposed by McNeil and Frey (2000) do not permit a direct comparison of the predictive performance of the competing forecasting methods either. Gneiting (2011) introduces the notion of elicitability, where elicitable broadly means that a measure is "properly" backtestable. He shows that in general, VaR is elicitable whereas ES is not. Specifically, Gneiting (2011) proves that the existence of convex level sets is a necessary condition for the elicitability of a risk measure and disproves the existence of convex level sets for the ES. This provides a potential explanation for the lack of literature on backtesting ES. As a simple option, the Basel Committee on Banking Supervision (2011) suggests to backtest the VaR instead of the ES - if VaR is inaccurate, the corresponding ES can hardly be correct.

Gneiting and Ranjan (2011), however, propose to compare density forecasts with emphasis on different regions of interest, such as the tails of the distributions. Given that with the ES the goal is to accurately predict a functional focused on the tails, this approach seems promising (Ziegel, 2014).

Testing density forecasts means that the object of interest is shifted from a particular risk measure to the entire profit and loss distribution, respectively its entire tail.

7.1.2 Backtesting the Entire Distribution

Instead of focusing on particular risk measures from the profit and loss distribution such as the Value-at-Risk or the Expected Shortfall, one might instead decide to backtest the entire forecasted profit and loss distribution from the risk model. This has the benefit of potentially further increasing the power to reject bad risk models. The idea of testing the entire predictive distribution goes back to Berkowitz (2001) and has the appeal that it makes use of the entire predictive distribution, compared to the approaches in section 7.1.1, which effectively throw away information. The attractiveness of the approach of Berkowitz is also based on the use of the probability integral transformation, which allows conducting much more powerful tests than otherwise possible. In time t the portfolio risk forecast models produce a cumulative profit and loss distribution forecast $F_t(\cdot)$ for $t+1$. At the end of $t+1$, with the realized portfolio return y_{t+1}, the risk model's probability of observing a return below the actual one can be computed with

$$\hat{p}_{t+1} = F_t(y_{t+1}). \tag{7.16}$$

Under the null hypothesis that the forecast model is accurate, the time series of observed probabilities $\hat{p}_{t+1}$ should then be independently and identically uniform (0,1) distributed:

$$H_0 : \hat{p}_{t+1} \overset{i.i.d.}{\sim} U(0,1). \tag{7.17}$$

A visual evaluation of the distribution can be made by constructing a histogram of $\hat{p}_{t+1}$ to determine if the distribution is reasonably flat (Christoffersen, 2012). While visual evaluations are less precise in comparison to statistical tests, they are constructive in the sense that they can provide guidance as to why and where a statistical test is rejected. The model adequacy can be statistically examined by testing whether $\hat{p}_{t+1}$ is uniform as predicted. This uniformity is tested by conducting Pearson's χ^2-test, which compares the observed with the expected number of values in different sub-intervals of the unit interval. The test statistic for k sub-intervals is given by

$$Q = \sum_{i=1}^{k} \frac{(N_{(l_i,u_i)} - N(u_i - l_i))^2}{N(u_i - l_i)}, \tag{7.18}$$

where $N_{(l_i,u_i)}$ refers to the number of observations in the i^{th} sub-interval and N refers to the total number of observations being used to construct the test. Also, l_i and u_i refer to the lower and upper bound of each sub-interval. The test statistic is approximately distributed according to the χ^2 distribution with $k-1$ degrees of freedom.

In order to apply powerful normality tests, Berkowitz (2001) suggests to further transform the i.i.d. uniform $\hat{p}_{t+1}$ into standard normally distributed variables under the null by using the inverse cumulative normal distribution function Φ^{-1}:

$$\begin{aligned} H_0: &\quad \hat{p}_{t+1} \overset{i.i.d.}{\sim} U(0,1) \Leftrightarrow \\ H_0: &\quad \hat{z}_{t+1} = \Phi^{-1}(\hat{p}_{t+1}) = \Phi^{-1}(F_t(y_{t+1})) \overset{i.i.d.}{\sim} \mathcal{N}(0,1). \end{aligned} \tag{7.19}$$

To investigate whether the $\hat{z}_{t+1}$ are i.i.d. normally distributed, the Kolmogorov-Smirnov (KS) test and the Anderson-Darling (AD) test, introduced in (6.3) and (6.5), are employed.

7.1.3 Backtesting the Entire Lower Tail

While it is useful to backtest the estimate of the whole predictive distribution to obtain additional information, the primary interest of financial risk management lies in the lower tail of the profit and loss distribution (McNeil et al., 2005). Testing the entire density as in section 7.1.2 might lead to a rejection of risk models which capture the lower tail of the profit and loss distribution well, but are not accurate for the rest of the distribution. Instead, Christoffersen (2012) proposes to construct a test that directly focuses on assessing the risk model's capability to capture the lower tail of the distribution, which contains the largest losses. Christoffersen (2012) restricts the attention to the tail of the distribution below the 10% quantile. His test consists of examining whether the probability integral transformations below this threshold are themselves uniform, constructing a rescaled probability integral transformed variable as

$$\hat{p}^*_{t+1} = \begin{cases} 10 \times F_t(y_{t+1}) & \text{if } y_t \leq -VaR_t \\ \text{Else not defined.} \end{cases} \tag{7.20}$$

The null hypothesis that the risk model provides the correct tail distribution is given by

$$H_0 : \hat{p}^*_{t+1} \overset{i.i.d.}{\sim} U(0,1) \tag{7.21}$$

or equivalently

$$H_0 : \hat{z}^*_{t+1} = \Phi^{-1}(\hat{p}^*_{t+1}) \overset{i.i.d.}{\sim} \mathcal{N}(0,1). \tag{7.22}$$

To do formal statistical testing, the alternative hypothesis is constructed as

$$\hat{z}^*_{t+1} = b_0 + b_1 \hat{z}^*_t + \sigma z_{t+1}, \quad \text{with} \quad z_{t+1} \overset{i.i.d.}{\sim} \mathcal{N}(0,1). \tag{7.23}$$

Then the log-likelihood of a sample of T observations of $\tilde{z}^*_{t+1}$ under the alternative hypothesis conditioned on an initial observation is

$$\log \mathcal{L} = -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log(\sigma^2) - \sum_{t=1}^{T}\left(\frac{(\tilde{z}^*_{t+1} - b_0 - b_1 \tilde{z}^*_t)^2}{2\sigma^2}\right). \tag{7.24}$$

With the parameter estimates $\hat{b}_0, \hat{b}_1, \hat{\sigma}^2$ obtained by the maximum likelihood estimation, the likelihood ratio test of a correct lower tail distribution is given by

$$LR_{lt} = -2\left(\log \mathcal{L}(0,0,1) - \log \mathcal{L}(\hat{b}_0, \hat{b}_1, \hat{\sigma}^2)\right), \tag{7.25}$$

which is asymptotically χ^2 distributed with three degrees of freedom (Christoffersen, 2012).

7.2 Overall Forecast Performance

This section presents the results of the backtest evaluation methods presented in the previous section sorted after portfolio. The predictive power of the different models is evaluated using the entire sample period. To put the backtest results of the presented models into perspective, they are compared to the results of a multivariate normal model which serves as a benchmark. Multivariate normal means that the model makes use of the same univariate GARCH processes but assumes the resulting standardized residuals to be distributed according to a multivariate normal distribution. In this model, each series of standardized residuals of the univariate filters is hence normally distributed and the dependence structure between the different series is described by a static Gaussian copula.

7.2.1 Equity Index Portfolio

Table 7.2 lists the backtest results for the benchmark multivariate normal model while 7.3 shows the results for the static copula models applied to the equity index data. The risk measure tests are conducted on three levels of confidence α: 90%, 95% and 99%. The hit ratios of the static models are materially different. On the three levels of α, all static mixture models' hit ratios are out of an acceptable range representing a multiple of the ones of the standalone copulas. This firstly stands in contrast to the theoretically promising setup of those models and secondly in contrast to their in-sample model fit. In fact, the static Student-t-Clayton mixture and the Gaussian-Clayton mixture, which were found to have the best and third-best in-sample fit according to both information criteria (see section 6), together with the Frank-Clayton mixture display the worst hit ratios among the static models. The results of the expected shortfall ratio substantiate the inferiority of the static mixture models compared to the standalone copulas. Naturally the Basel three-zones framework classifies these models as *red*. These results are consistent with the evidence for static bivariate copulas provided by Weiss (2011, p.186), who finds that mixture copulas yield "by far the worst VaR- and ES-estimates". In fact, the predictive power of the static mixture models proves to be considerably lower than the one of the multivariate normal benchmark model. One explanation for these results is that it is difficult to find numerically stable parameters for the static Archimedean copulas in the mixture due to the time variation in the dependence structure. While the violations of the mixture models' VaR forecasts are independent over time, they all clearly fail the unconditional coverage and the joint test. The inferiority of the static mixture models is further reflected by the test results for the entire forecasted profit and loss distributions.

Whilst the test statistics document the magnitude of the test failure, the histograms of the probability integral transforms of the historical returns taken with respect to the forecasted profit and loss distribution depicted in figure 7.2 allow to identify which parts of the distribution are particularly inaccurate. Figure 7.2 shows the histograms of the normalized frequency of probability integral transformed portfolio returns, divided into 50 bins of equal size. The overlaid dashed lines indicate the 95% percent confidence interval. Note that if a model's forecasts of the profit and loss distribution were accurate, all bars in a histogram would be equally high at exactly one. Clearly, none of the static mixture models yields suitable forecasts for either tail of the return distribution. The large bars for the lowest and the highest 2% quantiles of their histograms indicate that the copula mixture models severely underestimate the probability of joint extreme returns of the equity indices in the portfolio. The strong underestimation of the joint tails of the index returns entails an overestimation of the probability mass in the center of the distribution, manifested by multiple bars below the lower confidence level in the center of the histogram.

The Frank copula is the fourth static model that is classified as *red* by the Basel traffic light system yielding an unsatisfactory hit ratio and a relatively low ES ratio. While the violations of the VaR forecasted with the Frank copula model are independent over time, the model fails the joint test due to its inaccurate unconditional coverage. The model further fails the density tests altogether owing to its incapacity to forecast the lower tail of the portfolio profit and loss distribution, visualized by the bar of the lowest 2% quantile in the histogram in figure 7.2. Lacking the ability to model lower tail dependence, the Frank copula function proves to be incapable of adequately capturing the dependence structure with its single parameter θ_F.

The Clayton, Gaussian and Student-t copula models all reach a *yellow* traffic light classification. While the three dependence functions (just) pass the independence coverage test for all α levels, the Clayton is the only static model to pass the unconditional coverage test and the joint test for $\alpha = 99\%$. Interestingly, the hit ratios of the Gaussian model are identical to the ones for the Student-t. The capability of the Student-t copula to capture tail dependence hence does not produce better VaR forecasts, but yields a superior prognosis of the expected shortfall as documented by the ES ratio on $\alpha = (99\%, 95\%)$ being closer to 1 and a lower test statistic for the Lower Tail test compared to the Gaussian model results. The χ^2-test shows that the Gaussian copula is the most suitable static model to forecast the profit and loss distribution in its entirety. However, the forecasts are not accurate enough to pass neither the Kolmogorov-Smirnov nor the Anderson-Darling test. Comparing the 99% hit ratios of the Gaussian and the multivariate benchmark model shows the importance of incorporating fat tails and skewness in the marginal distributions, as the hit ratio of the latter model exceeds the former's by 1.9%.

Among the three *yellow* static models, the asymmetric Clayton copula model is most suitable to forecast the joint extreme negative returns of the portfolio constituents based on its hit ratio being closest to the expectation of 1%. Compared to the multivariate normal benchmark, the static Clayton model's forecasts for the 99% VaR are a staggering 2.6% closer to the expected hit ratio of 1%. The Clayton model's output further yields the most accurate forecasts for the entire lower tail of the portfolio return distribution as it is the only static model to pass the Lower Tail test. Note that in the in-sample analysis (see section 6), the static Clayton copula yields the second poorest model fit, while the Gaussian-Clayton and the Student-t-Clayton mixture copulas were ranked top three. The backtesting results at hand show that the ranking of the in-sample model fit does not contain much information about the out-of-sample predictive power.

Next, the results of the regime switching models, listed in table 7.4 are analyzed. Figure 7.3 indicates that four out of seven regime switching copula models have difficulties to forecast both tails of the portfolio profit and loss distribution: both regime switching models with three regimes and both structures where one regime is modeled by a mixture copula display large bars at the lowest and highest quantiles of the histograms. All four models yield enormous hit ratios, low ES ratios, classify as *red* according to the Basel regulatory framework and fail the unconditional coverage and the joint tests on all levels of α. Adding a third regime hence does not improve the forecasts; to the contrary, the G/T/C yields significantly worse results than the G/T. The static mixtures prove to be unsuccessful in forecasting also as a part of the regime switching setup.

The two-regime models G/G, G/T and T/T fare considerably better, all attaining a *yellow* classification. However, with hit ratios being substantially larger than the expected value, they fail the unconditional coverage and the joint test on all levels of α. While in particular the G/G regime switching model seems to produce acceptably accurate forecasts for the center of the profit and loss distribution based on the histogram in figure 7.3 and based on both χ^2 and KS test results, all regime switching models fail the Lower Tail test. The best fitting regime switching model according to the in-sample analysis (T/GC) produces some of the worst profit and loss distribution forecasts among the regime switching models. However, ranks two and three according to the in-sample information criteria are attributed to the G/G and the G/T copulas, which in fact belong to the regime switching models with the best out-of-sample forecast power.

A comparison of the backtesting results of the Gaussian/Gaussian regime switching model with those of the static Gaussian copula shows that the former does produce slightly more accurate forecasts in terms of hit ratio and expected shortfall ratio on all levels of α. The improved results for the Lower Tail test further indicate that the G/G issues better forecasts than the static Gaussian. Note that the regime switching Gaussian copula is the only model among both static and regime switching structures to pass the Kolmogorov-Smirnov test. According to the test results of the χ^2-test, the T/T regime switching copula yields better profit and loss density forecasts compared to the static Student-t copula. However, neither model passes the unconditional coverage or joint test and the hit ratios as well as the ES ratios are virtually the same which menas, underpinned by an even worse Lower Tail test statistic of the Markov switching setup, that one may spare the computational effort for the two-state Student-t model.

The results of the backtests with the dynamic models are shown in table 7.5 and figure 7.4. Even though the results for the dynamic Frank copula are better in nearly all backtest measures compared to its static counterpart, they are far from satisfactory, which means that the standalone Frank copula is neither in static nor dynamic form a suitable model for the equity index returns. The histogram of the dynamic Frank copula's results in figure 7.4 illustrates that the incapability to capture the dependencies of the lower extreme returns is the cause of the dynamic Frank copula's failure. This is substantiated by its failure to pass any of the density tests.

With a substantially elevated bar for the highest 2% quantile in the histogram, the dynamic Clayton is not a good model to forecast the upper part of the portfolio return distribution. The strengths of the dynamic Clayton model lie in its capability to capture lower tail dependence which results in the best forecasts for the lower tail of the equity portfolio's return distribution showing the highest p-value for the Lower Tail test among all static, regime switching and dynamic copula models. The dynamic version of the Clayton further provides lower hit ratios than the static Clayton for all tested α; in fact, it yields the overall lowest hit ratio on $\alpha = 99\%$ for the equity portfolio. The dynamic Clayton is furthermore the only model to qualify as *green* in the Basel three-zones approach.

Note that all dynamic mixture copulas yield substantially better results compared to their static counterparts. While the static mixtures proved to be virtually useless in terms of forecast power, the dynamic Gaussian-Clayton mixture and the Dynamic Student-t-Clayton mixture classify as *yellow*. The dynamic Student-t-Clayton mixture copula even yields the second-best hit ratio for the equity data on $\alpha = 99\%$ and passes the joint test on this level of significance. This shows the fundamental importance of accounting for time variation in mixture copula models. In the in-sample analysis of the dynamic models, the dynamic Student-t-Clayton mixture stood out with the lowest AIC and BIC values indicating a good fit which is underpinned by the results of the backtesting procedure. The dynamic Clayton copula, however, has the second poorest in-sample fit among the dynamic models but yields the best forecasts of the lower tail of the equity portfolio's profit and loss distribution. This shows that the in-sample model fit criteria might be too focused on the center of the distribution whereas the risks are located in the lower tail.

Table 7.2
Static multivariate normal out-of-sample backtest results

	α	Equities	Commodities	Multi Asset Classes
Hit Ratio	99%	0.045	0.037	0.046
	95%	0.108	0.078	0.101
	90%	0.165	0.122	0.152
ES Ratio	99%	0.911	0.902	0.806
	95%	0.917	0.918	0.891
	90%	0.942	0.943	0.912
Traffic Light		Red	Red	Red
Ind. Cov.	99%	1.241	2.567	1.401
		(0.265)	(0.109)	(0.237)
	95%	5.947	3.806	0.596
		(0.015)	(0.051)	(0.440)
	90%	2.948	3.131	1.195
		(0.086)	(0.077)	(0.274)
Unc. Cov.	99%	51.655	34.351	48.035
		(0.000)	(0.000)	(0.000)
	95%	41.689	11.232	30.204
		(0.000)	(0.001)	(0.000)
	90%	31.599	3.877	18.783
		(0.000)	(0.049)	(0.000)
Joint Test	99%	52.897	36.918	49.436
		(0.000)	(0.000)	(0.000)
	95%	47.636	15.038	30.801
		(0.000)	(0.001)	(0.000)
	90%	34.547	7.008	19.978
		(0.000)	(0.030)	(0.000)
χ^2-Test		57.040	42.118	53.652
		(0.004)	(0.088)	(0.010)
AD Test		8.706	5.441	8.703
		(0.000)	(0.002)	(0.000)
KS Test		0.068	0.056	0.057
		(0.001)	(0.014)	(0.021)
Lower Tail		93.783	97.262	173.604
		(0.000)	(0.000)	(0.000)

This table reports the backtest evaluation results for the multivariate normal benchmark model applied to the three index data sets. α denotes the significance level of VaR(α). The hit ratio reflects the percentage of times when the portfolio return exceeds VaR(α). ES ratio shows whether the mean of the returns when VaR(α) is violated corresponds to the average expected shortfall in these weeks. The traffic light is the model classification of the Basel regulatory framework. The mid and lower panel lists test statistics and p-values (in parentheses) for multiple backtesting evaluation tests. Independence (unconditional) coverage is abbreviated with Ind. Cov. (Unc. Cov.). Joint Test is the joint test for conditional coverage. The lower panel reports the test statistics with p-values in parentheses of density forecast evaluation tests. χ^2-Test is Pearson's χ^2-test with 10 evenly spaced bins. AD and KS are the tests of Anderson-Darling and Kolmogorov-Smirnov. Lower Tail is the test of Christoffersen (2012) assessing the models' ability to forecast the entire lower tail (losses below the 10% quantile) of the P&L distribution.

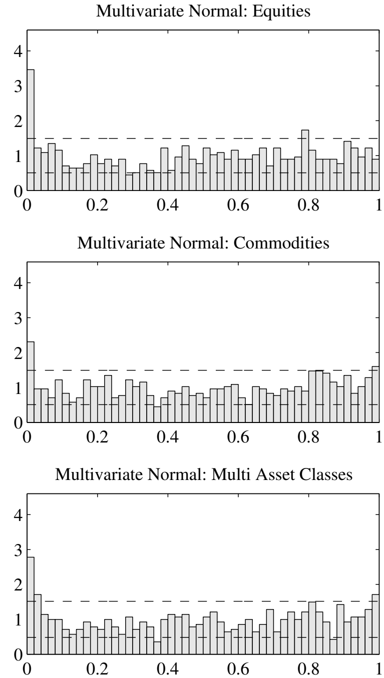

Figure 7.1: Relative frequency of occurrence of the probability integral transforms of the three portfolio returns taken with respect to the multivariate normal models' forecasted probability density distribution. The histogram is segmented into 50 bins of equal size. The overlaid dashed lines indicate the 95% confidence interval for the heights of the individual bins under the null hypothesis that the probability integral transforms are $(0, 1)$ uniform distributed.

Table 7.3
Static models out-of-sample backtest results: Equity portfolio

	α	F	C	G	T	FC	GC	TC
Hit Ratio	99%	0.078	0.019	0.026	0.026	0.186	0.185	0.187
	95%	0.121	0.101	0.097	0.097	0.250	0.258	0.258
	90%	0.169	0.178	0.165	0.165	0.295	0.299	0.303
ES Ratio	99%	0.855	0.962	0.910	0.941	0.732	0.726	0.728
	95%	0.821	1.063	0.968	0.997	0.680	0.693	0.691
	90%	0.850	1.078	1.013	1.028	0.658	0.663	0.669
Traffic Light		Red	Yellow	Yellow	Yellow	Red	Red	Red
Ind. Cov.	99%	2.328	0.589	2.702	2.702	2.778	2.344	4.098
		(0.127)	(0.443)	(0.100)	(0.100)	(0.096)	(0.126)	(0.043)
	95%	4.678	6.571	4.603	6.227	1.065	0.401	0.200
		(0.031)	(0.010)	(0.032)	(0.013)	(0.302)	(0.526)	(0.654)
	90%	2.089	4.942	2.948	2.948	0.339	0.379	0.094
		(0.148)	(0.026)	(0.086)	(0.086)	(0.560)	(0.538)	(0.759)
Unc. Cov.	99%	148.28	5.285	13.458	13.458	599.12	592.89	605.37
		(0.000)	(0.022)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	59.577	33.729	29.289	29.289	351.11	373.50	373.50
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	90%	35.153	44.080	31.599	31.599	229.01	237.02	245.14
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Joint Test	99%	150.60	5.874	16.160	16.160	601.90	595.24	609.46
		(0.000)	(0.053)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	64.255	40.301	33.892	35.516	352.17	373.90	373.70
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	90%	37.242	49.022	34.547	34.547	229.35	237.40	245.23
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
χ^2-Test		126.76	76.345	27.349	48.076	172.43	182.79	149.24
		(0.000)	(0.000)	(0.701)	(0.026)	(0.000)	(0.000)	(0.000)
AD Test		20.297	28.984	8.248	7.652	298.13	297.89	301.21
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
KS Test		0.077	0.094	0.068	0.071	0.211	0.211	0.216
		(0.000)	(0.000)	(0.002)	(0.001)	(0.000)	(0.000)	(0.000)
Lower Tail		359.10	11.174	52.457	33.593	1325.6	1287.8	1309.5
		(0.000)	(0.011)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)

This table reports the backtest evaluation results for the static copula models applied to the equity index data set. The copula models are abbreviated with F (Frank), C (Clayton), G (Gaussian), T (Student-t), FC (Frank-Clayton mixture), GC (Gaussian-Clayton mixture) and TC (Student-t-Clayton mixture). α denotes the confidence level of VaR(α). The hit ratio reflects the percentage of times when the portfolio return exceeds VaR(α). ES ratio shows whether the mean of the returns when VaR(α) is violated corresponds to the average expected shortfall in these weeks. The traffic light is the model classification of the Basel regulatory framework. The mid and lower panel lists test statistics and p-values (in parentheses) for multiple backtesting evaluation tests. Independence (unconditional) coverage is abbreviated with Ind. Cov. (Unc. Cov.). Joint Test is the joint test for conditional coverage. The lower panel reports the test statistics with p-values in parentheses of density forecast evaluation tests. χ^2-Test A is Pearson's χ^2-test with 10 evenly spaced bins. AD and KS are the tests of Anderson-Darling and Kolmogorov-Smirnov. Lower Tail is the test of Christoffersen (2012) assessing the models' ability to forecast the entire lower tail (losses below the 10% quantile) of the P&L distribution.

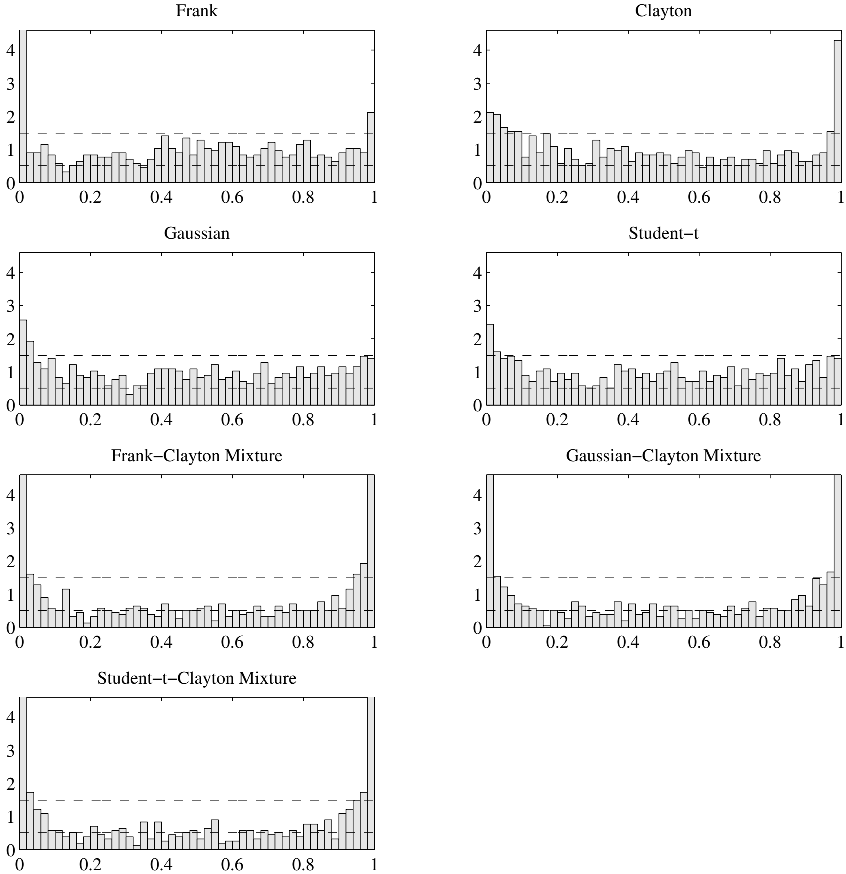

Figure 7.2: Relative frequency of occurrence of the probability integral transforms of the equity index portfolio returns taken with respect to the static copula models' forecasted probability density distribution. The histogram is segmented into 50 bins of equal size. The overlaid dashed lines indicate the 95% confidence interval for the heights of the individual bins under the null hypothesis that the probability integral transforms are $(0, 1)$ uniform distributed.

Table 7.4
Regime switching models out-of-sample backtest results: Equity portfolio

	α	G/G	G/T	T/T	T/GC	G/TC	G/T/C	G/C/F
Hit Ratio	99%	0.024	0.027	0.026	0.119	0.056	0.185	0.182
	95%	0.092	0.088	0.099	0.177	0.138	0.254	0.253
	90%	0.147	0.154	0.164	0.228	0.208	0.305	0.299
ES Ratio	99%	0.936	0.955	0.942	0.767	0.805	0.726	0.710
	95%	0.990	0.971	1.010	0.735	0.880	0.685	0.674
	90%	0.999	1.006	1.022	0.757	0.921	0.680	0.666
Traffic Light		Yellow	Yellow	Yellow	Red	Red	Red	Red
Ind. Cov.	99%	3.041	6.019	6.290	9.923	9.254	3.094	2.941
		(0.081)	(0.014)	(0.012)	(0.002)	(0.002)	(0.079)	(0.086)
	95%	4.593	5.826	2.894	4.346	1.484	0.859	0.701
		(0.032)	(0.016)	(0.089)	(0.037)	(0.223)	(0.354)	(0.403)
	90%	6.204	7.803	3.269	6.258	0.578	0.367	0.379
		(0.013)	(0.005)	(0.071)	(0.012)	(0.447)	(0.545)	(0.538)
Unc. Cov.	99%	11.595	15.424	13.458	300.363	81.570	592.893	580.489
		(0.001)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	23.779	19.966	30.740	164.638	88.645	362.242	358.517
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	90%	17.277	21.952	30.451	108.701	79.286	250.612	237.021
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Joint Test	99%	14.636	21.442	19.748	310.286	90.824	595.987	583.430
		(0.001)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	28.372	25.793	33.635	168.983	90.129	363.101	359.217
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	90%	23.480	29.755	33.720	114.959	79.864	250.979	237.401
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
χ^2-Test		31.129	33.684	23.663	86.816	40.081	160.511	172.617
		(0.409)	(0.339)	(0.856)	(0.000)	(0.378)	(0.000)	(0.000)
AD Test		5.086	6.274	7.599	102.487	38.557	299.779	300.721
		(0.003)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
KS Test		0.055	0.061	0.065	0.136	0.108	0.212	0.214
		(0.017)	(0.006)	(0.003)	(0.000)	(0.000)	(0.000)	(0.000)
Lower Tail		35.271	37.175	37.432	714.824	276.925	1271.40	1317.26
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)

This table reports the backtest evaluation results for the regime switch copula models applied to the equity index data set. The copula models are abbreviated with G (Gaussian), T (Student-t), F (Frank), C (Clayton), GC (Gaussian-Clayton mixture) and TC (Student-t-Clayton mixture). G/T/C and G/C/F are three-regime models. α denotes the confidence level of VaR(α). The hit ratio reflects the percentage of times when the portfolio return exceeds VaR(α). ES ratio shows whether the mean of the returns when VaR(α) is violated corresponds to the average expected shortfall in these weeks. The traffic light is the model classification of the Basel regulatory framework. The mid and lower panel lists test statistics and p-values (in parentheses) for multiple backtesting evaluation tests. Independence (unconditional) coverage is abbreviated with Ind. Cov. (Unc. Cov.). Joint Test is the joint test for conditional coverage. The lower panel reports the test statistics with p-values in parentheses of density forecast evaluation tests. χ^2-Test is Pearson's χ^2-test with 10 evenly spaced bins. AD and KS are the tests of Anderson-Darling and Kolmogorov-Smirnov. Lower Tail is the test of Christoffersen (2012) assessing the models' ability to forecast the entire lower tail (losses below the 10% quantile) of the P&L distribution.

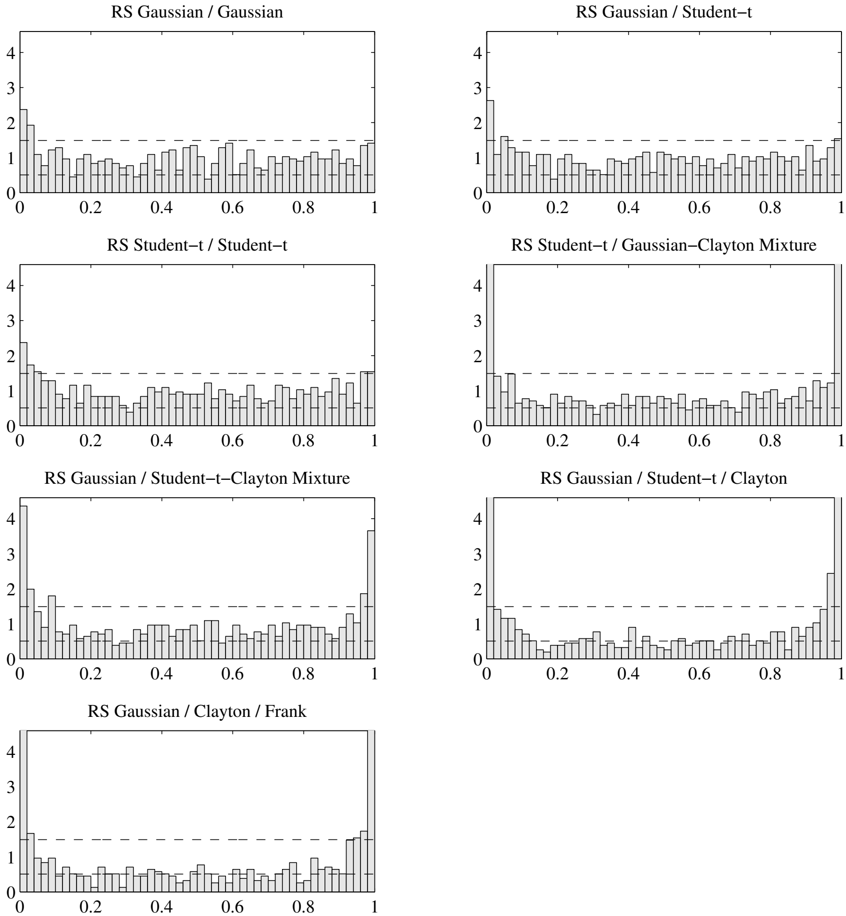

Figure 7.3: Relative frequency of occurrence of the probability integral transforms of the equity index portfolio returns taken with respect to the regime switching (RS) copula models' forecasted probability density distribution. The histogram is segmented into 50 bins of equal size. The overlaid dashed lines indicate the 95% confidence interval for the heights of the individual bins under the null hypothesis that the probability integral transforms are $(0, 1)$ uniform distributed.

Table 7.5
Dynamic models out-of-sample backtest results: Equity portfolio

	α	DF	DC	DG	DT	DFC	DGC	DTC
Hit Ratio	99%	0.072	0.015	0.026	0.024	0.041	0.023	0.017
	95%	0.114	0.099	0.097	0.095	0.100	0.095	0.096
	90%	0.158	0.164	0.160	0.162	0.164	0.156	0.155
ES Ratio	99%	0.850	0.928	0.915	0.950	0.977	1.356	0.877
	95%	0.828	1.086	0.982	0.972	0.924	1.028	1.021
	90%	0.854	1.070	1.001	1.003	0.970	1.025	1.026
Traffic Light		Red	Green	Yellow	Yellow	Red	Yellow	Yellow
Ind. Cov.	99%	2.286	1.862	6.290	0.497	7.203	0.626	0.441
		(0.131)	(0.172)	(0.012)	(0.481)	(0.007)	(0.429)	(0.507)
	95%	3.959	2.894	3.200	5.406	3.871	1.492	3.522
		(0.047)	(0.089)	(0.074)	(0.020)	(0.049)	(0.222)	(0.061)
	90%	5.131	3.269	4.333	2.223	1.720	4.432	2.843
		(0.024)	(0.071)	(0.037)	(0.136)	(0.190)	(0.035)	(0.092)
Unc. Cov.	99%	127.451	1.962	13.458	11.595	42.710	9.840	2.917
		(0.000)	(0.161)	(0.000)	(0.001)	(0.000)	(0.002)	(0.088)
	95%	50.316	30.740	29.289	26.474	32.221	26.474	27.867
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	90%	24.995	30.451	27.121	28.212	30.451	23.962	22.947
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Joint Test	99%	129.737	3.824	19.748	12.092	49.913	10.466	3.358
		(0.000)	(0.148)	(0.000)	(0.002)	(0.000)	(0.005)	(0.187)
	95%	54.274	33.635	32.489	31.881	36.091	27.966	31.389
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	90%	30.126	33.720	31.454	30.435	32.171	28.394	25.790
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
χ^2-Test		123.620	67.379	33.151	37.737	35.672	26.825	30.809
		(0.000)	(0.001)	(0.411)	(0.223)	(0.390)	(0.632)	(0.425)
AD Test		16.561	24.169	7.239	7.928	13.480	5.935	6.319
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.001)
KS Test		0.071	0.088	0.066	0.065	0.065	0.058	0.064
		(0.001)	(0.000)	(0.002)	(0.003)	(0.003)	(0.010)	(0.003)
Lower Tail		331.268	8.352	39.353	34.571	107.550	18.414	18.510
		(0.000)	(0.039)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)

This table reports the backtest evaluation results for the dynamic copula models applied to the equity index data set. The models are abbreviated with D (Dynamic), F (Frank), C (Clayton), G (Gaussian), T (Student-t), FC (Frank-Clayton mixture), GC (Gaussian-Clayton mixture) and TC (Student-t-Clayton mixture). α denotes the confidence level of VaR(α). The hit ratio reflects the percentage of times when the portfolio return exceeds VaR(α). ES ratio shows whether the mean of the returns when VaR(α) is violated corresponds to the average expected shortfall in these weeks. The traffic light is the model classification of the Basel regulatory framework. The mid and lower panel lists test statistics and p-values (in parentheses) for multiple backtesting evaluation tests. Independence (unconditional) coverage is abbreviated with Ind. Cov. (Unc. Cov.). Joint Test is the joint test for conditional coverage. The lower panel reports the test statistics with p-values in parentheses of density forecast evaluation tests. χ^2-Test is Pearson's χ^2-test with 10 evenly spaced bins. AD and KS are the tests of Anderson-Darling and Kolmogorov-Smirnov. Lower Tail is the test of Christoffersen (2012) assessing the models' ability to forecast the entire lower tail (losses below the 10% quantile) of the P&L distribution.

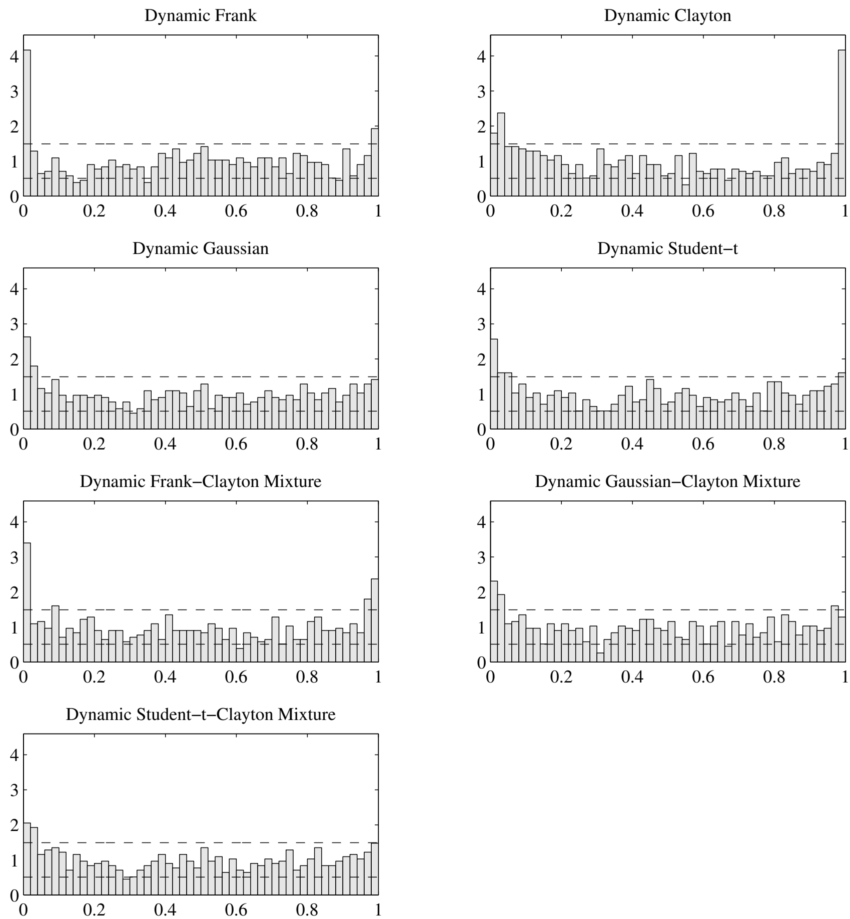

Figure 7.4: Relative frequency of occurrence of the probability integral transforms of the equity index portfolio returns using the dynamic copula models' forecasted probability density distribution. The histogram is segmented into 50 bins of equal size. The overlaid dashed lines indicate the 95% confidence interval for the heights of the individual bins under the null hypothesis that the probability integral transforms are $(0, 1)$ uniform distributed.

7.2.2 Commodity Futures Index Portfolio

This section discusses the outcomes of the backtesting procedure for the commodity data over the entire out-of-sample time frame from 1998 until 2013. The results for the static models are listed in table 7.6 and figure 7.5 depicts the histograms of the probability integral transforms of the empirical portfolio returns taken with respect to the forecasted profit and loss distribution.

According to the hit ratios and the expected shortfall ratios, the static mixture copulas are ranked in the rearmost positions. None of the mixture passes the unconditional coverage and the joint test and all static mixture models classify as *red*. Figure 7.5 shows the static mixtures' failure to adequately forecast both lower and upper tail of the profit and loss distribution. Despite the Frank-Clayton and the Student-t-Clayton mixture copulas passing both χ^2 tests, the comparatively high test statistics of the Anderson-Darling, Kolmogorov-Smirnov and Christoffersen's Lower Tail test confirm the impression given by figure 7.5, that for a commodity futures index portfolio the static mixture copulas are not suitable models to forecast neither for the entire profit and loss distribution nor its lower tail, which is identical to the findings for the equity data. The static Frank and the Gaussian copula are two further models with no predictive power of the commodity data's return distribution. Both traffic lights are *red* and even though the violation of their VaR forecasts are independent in time according to the independence coverage tests, they both fail the remaining tests altogether.

The backtest comparison of the static Gaussian model with the multivariate normal benchmark model indicates that modeling the standardized residuals with a skewed-t distribution does not improve the hit ratios on any of the three confidence levels. The benefit of the more elaborate marginals comes forward with the improved expected shortfall ratios, which are particularly meaningful on $\alpha = 99\%$ and $\alpha = 95\%$ since the two compared models have identical hit ratios for these confidence levels.

Among the two *yellow* models Clayton and Student-t, the former is substantially more successful in forecasting the lower tail of the portfolio's return distribution, documented by the hit ratios and ES ratios on $\alpha = 99\%$ and $\alpha = 95\%$ and by the fact that the Clayton is the only static model to pass the Lower Tail test.

While incorporating skewness and kurtosis in the marginal distributions does not improve the hit ratios for the commodity data (documented by the comparison of the Gaussian copula with multivariate normal model results), the capability of modeling lower tail dependence (as in the Student-t and Clayton copula) results in substantially better hit ratios on the 99% confidence level. Applying the static Clayton copula setup instead of the multivariate normal model reduces the 99% hit ratio by more than 50%. The χ^2-test even indicates that the Clayton copula is the best static model to forecast the entire profit and loss distribution of the commodity futures index portfolio.

The backtesting results of the regime switching models are listed in table 7.7. Forecasting the profit and loss distribution of the commodity futures index portfolio by means of the regime switching copulas with three states yields worse results than with two-state models, analogous to the results for the equity data. The histograms of the three-state models in figure 7.6 visualize their incapability to forecast either tail of the portfolio return distribution. The inclusion of a mixture copula to characterize one of the two regimes does not produce better forecasts compared to the regime switching structures with two standalone copulas.

The G/G and the T/T regime switching copulas are the only two Markov-chain models to achieve a *yellow* rating. The former's hit ratios compared to those of the static Gaussian model show that allowing for regime switches results in preferable VaR forecasts. The results of the χ^2 tests further suggest that the two-state Gaussian setup yields better forecasts for the entire return distribution than the one-state Gaussian model and the multivariate normal benchmark. The regime switching copula models do not shine when it comes to forecast the lowest 1% and 5% quantiles of the commodities' profit and loss distribution as they fail the unconditional coverage test and the joint test on the according α levels and further do not pass the Lower Tail test. The backtesting outcomes of the G/G and the G/T models on $\alpha = 90\%$, however, allude to their predictive power for the VaR on the according level. Note that the information criteria for the in-sample fit relegate the latter two models to the rear positions among the regime switching models whilst the models with the best in-sample fit perform rather poorly in the backtesting procedure which again points to a limited use of the in-sample rankings to gauge the predictive power.

Next, the performance of the dynamic models is shifted into focus. The according results are listed in table 7.8. Comparing the backtest performance of the dynamic copulas to those of their static counterparts (see table 7.6) shows that allowing for time variation in the dependence structure improves most of the backtest results for all the copula models under consideration. Four models which are classified as *red* in their static versions rank as *yellow* under the Basel regulatory framework in their dynamic specification. The dynamic Frank copula, however, does not produce materially different results than its static version, indicating that the Frank copula is in neither static nor dynamic form an appropriate dependence model to forecast the commodity portfolio's returns.

Adding the Clayton to the Frank copula in the form of a dynamic convex combination of both models improves the performance to the degree that the dynamic Frank-Clayton mixture reaches a *yellow* traffic light ranking, but still yields the second poorest results among the dynamic models. Figure 7.7 depicts the histograms of the portfolio returns' probability integral transforms with respect to the dynamic models forecasted profit and loss distributions. Clearly, the forecasted return distribution of the dynamic Frank-Clayton mixture copula concentrates too much probability mass in the center of the distribution, which results in the high bars at lowest and highest quantiles of the histogram.

The model with the highest predictive power for the negative extreme returns of the commodity portfolio is the dynamic Clayton copula. Even though the dynamic version only improves the VaR forecasts on the 90% confidence level compared to the static Clayton copula, the dynamic version passes the Lower Tail test with a p-value twice as large. This outperformance of the multivariate Clayton copula (both static and dynamic) in forecasting the lowest portfolio return quantiles is due to its capability to model lower tail dependence. The histogram in figure 7.7 visualizes the Clayton's superiority in this regard with the low bar for the 2% quantile. This highlights the importance of modeling lower tail dependence to forecast the risk of a commodity futures index portfolio. The second poorest in-sample fit of the dynamic Clayton copula among the dynamic models according to both information criteria questions the usefulness of AIC and BIC rankings to identify a powerful risk forecasting model.

Table 7.6
Static models out-of-sample backtest results: Commodity portfolio

	α	F	C	G	T	FC	GC	TC
Hit Ratio	99%	0.056	0.018	0.037	0.029	0.099	0.100	0.097
	95%	0.100	0.077	0.078	0.078	0.172	0.172	0.164
	90%	0.144	0.141	0.127	0.122	0.227	0.233	0.228
ES Ratio	99%	0.847	1.025	0.989	0.956	0.794	0.793	0.803
	95%	0.839	1.044	0.935	0.964	0.804	0.800	0.787
	90%	0.857	1.060	0.975	0.973	0.807	0.825	0.819
Traffic Light		Red	Yellow	Red	Yellow	Red	Red	Red
Ind. Cov.	99%	2.281	5.264	2.567	4.780	1.792	3.698	3.200
		(0.131)	(0.022)	(0.109)	(0.029)	(0.181)	(0.054)	(0.074)
	95%	2.605	1.369	2.328	2.197	3.049	2.269	3.269
		(0.107)	(0.242)	(0.127)	(0.138)	(0.081)	(0.132)	(0.071)
	90%	5.009	4.625	2.950	4.298	0.042	0.547	0.100
		(0.025)	(0.032)	(0.086)	(0.038)	(0.837)	(0.460)	(0.752)
Unc. Cov.	99%	81.570	4.028	34.351	19.644	220.606	225.388	215.854
		(0.000)	(0.045)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	32.221	10.295	11.232	11.232	153.523	153.523	137.393
		(0.000)	(0.001)	(0.001)	(0.001)	(0.000)	(0.000)	(0.000)
	90%	14.715	13.111	5.840	3.877	106.751	116.647	108.701
		(0.000)	(0.000)	(0.016)	(0.049)	(0.000)	(0.000)	(0.000)
Joint Test	99%	83.851	9.292	36.918	24.424	222.399	229.086	219.053
		(0.000)	(0.010)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	34.826	11.664	13.559	13.429	156.572	155.792	140.662
		(0.000)	(0.003)	(0.001)	(0.001)	(0.000)	(0.000)	(0.000)
	90%	19.724	17.736	8.790	8.175	106.793	117.193	108.801
		(0.000)	(0.000)	(0.012)	(0.017)	(0.000)	(0.000)	(0.000)
χ^2-Test		78.975	36.276	54.768	53.183	64.295	72.233	65.630
		(0.000)	(0.363)	(0.005)	(0.008)	(0.015)	(0.003)	(0.011)
AD Test		14.708	17.319	5.859	5.605	118.621	123.705	118.060
		(0.000)	(0.000)	(0.001)	(0.001)	(0.000)	(0.000)	(0.000)
KS Test		0.065	0.079	0.061	0.063	0.175	0.174	0.175
		(0.002)	(0.000)	(0.006)	(0.004)	(0.000)	(0.000)	(0.000)
Lower Tail		259.651	4.548	66.008	52.095	474.543	518.550	427.286
		(0.000)	(0.208)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)

This table reports the backtest evaluation results for the static copula models applied to the commodity index data set. The copula models are abbreviated with F (Frank), C (Clayton), G (Gaussian), T (Student-t), FC (Frank-Clayton mixture), GC (Gaussian-Clayton mixture) and TC (Student-t-Clayton mixture). α denotes the confidence level of VaR(α). The hit ratio reflects the percentage of times when the portfolio return exceeds VaR(α). ES ratio shows whether the mean of the returns when VaR(α) is violated corresponds to the average expected shortfall in these weeks. The traffic light is the model classification of the Basel regulatory framework. The mid and lower panel lists test statistics and p-values (in parentheses) for multiple backtesting evaluation tests. Independence (unconditional) coverage is abbreviated with Ind. Cov. (Unc. Cov.). Joint Test is the joint test for conditional coverage. The lower panel reports the test statistics with p-values in parentheses of density forecast evaluation tests. χ^2-Test is Pearson's χ^2-test with 10 evenly spaced bins. AD and KS are the tests of Anderson-Darling and Kolmogorov-Smirnov. Lower Tail is the test of Christoffersen (2012) assessing the models' ability to forecast the entire lower tail (losses below the 10% quantile) of the P&L distribution.

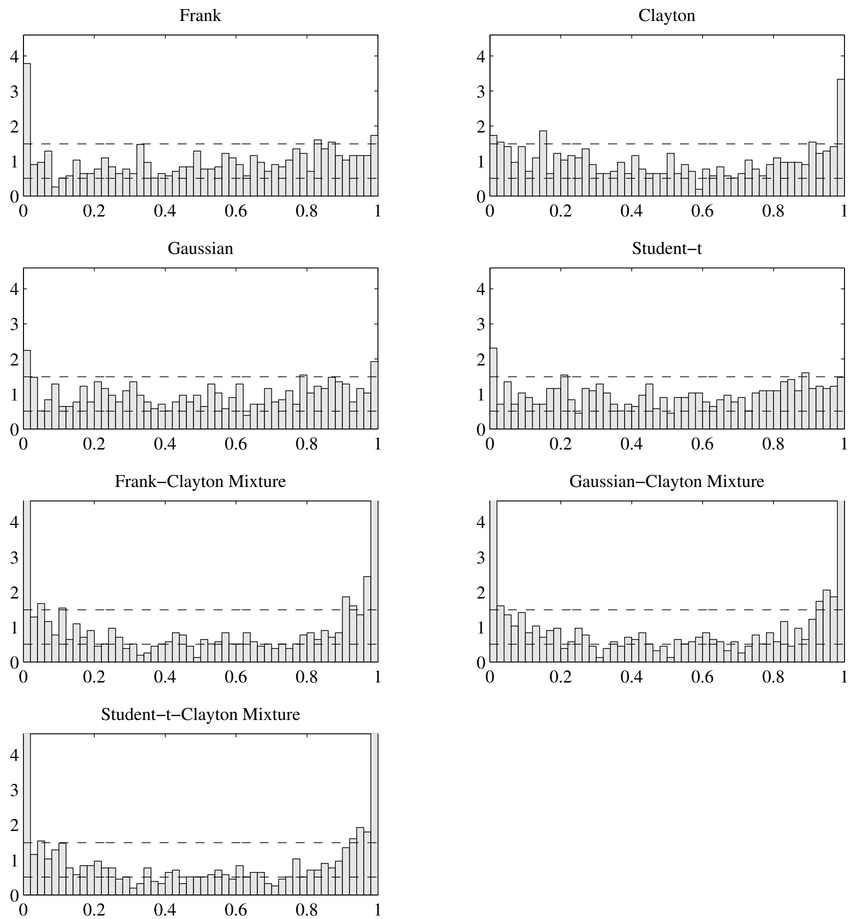

Figure 7.5: Commodity portfolio: Relative frequency of occurrence of the probability integral transforms of the commodity futures index portfolio returns using the static copula models' forecasted probability density distribution. The histogram is segmented into 50 bins of equal size. The overlaid dashed lines indicate the 95% confidence interval for the heights of the individual bins under the null hypothesis that the probability integral transforms are $(0, 1)$ uniform distributed.

Table 7.7
Regime switching models out-of-sample backtest results: Commodity portfolio

	α	G/G	G/T	T/T	T/GC	G/TC	G/T/C	G/C/F
Hit Ratio	99%	0.029	0.036	0.031	0.082	0.053	0.094	0.095
	95%	0.077	0.078	0.078	0.141	0.092	0.164	0.168
	90%	0.114	0.115	0.126	0.195	0.145	0.232	0.231
ES Ratio	99%	0.922	0.927	0.973	0.832	0.840	0.766	0.787
	95%	0.930	0.940	0.954	0.799	0.836	0.779	0.797
	90%	0.929	0.928	0.975	0.826	0.905	0.823	0.822
Traffic Light		Yellow	Red	Yellow	Red	Red	Red	Red
Ind. Cov.	99%	4.780	5.754	4.343	2.845	3.157	1.597	1.492
		(0.029)	(0.016)	(0.037)	(0.092)	(0.076)	(0.206)	(0.222)
	95%	2.600	2.328	2.328	2.569	3.140	3.269	2.359
		(0.107)	(0.127)	(0.127)	(0.109)	(0.076)	(0.071)	(0.125)
	90%	3.959	3.613	1.442	1.047	3.534	0.190	0.109
		(0.047)	(0.057)	(0.230)	(0.306)	(0.060)	(0.663)	(0.741)
Unc. Cov.	99%	19.644	31.706	21.891	161.202	71.123	201.772	206.436
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	10.295	11.232	11.232	93.153	23.779	137.393	145.375
		(0.001)	(0.001)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)
	90%	1.656	1.964	5.314	62.910	15.548	114.639	112.645
		(0.198)	(0.161)	(0.021)	(0.000)	(0.000)	(0.000)	(0.000)
Joint Test	99%	24.424	37.460	26.234	164.047	74.280	203.369	207.928
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	12.894	13.559	13.559	95.723	26.919	140.662	147.734
		(0.002)	(0.001)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)
	90%	5.615	5.578	6.756	63.958	19.082	114.829	112.754
		(0.060)	(0.061)	(0.034)	(0.000)	(0.000)	(0.000)	(0.000)
χ^2-Test		33.755	49.988	34.938	44.981	58.597	84.563	91.784
		(0.383)	(0.029)	(0.286)	(0.271)	(0.010)	(0.000)	(0.000)
AD Test		4.773	5.478	6.287	82.530	15.924	121.344	121.996
		(0.004)	(0.002)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)
KS Test		0.058	0.057	0.066	0.142	0.074	0.179	0.177
		(0.010)	(0.012)	(0.002)	(0.000)	(0.000)	(0.000)	(0.000)
Lower Tail		65.859	60.530	35.474	344.719	212.651	562.397	519.691
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)

This table reports the backtest evaluation results for the regime switch copula models applied to the commodity futures index data set. The copula models are abbreviated with G (Gaussian), T (Student-t), F (Frank), C (Clayton), GC (Gaussian-Clayton mixture) and TC (Student-t-Clayton mixture). G/T/C and G/C/F are three-regime models. α denotes the confidence level of VaR(α). The hit ratio reflects the percentage of times when the portfolio return exceeds VaR(α). ES ratio shows whether the mean of the returns when VaR(α) is violated corresponds to the average expected shortfall in these weeks. The traffic light is the model classification of the Basel regulatory framework. The mid and lower panel lists test statistics and p-values (in parentheses) for multiple backtesting evaluation tests. Independence (unconditional) coverage is abbreviated with Ind. Cov. (Unc. Cov.). Joint Test is the joint test for conditional coverage. The lower panel reports the test statistics with p-values in parentheses of density forecast evaluation tests. χ^2-Test is Pearson's χ^2-test with 10 evenly spaced bins. AD and KS are the tests of Anderson-Darling and Kolmogorov-Smirnov. Lower Tail is the test of Christoffersen (2012) assessing the models' ability to forecast the entire lower tail (losses below the 10% quantile) of the P&L distribution.

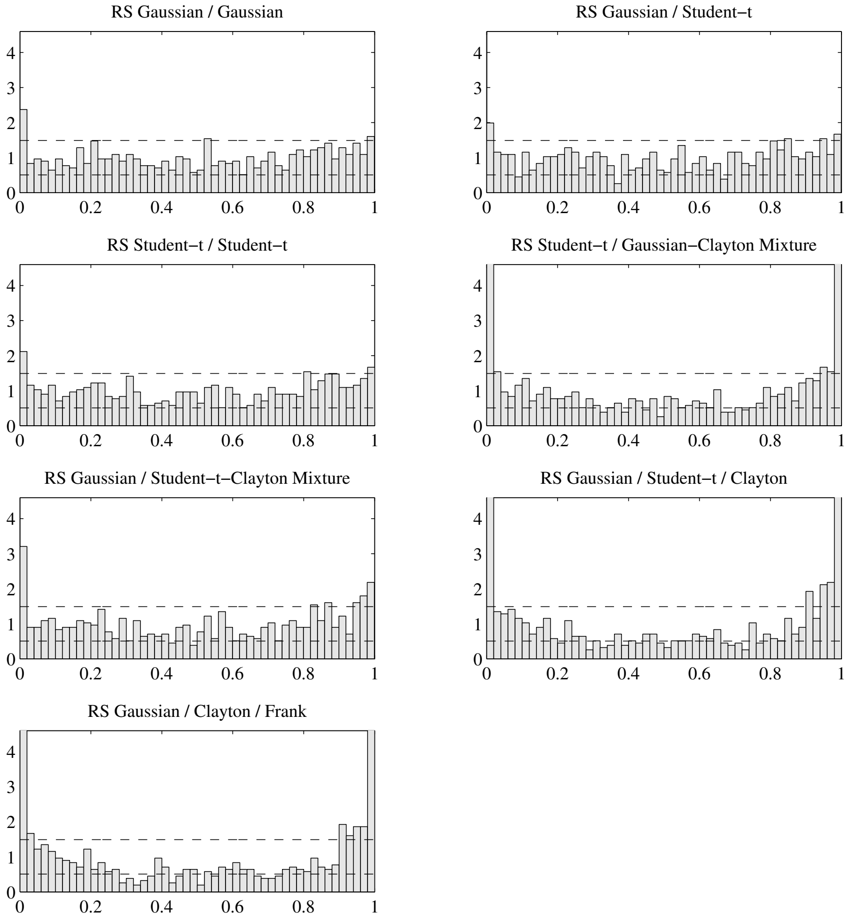

Figure 7.6: Commodity portfolio: Relative frequency of occurrence of the probability integral transforms of the commodity futures index portfolio returns using the regime switching (RS) copula models' forecasted probability density distribution. The histogram is segmented into 50 bins of equal size. The overlaid dashed lines indicate the 95% confidence interval for the heights of the individual bins under the null hypothesis that the probability integral transforms are $(0, 1)$ uniform distributed.

Table 7.8
Dynamic models out-of-sample backtest results: Commodity portfolio

	α	DF	DC	DG	DT	DFC	DGC	DTC
Hit Ratio	99%	0.053	0.018	0.029	0.024	0.033	0.026	0.026
	95%	0.104	0.078	0.079	0.072	0.082	0.074	0.078
	90%	0.136	0.135	0.122	0.118	0.138	0.119	0.122
ES Ratio	99%	0.839	1.065	0.972	0.954	0.953	0.950	0.983
	95%	0.863	1.049	0.965	0.939	0.942	1.072	0.975
	90%	0.851	1.052	0.974	0.960	0.981	1.011	0.984
Traffic Light		Red	Yellow	Yellow	Yellow	Yellow	Yellow	Yellow
Ind. Cov.	99%	3.157	5.264	4.780	6.870	3.557	2.702	6.290
		(0.076)	(0.022)	(0.029)	(0.009)	(0.059)	(0.100)	(0.012)
	95%	1.835	0.395	3.305	3.687	1.609	3.039	2.328
		(0.176)	(0.530)	(0.069)	(0.055)	(0.205)	(0.081)	(0.127)
	90%	1.941	4.201	2.136	1.982	2.253	2.677	2.136
		(0.164)	(0.040)	(0.144)	(0.159)	(0.133)	(0.102)	(0.144)
Unc. Cov.	99%	71.123	4.028	19.644	11.595	26.639	13.458	13.458
		(0.000)	(0.045)	(0.000)	(0.001)	(0.000)	(0.000)	(0.000)
	95%	36.831	11.232	12.204	6.914	14.254	8.530	11.232
		(0.000)	(0.001)	(0.000)	(0.009)	(0.000)	(0.003)	(0.001)
	90%	10.159	9.475	3.877	2.656	11.592	3.038	3.877
		(0.001)	(0.002)	(0.049)	(0.103)	(0.001)	(0.081)	(0.049)
Joint Test	99%	74.280	9.292	24.424	18.465	30.196	16.160	19.748
		(0.000)	(0.010)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	38.666	11.627	15.509	10.600	15.863	11.569	13.559
		(0.000)	(0.003)	(0.000)	(0.005)	(0.000)	(0.003)	(0.001)
	90%	12.099	13.676	6.013	4.638	13.845	5.716	6.013
		(0.002)	(0.001)	(0.049)	(0.098)	(0.001)	(0.057)	(0.049)
χ^2-Test		71.852	65.422	38.637	38.480	42.313	44.958	31.116
		(0.000)	(0.002)	(0.163)	(0.200)	(0.128)	(0.064)	(0.410)
AD Test		13.972	15.284	4.910	4.745	14.748	4.108	4.801
		(0.000)	(0.000)	(0.003)	(0.004)	(0.000)	(0.008)	(0.004)
KS Test		0.063	0.079	0.058	0.059	0.087	0.057	0.060
		(0.004)	(0.000)	(0.009)	(0.008)	(0.000)	(0.013)	(0.007)
Lower Tail		302.755	2.295	41.068	36.010	57.616	37.133	21.853
		(0.000)	(0.513)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)

This table reports the backtest evaluation results for the dynamic copula models applied to the commodity index data set. The models are abbreviated with D (Dynamic), F (Frank), C (Clayton), G (Gaussian), T (Student-t), FC (Frank-Clayton mixture), GC (Gaussian-Clayton mixture) and TC (Student-t-Clayton mixture). α denotes the confidence level of VaR(α). The hit ratio reflects the percentage of times when the portfolio return exceeds VaR(α). ES ratio shows whether the mean of the returns when VaR(α) is violated corresponds to the average expected shortfall in these weeks. The traffic light is the model classification of the Basel regulatory framework. The mid and lower panel lists test statistics and p-values (in parentheses) for multiple backtesting evaluation tests. Independence (unconditional) coverage is abbreviated with Ind. Cov. (Unc. Cov.). Joint Test is the joint test for conditional coverage. The lower panel reports the test statistics with p-values in parentheses of density forecast evaluation tests. χ^2-Test is Pearson's χ^2-test with 10 evenly spaced bins. AD and KS are the tests of Anderson-Darling and Kolmogorov-Smirnov. Lower Tail is the test of Christoffersen (2012) assessing the models' ability to forecast the entire lower tail (losses below the 10% quantile) of the P&L distribution.

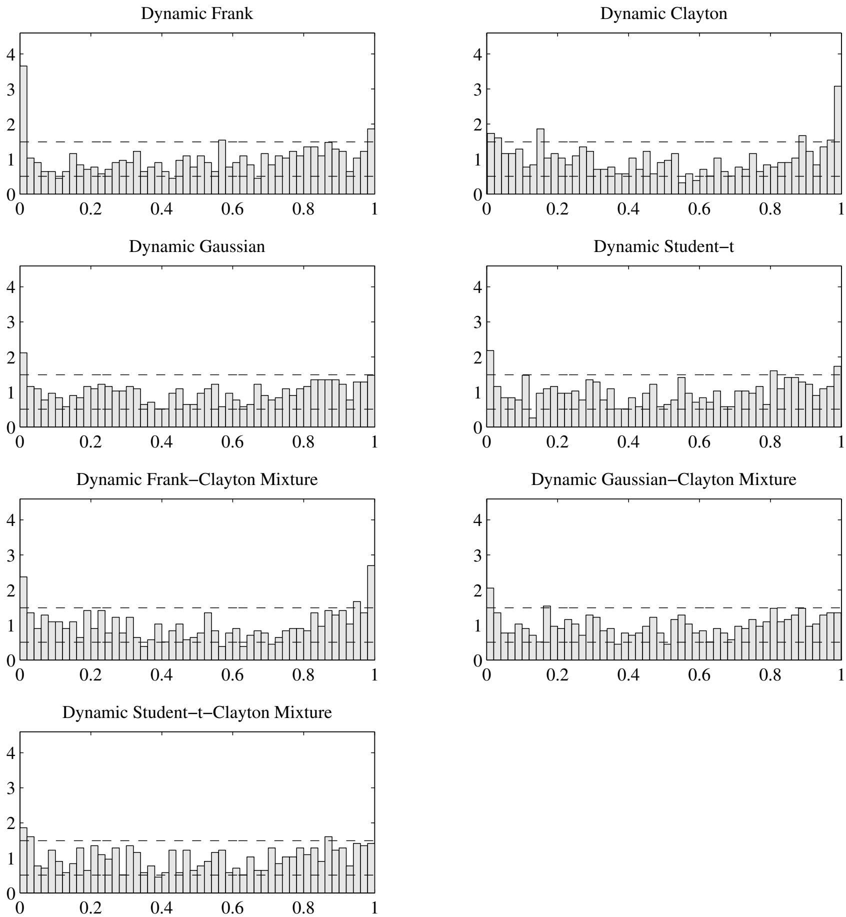

Figure 7.7: Commodity portfolio: Relative frequency of occurrence of the probability integral transforms of the commodity futures index portfolio returns using the dynamic copula models' forecasted probability density distribution. The histogram is segmented into 50 bins of equal size. The overlaid dashed lines indicate the 95% confidence interval for the heights of the individual bins under the null hypothesis that the probability integral transforms are $(0, 1)$ uniform distributed.

7.2.3 Multi Asset Classes Index Portfolio

This section discusses the results of the backtests for the multi asset classes portfolio starting with the performance analysis of the static copula models, which are listed in table 7.9. Figure 7.8 depicts the histograms of the probability integral transformed portfolio returns taken with regard to the according model's forecasted profit and loss distribution and shows that none of the static mixture copulas is capable of producing adequate forecasts for the tails of the portfolio's return distribution. The significantly elevated bars at the lowest 2% quantiles in the mixtures' histograms point to a severe underestimation of the probability of joint negative extreme returns of the portfolio constituents. This is substantiated by the high levels of the mixture models' hit ratios, their low ES ratios and the large test statistics for the unconditional coverage, joint and Lower Tail tests. Unsurprisingly, they are rated *red* according to the Basel regulatory framework's traffic light system.

The Frank copula, a further static model with a very high bar for the lowest 2% quantile in figure 7.8, also shows overall evaluation results indicating poor predictive power. The comparison of the static Gaussian with the multivariate normal benchmark model (see table 7.2) shows that incorporating skewness in the marginal models improves the VaR forecasts for $\alpha = 99\%, 95\%$ and yields a lower test statistic for the Lower Tail test. The substantially higher p-value of the χ^2-test further suggests that the Gaussian copula model provides better forecasts for the entire profit and loss distribution compared to the multivariate normal model.

The model with the highest predictive power for the lower tail of the profit and loss distribution among the static copulas is the Clayton copula. The asymmetric Archimedean model yields the best hit ratios on the 99% and 95% significance level. The histogram in figure 7.8 shows the comparatively low bar for the first 2% quantile of the Clayton model's probability integral transforms, which visualizes the fatter lower tail of the Clayton's forecasted profit and loss distribution.

However, all of the static copulas classify as *red* according to the Basel regulatory framework. The static Gaussian and the static Student-t copula may provide suitable forecasts for a major part of the profit and loss distribution since they pass both the χ^2 and the Kolmogorov-Smirnov test, but their predictions for the lower tail are far from accurate even though substantially better than the multivariate normal model on the 99% and 95% confidence levels.

The results of the regime switching models are focused on next. The comparison of the Gaussian/Gaussian regime switching model with the static Gaussian reveals that the advantage of the two-state specification is of minor nature. The latter is also branded as *red* and whilst yielding no improvement for the VaR forecasts on $\alpha = 99\%$, its hit ratios on the 95% and 90% confidence levels are more accurate in comparison to the ones of the static Gaussian. The performance of the Gaussian/Student-t model is similar to the one of the Gaussian/Gaussian model with identical results for the hit ratios and coverage test on the lower two significance levels.

The χ^2 and the Kolmogorov-Smirnov test results suggest that all two-state regime switching models with standalone copulas produce acceptable forecasts for the entire profit and loss distribution which is visualized in figure 7.9 by the flattest histograms among the Markov chain models for the G/G, G/T and T/T model. The large bars for the lowest and highest quantiles in the histograms of the other four regime switching models document that they underestimate the probability of joint extreme returns of the indices in the portfolio.

The amalgamation of the asymmetric Clayton copula with neither the Gaussian nor the Student-t regime of the G/T copula to form the two regime switching structures with mixture copulas has an improving effect on the results. To the contrary, practically all backtesting results are inferior. This shows that the static mixtures models, despite their promising in-sample fit, have poor predictive power also as a part of the regime switching structures.

In addition, the models with three regimes produce even more imprecise forecasts displaying the worst results for all backtests. Overall, the two-state models containing singular copulas are suitable to forecast the majority of the profit and loss distribution based on them passing the Kolmogorov-Smirnov and the χ^2 test. The lower tail of the portfolio return distribution however is not forecasted appropriately by any of the regime switching models, as they all classify as *red* and fail the unconditional coverage and the joint tests.

Next, the dynamic models' forecast performance is analyzed. The additional flexibility of the dynamic copulas to capture time variations in the dependence structure results in improved forecasts for the lower tail of the profit and loss distribution for all models except the Frank copula, as documented in table 7.11. The Frank copula proves to be unsuitable to forecast the multi asset classes portfolio's returns in both its static and its dynamic form.

Figure 7.10 visualizes the improvement of the dynamic mixture models' forecasts for the tails of the portfolio return distribution with substantially smaller bars for the outermost quantiles compared to the histograms of their static counterparts in figure 7.8. With the dynamic model specifications, the hit ratios and ES ratios of the mixture models are a major step closer to the expected values. In contrast to their results as static models, all dynamic mixtures pass the χ^2 and the Kolmogorov-Smirnov test. This demonstrates that the dynamic mixture copula models have more predictive power with regards to the entire profit and loss distribution of the portfolio than their static versions, which underpins the importance of allowing for time variation in copula mixture models.

The comparison of the backtest results of the dynamic Gaussian with the dynamic Gaussian-Clayton mixture reveals that the additional capability to capture asymmetries in the dependence structure yields better forecasts for both VaR and ES on the 99% and 95% levels of α. However, the dynamic Gaussian-Clayton mixture overforecasts the expected shortfall on $\alpha = 90\%$

and while the dynamic Gaussian's forecasts are smaller than the realized loss, its according ES ratio is closer to the expectation of 1.

Incorporating asymmetries into the dynamic Student-t copula with the dynamic Student-t-Clayton mixture improves the VaR forecasts on the 95% significance level and the forecasts for the expected shortfall on both $\alpha = 95\%$ and $\alpha = 99\%$. The best VaR forecasts are produced by the dynamic Clayton copula model, which stands out as the only model attaining a *yellow* classification according to the Basel regulatory framework among all dynamic models, even among all tested models for the multi asset classes portfolio. While the Clayton copula is not appropriate to forecast the entire profit and loss distribution neither in its static nor in its dynamic version, as both fail the χ^2 test and the Kolmogorov-Smirnov test, it is the model with the most predictive power for the lowest 10% quantile of the return distribution of the multi asset classes portfolio. Compared to the static form, the backtest results show that allowing for time variation in the Clayton copula increases the accuracy of the VaR forecasts on all three significance levels.

Table 7.9
Static models out-of-sample backtest results: Multi asset classes portfolio

	α	F	C	G	T	FC	GC	TC
Hit Ratio	99%	0.060	0.026	0.036	0.036	0.124	0.131	0.123
	95%	0.120	0.084	0.091	0.101	0.192	0.192	0.192
	90%	0.160	0.157	0.152	0.151	0.246	0.255	0.246
ES Ratio	99%	0.773	0.953	0.869	0.915	0.760	0.779	0.763
	95%	0.828	1.023	0.928	0.993	0.746	0.741	0.746
	90%	0.830	1.054	0.955	0.964	0.762	0.786	0.755
Traffic Light		Red	Red	Red	Red	Red	Red	Red
Ind. Cov.	99%	2.369	3.066	3.450	3.450	1.260	0.451	2.370
		(0.124)	(0.080)	(0.063)	(0.063)	(0.262)	(0.502)	(0.124)
	95%	0.140	3.453	1.953	0.596	2.183	1.014	1.545
		(0.708)	(0.063)	(0.162)	(0.440)	(0.140)	(0.314)	(0.214)
	90%	2.087	1.190	0.659	0.817	0.008	0.240	0.087
		(0.149)	(0.275)	(0.417)	(0.366)	(0.927)	(0.624)	(0.768)
Unc. Cov.	99%	82.099	12.112	28.015	28.015	287.602	314.319	282.337
		(0.000)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	52.476	14.349	20.358	30.204	179.683	179.683	179.683
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	90%	23.870	21.771	18.783	17.831	124.188	137.416	124.188
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Joint Test	99%	84.468	15.178	31.465	31.465	288.863	314.770	284.707
		(0.000)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	52.616	17.802	22.311	30.801	181.867	180.697	181.228
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	90%	25.957	22.961	19.442	18.648	124.196	137.656	124.275
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
χ^2-Test		85.354	55.870	24.398	33.921	72.631	71.009	44.386
		(0.000)	(0.008)	(0.830)	(0.375)	(0.003)	(0.005)	(0.413)
AD Test		15.085	18.196	8.599	8.971	133.316	135.057	133.707
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
KS Test		0.074	0.067	0.056	0.058	0.161	0.162	0.163
		(0.001)	(0.004)	(0.022)	(0.016)	(0.000)	(0.000)	(0.000)
Lower Tail		265.505	18.459	79.472	70.767	620.521	633.204	629.283
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)

This table reports the backtest evaluation results for the static copula models applied to the multi asset classes data set. The copula models are abbreviated with F (Frank), C (Clayton), G (Gaussian), T (Student-t), FC (Frank-Clayton mixture), GC (Gaussian-Clayton mixture) and TC (Student-t-Clayton mixture). α denotes the confidence level of VaR(α). The hit ratio reflects the percentage of times when the portfolio return exceeds VaR(α). ES ratio shows whether the mean of the returns when VaR(α) is violated corresponds to the average expected shortfall in these weeks. The traffic light is the model classification of the Basel regulatory framework. The mid and lower panel lists test statistics and p-values (in parentheses) for multiple backtesting evaluation tests. Independence (unconditional) coverage is abbreviated with Ind. Cov. (Unc. Cov.). Joint Test is the joint test for conditional coverage. The lower panel reports the test statistics with p-values in parentheses of density forecast evaluation tests. χ^2-Test is Pearson's χ^2-test with 10 evenly spaced bins. AD and KS are the tests of Anderson-Darling and Kolmogorov-Smirnov. Lower Tail is the test of Christoffersen (2012) assessing the models' ability to forecast the entire lower tail (losses below the 10% quantile) of the P&L distribution.

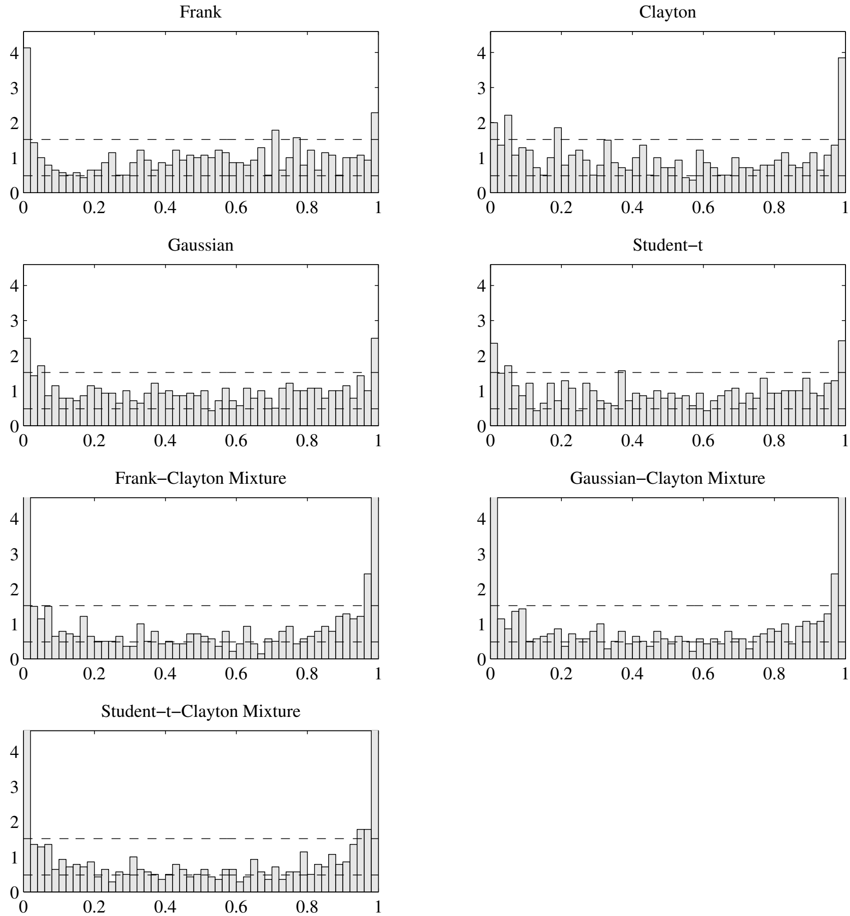

Figure 7.8: Multi asset classes: Relative frequency of occurrence of the probability integral transforms of the multi asset classes portfolio returns taken with respect to the static copula models' forecasted probability density distribution. The histogram is segmented into 50 bins of equal size. The overlaid dashed lines indicate the 95% confidence interval for the heights of the individual bins under the null hypothesis that the probability integral transforms are $(0, 1)$ uniform distributed.

Table 7.10
Regime switching models out-of-sample backtest results: Multi asset classes portfolio

	α	G/G	G/T	T/T	T/GC	G/TC	G/T/C	G/C/F
Hit Ratio	99%	0.036	0.036	0.031	0.083	0.063	0.125	0.127
	95%	0.085	0.085	0.093	0.128	0.128	0.198	0.192
	90%	0.145	0.150	0.150	0.197	0.174	0.248	0.246
ES Ratio	99%	0.870	0.881	0.857	0.807	0.844	0.767	0.782
	95%	0.926	0.922	0.961	0.775	0.870	0.759	0.757
	90%	0.957	0.973	0.976	0.861	0.858	0.760	0.763
Traffic Light		Red	Red	Red	Red	Red	Red	Red
Ind. Cov.	99%	3.450	3.450	4.723	1.171	1.715	1.065	0.377
		(0.063)	(0.063)	(0.030)	(0.279)	(0.190)	(0.302)	(0.539)
	95%	3.115	0.203	1.708	3.234	0.724	0.733	0.281
		(0.078)	(0.652)	(0.191)	(0.072)	(0.395)	(0.392)	(0.596)
	90%	0.477	1.637	0.992	0.907	0.257	0.046	0.013
		(0.490)	(0.201)	(0.319)	(0.341)	(0.612)	(0.830)	(0.911)
Unc. Cov.	99%	28.015	28.015	20.626	146.829	89.561	292.894	298.212
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	15.479	15.479	21.665	64.338	64.338	191.904	179.683
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	90%	14.248	16.902	16.902	58.501	35.620	126.355	124.188
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Joint Test	99%	31.465	31.465	25.348	148.000	91.276	293.959	298.589
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	18.594	15.682	23.373	67.572	65.063	192.637	179.965
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	90%	14.725	18.538	17.894	59.408	35.877	126.401	124.201
		(0.001)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
χ^2-Test		27.621	23.925	29.093	40.273	35.299	55.974	50.309
		(0.641)	(0.847)	(0.564)	(0.414)	(0.454)	(0.089)	(0.207)
AD Test		7.028	7.941	7.748	52.912	29.154	136.227	132.362
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
KS Test		0.055	0.056	0.057	0.102	0.084	0.159	0.161
		(0.027)	(0.024)	(0.022)	(0.000)	(0.000)	(0.000)	(0.000)
Lower Tail		61.145	68.402	53.432	337.326	216.795	645.351	600.885
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)

This table reports the backtest evaluation results for the regime switch copula models applied to the multi asset classes data set. The copula models are abbreviated with G (Gaussian), T (Student-t), F (Frank), C (Clayton), GC (Gaussian-Clayton mixture) and TC (Student-t-Clayton mixture). G/T/C and G/C/F are three-regime models. α denotes the confidence level of VaR(α). The hit ratio reflects the percentage of times when the portfolio return exceeds VaR(α). ES ratio shows whether the mean of the returns when VaR(α) is violated corresponds to the average expected shortfall in these weeks. The traffic light is the model classification of the Basel regulatory framework. The mid and lower panel lists test statistics and p-values (in parentheses) for multiple backtesting evaluation tests. Independence (unconditional) coverage is abbreviated with Ind. Cov. (Unc. Cov.). Joint Test is the joint test for conditional coverage. The lower panel reports the test statistics with p-values in parentheses of density forecast evaluation tests. χ^2-Test is Pearson's χ^2-test with 10 evenly spaced bins. AD and KS are the tests of Anderson-Darling and Kolmogorov-Smirnov. Lower Tail is the test of Christoffersen (2012) assessing the models' ability to forecast the entire lower tail (losses below the 10% quantile) of the P&L distribution.

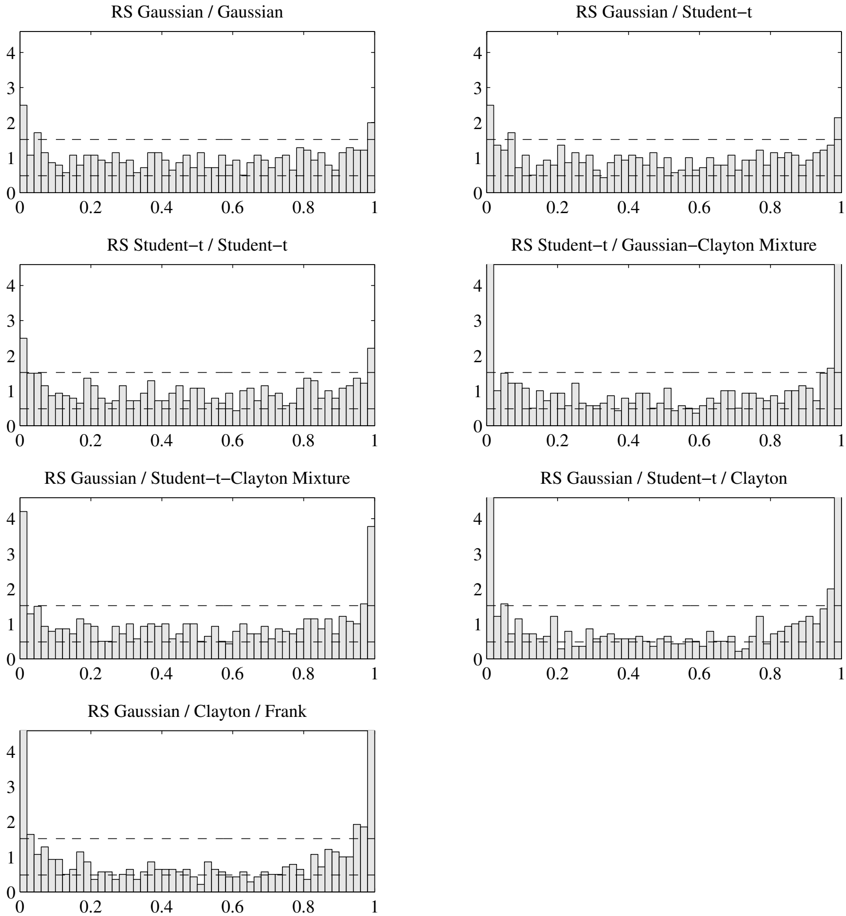

Figure 7.9: Multi Asset Classes: Relative frequency of occurrence of the probability integral transforms of the multi asset classes portfolio returns taken with respect to the regime switching (RS) copula models' forecasted probability density distribution. The histogram is segmented into 50 bins of equal size. The overlaid dashed lines indicate the 95% confidence interval for the heights of the individual bins under the null hypothesis that the probability integral transforms are $(0, 1)$ uniform distributed.

Table 7.11
Dynamic models out-of-sample backtest results: Multi asset classes portfolio

	α	DF	DC	DG	DT	DFC	DGC	DTC
Hit Ratio	99%	0.061	0.022	0.036	0.034	0.033	0.033	0.034
	95%	0.115	0.078	0.088	0.090	0.091	0.083	0.080
	90%	0.151	0.154	0.148	0.148	0.154	0.148	0.151
ES Ratio	99%	0.799	0.912	0.859	0.886	0.890	0.883	0.919
	95%	0.838	0.989	0.927	0.938	0.970	0.959	1.041
	90%	0.831	1.061	0.954	0.948	1.002	1.132	1.261
Traffic Light		Red	Yellow	Red	Red	Red	Red	Red
Ind. Cov.	99%	3.921	3.891	3.450	3.844	4.267	1.552	3.844
		(0.048)	(0.049)	(0.063)	(0.050)	(0.039)	(0.213)	(0.050)
	95%	0.407	0.779	1.351	1.151	0.969	1.171	1.590
		(0.523)	(0.377)	(0.245)	(0.283)	(0.325)	(0.279)	(0.207)
	90%	0.817	1.633	0.641	0.641	0.518	0.260	0.382
		(0.366)	(0.201)	(0.423)	(0.423)	(0.472)	(0.610)	(0.536)
Unc. Cov.	99%	85.805	8.519	28.015	25.464	23.000	23.000	25.464
		(0.000)	(0.004)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	46.906	10.204	17.848	19.085	20.358	13.256	11.183
		(0.000)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)
	90%	17.831	19.757	15.994	15.994	19.757	15.994	17.831
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Joint Test	99%	89.726	12.410	31.465	29.308	27.267	24.552	29.308
		(0.000)	(0.002)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	47.314	10.984	19.198	20.237	21.327	14.426	12.772
		(0.000)	(0.004)	(0.000)	(0.000)	(0.000)	(0.001)	(0.002)
	90%	18.648	21.390	16.635	16.635	20.276	16.254	18.213
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
χ^2-Test		96.927	52.061	30.634	29.344	34.861	33.713	40.885
		(0.000)	(0.019)	(0.485)	(0.551)	(0.380)	(0.338)	(0.110)
AD Test		12.930	15.792	7.732	7.557	12.664	7.160	7.240
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
KS Test		0.071	0.068	0.058	0.061	0.059	0.054	0.060
		(0.002)	(0.003)	(0.018)	(0.011)	(0.014)	(0.033)	(0.012)
Lower Tail		275.164	18.268	65.900	49.251	58.280	64.487	51.655
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)

This table reports the backtest evaluation results for the dynamic copula models applied to the multi asset classes index data set. The models are abbreviated with D (Dynamic), F (Frank), C (Clayton), G (Gaussian), T (Student-t), FC (Frank-Clayton mixture), GC (Gaussian-Clayton mixture) and TC (Student-t-Clayton mixture). α denotes the confidence level of VaR(α). The hit ratio reflects the percentage of times when the portfolio return exceeds VaR(α). ES ratio shows whether the mean of the returns when VaR(α) is violated corresponds to the average expected shortfall in these weeks. The traffic light is the model classification of the Basel regulatory framework. The mid and lower panel lists test statistics and p-values (in parentheses) for multiple backtesting evaluation tests. Independence (unconditional) coverage is abbreviated with Ind. Cov. (Unc. Cov.). Joint Test is the joint test for conditional coverage. The lower panel reports the test statistics with p-values in parentheses of density forecast evaluation tests. χ^2-Test is Pearson's χ^2-test with 10 evenly spaced bins. AD and KS are the tests of Anderson-Darling and Kolmogorov-Smirnov. Lower Tail is the test of Christoffersen (2012) assessing the models' ability to forecast the entire lower tail (losses below the 10% quantile) of the P&L distribution.

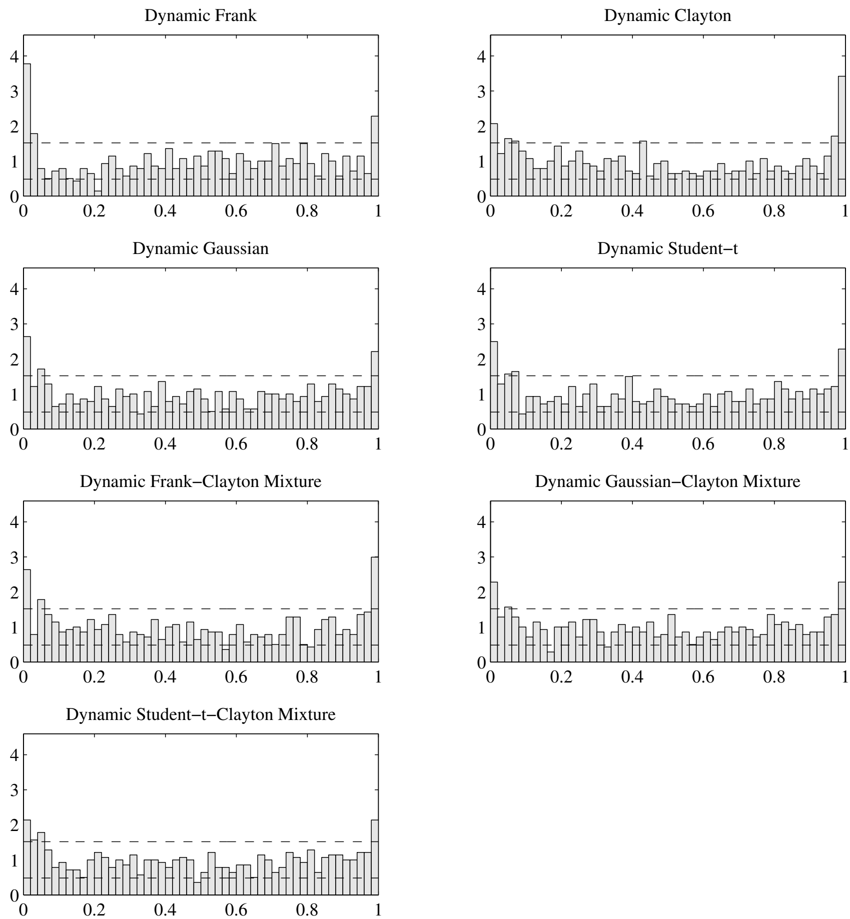

Figure 7.10: Multi asset classes portfolio: Relative frequency of occurrence of the probability integral transforms of the multi asset classes portfolio returns taken with respect to the dynamic copula models' forecasted probability density distribution. The histogram is segmented into 50 bins of equal size. The overlaid dashed lines indicate the 95% confidence interval for the heights of the individual bins under the null hypothesis that the probability integral transforms are $(0, 1)$ uniform distributed.

7.3 Crisis Forecast Performance

The standard risk management models employed by the financial industry have drawn wide criticism for their performance during the financial crisis (see e.g. Contreras, 2010; Das et al., 2013; Skoglund et al., 2010). Many financial institutions reported a sharp increase in the number of VaR violations in the unfolding of the crisis, which underpinned the perception that risk forecasting models perform well except in times of crises. While in the previous section, the scope of the backtest covered the entire time frame evaluating out-of-sample forecasts from 1998 until 2013, this section concentrates on the performance of the presented models during the last financial crisis. To ensure comparability, the same backtest procedures including the same univariate model specifications as in section 7.2 (see table 7.1) are used in this section. Following the Basel supervisory framework (2013), which demands backtests of the risk model based on the VaR measure on the 99% confidence level, this section focuses on the performance of the models on $\alpha = 99\%$.

Firstly, the crisis forecast performance analysis focuses on the out-of-sample accuracy of the risk forecasts of the different models during the crisis period from January 7, 2007 until January 5, 2011. The time frame therewith covers the unfolding of the financial turmoil from the disruptions in the subprime mortgage market to the virulent global financial crisis. Secondly, the changes in VaR violations of the presented models in reaction to the outbreak of the financial crisis are investigated over time.

Table 7.14 summarizes the results of the backtests for all presented copula models during the crisis period for the equity index portfolio. The therein reported measures refer to the 99% significance level with the exception of the Lower Tail test, which assesses the model's capability to forecast the density of the profit and loss distribution below the 10% quantile. According to the Basel three-zones approach, the Frank copula model does not produce acceptable forecasts during the crisis neither in its dynamic nor in its static form. Furthermore, all static mixture copulas, the dynamic Frank-Clayton mixture, the three-state regime switching models and the two-state models with mixture regimes are also labeled *red*, which means that the accuracy of their VaR forecasts is not acceptable either. The inaccuracy of these models' forecasts is substantiated by their failure to pass the unconditional coverage and the joint test as well as the Lower Tail test.

Eight models qualify as *yellow*, among which seven yield the identical hit ratio of 2.9%: the Gaussian copula in the static and dynamic version, the regime switching models G/G, G/T and T/T, the dynamic Student-t copula and the dynamic Gaussian-Clayton mixture. These seven models' VaR forecasts are violated almost 50% less than those of the multivariate normal benchmark (see table 7.13). While they are identical with regards to their VaR forecasts, their expected shortfall ratios reveal the differences in the accuracy of the average forecasted loss given a VaR violation: the least accurate ES ratio among the *yellow* models is found with the static Gaussian copula and the closest to the expected value of 1 is the ES ratio of the dynamic Gaussian-Clayton mixture followed by the dynamic Student-t copula. Note that the direct comparisons of the expected shortfall ratios is expressive in this case since the compared models all have the same hit ratio. As the ES ratio hinges on the hit ratio, a sole comparison of the former without considering the latter is not very meaningful.

Three copulas, however, are distinctly more accurate during the financial crisis than all other models as they are the only ones to score *green* according to the Basel framework: the static Clayton copula, the dynamic Clayton copula and the dynamic Student-t-Clayton mixture. Their results for the coverage tests and the joint test confirm their superiority, which is further substantiated by their results for the Lower Tail test. The dynamic Clayton copula clearly produces the most accurate VaR forecasts during the financial crisis with a hit ratio exactly equal to the theoretical expectation of 1%. The highest p-value for the Lower Tail test confirms the dynamic Clayton's superior capabilities to forecast the lowest decile of the equity portfolio's profit and loss distribution. While the results for the static Clayton highlight the importance of modeling lower tail dependence in a multivariate setting, the substantially increased forecast accuracy of the dynamic Clayton copula emphasizes the significance of also modeling time variation in the dependence structure of the equity index portfolio.

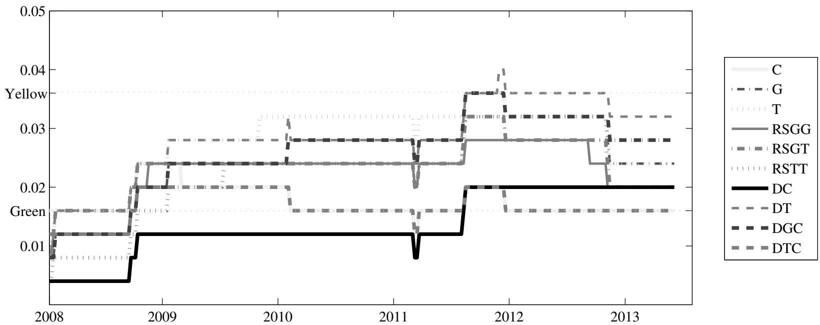

Figure 7.11: Equity portfolio: The plot shows the evolution of the hit ratios of several copula models starting from the outbreak of the financial crisis. Depicted are the hit ratios at the 99% level over a rolling window of 250 returns of all those models which classified as "green" by the Basel II framework at the beginning of 2008. Hit ratios below the dotted line labeled "Green" respectively "Yellow" are classified accordingly by the Basel regulatory framework. The models with ratios above the line "yellow" are categorized as "red" according to the Basel traffic light approach.

To investigate the models' reaction to the unfolding of the financial crisis, figure 7.11 visualizes the hit ratios over time. The plot shows the hit ratios at the 99% level, calculated on a rolling window basis using the preceding 250 returns in allusion to the Basel regulatory framework (Basel Committee on Banking Supervision, 2013). The plot includes the hit ratios of those models which qualify as *green* at the beginning of 2008. Naturally, the hit ratio of an ideal model would remain close to the expected hit ratio of 0.01 showing little to no reaction to the outbreak of the crisis. The three zones of the Basel framework are marked on the y-axis of the plot: models with a hit ratio in time t below the mark "Green" respectively "Yellow" classify accordingly, while models whose hit ratios in time t exceed "Yellow" are ranked *red*. All depicted models remain in the *green* zone until September 2008, when the hit ratios increase sharply in reaction to the bankruptcy of the investment bank Lehman Brothers and the collapse of the insurance firm AIG. With the exception of the dynamic Clayton copula, all depicted models turn from green to yellow in only a few weeks. The dynamic Clayton's hit ratio however shows the smallest reaction to the outbreak of the global crisis preserving its *green* status until the aggravation of the Eurozone crisis in mid-2011. At the same time, the static Gaussian and the static Student-t show the strongest immediate reaction to eruption of the crisis. The second and third best model according to the results in table 7.14 are easily identifiable in figure 7.11 as together with the dynamic Clayton their hit ratios are closest to the expected value of 0.01: the dynamic Student-t-Clayton mixture and the static Clayton. Whilst the dynamic Clayton model provides the most accurate forecasts during the global financial crisis remaining *green* until mid 2011 the other two models cope better with the aftermath of the Eurozone crisis.

The models' performance during the financial crisis for the commodity futures index portfolio is listed in table 7.15. None of the copula models attains a *green* classification according to the Basel regulatory framework. 19 out of 21 tested models rank *red*, which means that the accuracy of their VaR forecasts for the commodities between 2007 and 2011 is not acceptable. Even though all are labeled *red*, the differences in the results of the static Gaussian, G/G regime switching and dynamic Gaussian copula highlight the advantages of a time varying specification of the Gaussian copula as the static version shows the most inaccurate hit ratio among the three models. The G/G regime switching version and the dynamic version of the Gaussian copula produce VaR forecasts of identical accuracy, but the dynamic Gaussian's ES ratio being closer to 1 and its Lower Tail test statistic show that the dynamic version's forecast of the profit and loss distribution's density in the lowest quantiles is somewhat more accurate. For the Student-t copula, the ability to switch between two states (T/T) does not result in a better hit ratio compared to the static setup, the dynamic form however clearly beats the static and regime switching form in terms of hit ratio, ES ratio and Lower Tail test statistic.

The best two models both achieve a *yellow* classification: the static and the dynamic Clayton copula. The dominance of this asymmetric copula is manifested by far more accurate hit ratios compared to all other models. Furthermore, both static and dynamic Clayton are the only two copulas to pass the Lower Tail test, which shows the importance of capturing lower tail dependence. Indeed, ranks three and four in terms of hit ratio and Lower Tail test statistic are taken by the dynamic Gaussian-Clayton mixture and the Student-t Clayton mixture, whose Clayton component enables them to model asymmetries and lower tail dependence.

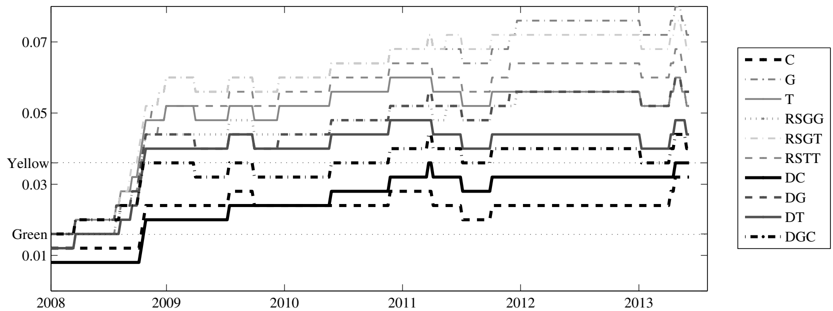

Figure 7.12: Commodity portfolio: The plot shows the evolution of the hit ratios of several copula models starting from the outbreak of the financial crisis. Depicted are the hit ratios at the 99% level over a rolling window of 250 returns of all those models which classified as "green" by the Basel II framework at the beginning of 2008. Hit ratios below the dotted line labeled "green" respectively "yellow" are classified accordingly by the Basel regulatory framework. The models with ratios above the line "yellow" are categorized as "red" according to the Basel traffic light approach.

The dynamic version of the Clayton copula does not increase the accuracy of commodity portfolio return forecasts compared to the static version. The former yields a less accurate hit ratio and further overforecasts the loss given a VaR violation. Figure 7.12 depicts the evolution of the hit ratios of a selection of models for the commodity index data from the outbreak to the aftermath of the last financial crisis. The depicted hit ratios are the ones of those models which qualified as *green* according to the Basel regulatory framework at the outbreak of the financial crisis: the dynamic Gaussian, the dynamic Student-t and the dynamic Clayton with their static counterparts, the regime switching models with two regimes: Gaussian/Gaussian, Gaussian/Student-t and Student-t/Student-t and finally the dynamic Gaussian-Clayton mixture. None of these models preserves its Basel traffic light category during the crisis, however, there are subtle differences in the reaction to the outbreak of the financial turmoil.

Only two of the ten models do not become classified as *red* in their immediate reaction to the beginning of the financial turmoil: The dynamic Clayton and the static Clayton copula. While the dynamic Clayton was better capable of handling the initial impacts of the crisis from 2008 to 2010, the static version shows the best hit ratio in the aftermath of the crisis. Note that these two models are also the closest to the expected hit ratio of 1% in the first half of the year 2008. The dynamic Gaussian-Clayton mixture also faces a deterioration from *green* to *yellow* and manages to maintain the *yellow* classification until the end of 2010, when it becomes red for most of the remainder of the observation period. All the other models in the figure deteriorated from an acceptable model (*green*) to an unacceptable one (*red*) in just a few months. The most pronounced deterioration during the outbreak of the crisis is shown by the Gaussian and the regime switching Gaussian/Student-t copula models, which prove to be least capable to appropriately forecast the portfolios' returns during the beginning crisis.

Of all the considered dependence structures, the three models which performed best in terms of hit ratio of the commodity futures portfolio during the times of crises are asymmetric models capable of capturing lower tail dependence. Note that the models whose hit ratio deteriorated drastically at the outbreak of the crisis are all (combinations of) elliptical copulas. The superiority of the static and dynamic Clayton copula established in the analysis of the backtesting results over the entire data set covering more than 16 years out-of-sample is thus confirmed also by the analysis of the models' reaction to the outbreak of the crisis.

Next, the focus is laid on the crisis performance of the copula models for the multi asset classes portfolio. The according results are listed in table 7.16 and show that among the static copulas only the Clayton attains a *yellow* traffic light, whilst the other static models are all rated *red* displaying high hit ratios. Despite the increased accuracy of the static Clayton's VaR forecasts, the copula fails the unconditional coverage test, the joint test and the Lower Tail test, as do all the other static models.

The Markov two-state versions of the elliptical copulas do not ameliorate the quality of the forecasts: the G/G regime switching model yields the same hit ratio as the static Gaussian and the T/T copula's VaR forecasts are identical to the ones of static Student-t copula. The lower ES ratios of the regime switching models furthermore show that they are less capable of predicting the loss given a VaR violation compared to their static counterparts. Like the static models, the complete range of regime switching models fail three of the four tests indicating that they are neither suitable to forecast the 99% VaR nor the density in the lowest decile of the multi asset classes portfolio's profit and loss distribution during the financial turmoil.

With the dynamic Clayton, the dynamic Frank-Clayton mixture and the dynamic Gaussian-Clayton mixture three out of seven dynamic models attain the *yellow* Basel traffic light categorization. Together with the static Clayton copula, these dependence models are distinctly better capable of forecasting the VaR of the multi asset classes portfolio. Note that none of the symmetrical models comes close to the results of the Archimedean Clayton copula neither in static nor in dynamic form. This demonstrates that asymmetric dependence which captures lower tail dependence is an crucial feature of a dependence model to accurately forecast the multivariate distribution of the portfolio's returns in times of crises.

The dynamic Clayton copula furthermore stands out compared to the static Clayton and in comparison to all other tested models firstly due to the most accurate hit ratio, secondly because it is the only copula to pass the unconditional coverage and the joint test and thirdly it is the sole model to pass the Lower Tail test. The dynamic Clayton is therewith the only model for which the null hypothesis of accurate forecasts of the entire lower tail of the multi asset classes portfolio's profit and loss distribution cannot be rejected.

While allowing for skewness and kurtosis in the marginal distributions reduces the 99% hit ratio from 7.7% (multivariate normal) to 6.2% (Gaussian copula), allowing for time variation as in the dynamic Gaussian model further decreases the hit ratio to 5.7%. Incorporating asymmetries in the dynamic dependence structure as in the dynamic Gaussian-Clayton mixture copula further lowers the hit ratio to 4.3%. The most accurate VaR forecasts are obtained by employing the asymmetric dynamic Clayton model, which yields a 99% hit ratio of 2.9%. Employing the dynamic Clayton copula model instead of the multivariate normal model hence reduces the number of VaR violations during the financial crisis by more than 62%.

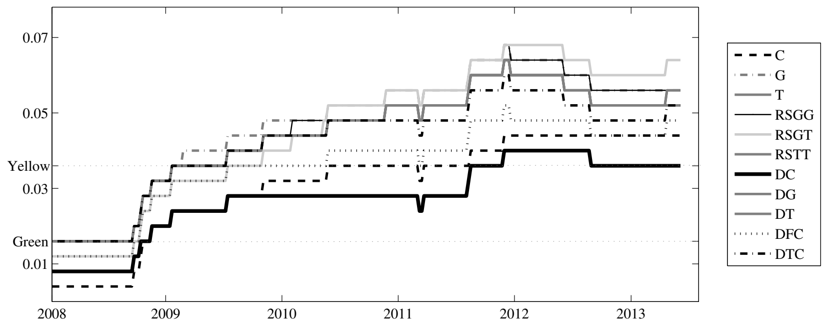

Figure 7.13: Multi asset classes portfolio: The plot shows the evolution of the hit ratios of several copula models starting from the outbreak of the financial crisis. Depicted are the hit ratios at the 99% level over a rolling window of 250 returns of all those models which classified as "green" by the Basel II framework at the beginning of 2008. Hit ratios below the dotted line labeled "green" respectively "yellow" are classified accordingly by the Basel regulatory framework. The models with ratios above the line "yellow" are categorized as "red" according to the Basel traffic light approach.

The dominance of the dynamic Clayton in terms of VaR forecast accuracy is depicted in figure 7.13. While the fastest deterioration among the depicted hit ratios is displayed by the Gaussian copula, the hit ratio of the dynamic Clayton clearly shows the least reaction to the outbreak of the global crisis and remains the most accurate until the end of the observation period. The static Clayton overforecasts the VaR before September 2008 and yields a higher hit ratio than its dynamic counterpart from 2010 until 2013.

It is remarkable that the multivariate Clayton copula model in both static and dynamic version produces VaR(99%)-forecasts which are more accurate than those of all other models at any point in time during the financial turmoil.

Table 7.12 relates the results of the models for the financial crisis period (upper panel) to the overall performance of the models documented in the previous section 7.2 (lower panel), by listing the top rankings of the models for all portfolios according to the accuracy of their VaR(99%) forecasts. The comparison shows, that the models' ranking during the financial crisis is largely consistent with the overall ranking. The Clayton copula stands out, as it is ranked first for all portfolios in either static (commodity portfolio) or dynamic form (equity and multi asset classes portfolio). In general, the table demonstrates that the fully dynamic copulas yield VaR forecasts of superior accuracy compared to both the regime switching and the static copulas. It further displays that asymmetric dynamic copulas generally dominate the top five rankings of the crisis and the overall performance.

Table 7.12
Out-of-sample forecast accuracy ranking

	1.	2.	3.	4.	5.
Financial Crisis					
Equities	DC	DTC	C	DGC	DT
Commodities	C	DC	DGC	DTC	DG
Multi Asset Classes	DC	C	DGC	DFC	DTC
Overall					
Equities	DC	DTC	C	DGC	DT
Commodities	C	DC	DT	DTC	DGC
Multi Asset Classes	DC	DFC	DGC	DTC	DT

This table presents the top rankings of the out-of-sample VaR(99%) forecast accuracy for the static, regime-switching and fully dynamic copulas for the three portfolios under consideration. The rankings for models with identical hit ratios are determined by the accuracy of their ES ratio. The upper panel refers to the financial crisis out-of-sample performance while the lower panel refers to the overall out-of-sample performance of the models (see section 7.2). The copulas are abbreviated: Gaussian (G), Student-t (T), Clayton (C), Frank-Clayton mixture (FC), Gaussian-Clayton mixture (GC) and Student-t-Clayton mixture (TC). The prefixed D denotes fully dynamic copula models.

Table 7.13
Financial crisis: Static multivariate normal model out-of-sample backtest results for $\alpha = 99\%$

	Equities	Commodities	Multi Asset Classes
Hit Ratio	0.057	0.067	0.077
ES Ratio	0.945	0.904	0.828
Traffic Light	Red	Red	Red
Ind. Cov.	0.137	3.542	0.054
	(0.711)	(0.060)	(0.817)
Unc. Cov.	22.608	30.132	38.270
	(0.000)	(0.000)	(0.000)
Joint Test	22.746	33.674	38.324
	(0.000)	(0.000)	(0.000)
Lower Tail	31.614	50.977	85.887
	(0.000)	(0.000)	(0.000)

This table reports the backtest evaluation results for the forecasting period from January 2007 until January 2011 for the static multivariate normal benchmark model applied to the three index portfolios. The confidence level of VaR is $\alpha = 99\%$. The hit ratio reflects the percentage of times when the portfolio return exceeds VaR(α). ES ratio shows whether the mean of the returns when VaR(α) is violated corresponds to the average expected shortfall in these weeks. The traffic light is the model classification of the Basel regulatory framework. Independence (unconditional) coverage is abbreviated with Ind. Cov. (Unc. Cov.). Joint Test is the test for conditional coverage. Lower Tail is the test of Christoffersen (2012) assessing the models' ability to forecast the lowest decile of the P&L distribution. P-values are given in parentheses.

Table 7.14
Financial crisis: Equity portfolio out-of-sample backtest results for $\alpha = 99\%$

Static	F	C	G	T	FC	GC	TC
Hit Ratio	0.077	0.019	0.029	0.033	0.201	0.211	0.206
ES Ratio	0.811	0.915	0.909	0.997	0.702	0.722	0.720
Traffic Light	Red	Green	Yellow	Yellow	Red	Red	Red
Ind. Cov.	0.025	0.157	0.356	1.545	3.866	2.515	4.721
	(0.875)	(0.692)	(0.550)	(0.214)	(0.049)	(0.113)	(0.030)
Unc. Cov.	38.270	1.391	4.910	7.220	180.470	193.447	186.929
	(0.000)	(0.238)	(0.027)	(0.007)	(0.000)	(0.000)	(0.000)
Joint Test	38.295	1.548	5.266	8.765	184.336	195.961	191.650
	(0.000)	(0.461)	(0.072)	(0.012)	(0.000)	(0.000)	(0.000)
Lower Tail	133.466	6.872	16.678	9.537	446.127	394.398	413.505
	(0.000)	(0.076)	(0.001)	(0.023)	(0.000)	(0.000)	(0.000)

Regime Switching	G/G	G/T	T/T	T/GC	G/TC	G/T/C	G/C/F
Hit Ratio	0.029	0.029	0.029	0.196	0.191	0.048	0.096
ES Ratio	0.964	0.925	0.943	0.667	0.679	0.804	0.814
Traffic Light	Yellow	Yellow	Yellow	Red	Red	Red	Red
Ind. Cov.	0.356	2.104	0.296	3.059	3.757	3.189	16.435
	(0.550)	(0.147)	(0.586)	(0.080)	(0.053)	(0.074)	(0.000)
Unc. Cov.	4.910	4.910	4.910	174.070	167.731	15.795	56.120
	(0.027)	(0.027)	(0.027)	(0.000)	(0.000)	(0.000)	(0.000)
Joint Test	5.266	7.014	5.206	177.129	171.488	18.984	72.555
	(0.072)	(0.030)	(0.074)	(0.000)	(0.000)	(0.000)	(0.000)
Lower Tail	11.819	9.272	14.274	480.834	408.501	52.227	113.822
	(0.008)	(0.026)	(0.003)	(0.000)	(0.000)	(0.000)	(0.000)

Dynamic	DF	DC	DG	DT	DFC	DGC	DTC
Hit Ratio	0.081	0.010	0.029	0.029	0.048	0.029	0.014
ES Ratio	0.855	0.769	0.935	0.983	1.009	0.996	0.818
Traffic Light	Red	Green	Yellow	Yellow	Red	Yellow	Green
Ind. Cov.	0.141	0.039	2.104	0.356	0.485	0.356	0.088
	(0.707)	(0.844)	(0.147)	(0.550)	(0.486)	(0.550)	(0.767)
Unc. Cov.	42.547	0.004	4.910	4.910	15.795	4.910	0.353
	(0.000)	(0.950)	(0.027)	(0.027)	(0.000)	(0.027)	(0.553)
Joint Test	42.688	0.043	7.014	5.266	16.280	5.266	0.441
	(0.000)	(0.979)	(0.030)	(0.072)	(0.000)	(0.072)	(0.802)
Lower Tail	118.078	4.043	9.350	11.086	36.622	6.590	6.376
	(0.000)	(0.257)	(0.025)	(0.011)	(0.000)	(0.086)	(0.095)

This table reports the backtest evaluation results for the forecasting period from January 2007 until January 2011 for all copula models applied to the equity index portfolio. The copula models are abbreviated with G (Gaussian), T (Student-t), F (Frank), C (Clayton), FC (Frank-Clayton mixture), GC (Gaussian-Clayton mixture) and TC (Student-t-Clayton mixture). The prefixed D denotes dynamic copulas. The confidence level of VaR is $\alpha = 99\%$. The hit ratio reflects the percentage of times when the portfolio return exceeds VaR(α). ES ratio shows whether the mean of the returns when VaR(α) is violated corresponds to the average expected shortfall in these weeks. The traffic light is the model classification of the Basel regulatory framework. Independence (unconditional) coverage is abbreviated with Ind. Cov. (Unc. Cov.). Joint Test is the test for conditional coverage. Lower Tail is the test of Christoffersen (2012) assessing the models' ability to forecast the lowest decile of the P&L distribution. P-values are given in parentheses.

Table 7.15
Financial crisis: Commodity portfolio backtest results for $\alpha = 99\%$

Static	F	C	G	T	FC	GC	TC
Hit Ratio	0.086	0.024	0.072	0.062	0.120	0.124	0.115
ES Ratio	0.807	1.050	1.010	0.978	0.702	0.723	0.716
Traffic Light	Red	Yellow	Red	Red	Red	Red	Red
Ind. Cov.	3.459	9.187	2.859	4.326	0.509	0.299	0.779
	(0.063)	(0.002)	(0.091)	(0.038)	(0.475)	(0.584)	(0.377)
Unc. Cov.	46.951	2.944	34.129	26.288	80.901	86.145	75.749
	(0.000)	(0.086)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Joint Test	50.410	12.131	36.989	30.614	81.411	86.444	76.528
	(0.000)	(0.002)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Lower Tail	162.158	5.792	39.675	32.527	221.283	215.249	196.192
	(0.000)	(0.122)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Regime Switching	G/G	G/T	T/T	T/GC	G/TC	G/T/C	G/C/F
Hit Ratio	0.053	0.072	0.067	0.120	0.115	0.086	0.095
ES Ratio	0.956	0.952	1.015	0.713	0.677	0.928	0.787
Traffic Light	Red	Red	Red	Red	Red	Red	Red
Ind. Cov.	6.258	5.992	3.542	0.509	0.779	3.459	1.492
	(0.012)	(0.014)	(0.060)	(0.475)	(0.377)	(0.063)	(0.222)
Unc. Cov.	19.105	34.129	30.132	80.901	75.749	46.951	206.436
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Joint Test	25.363	40.122	33.674	81.411	76.528	50.410	207.928
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Lower Tail	41.192	49.911	26.440	254.503	259.490	61.174	519.691
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Dynamic	DF	DC	DG	DT	DFC	DGC	DTC
Hit Ratio	0.081	0.029	0.053	0.048	0.062	0.038	0.043
ES Ratio	0.798	1.144	1.002	1.009	0.993	0.986	1.035
Traffic Light	Red	Yellow	Red	Red	Red	Yellow	Yellow
Ind. Cov.	4.203	7.440	6.258	7.451	4.770	4.924	8.837
	(0.040)	(0.006)	(0.012)	(0.006)	(0.029)	(0.026)	(0.003)
Unc. Cov.	42.547	4.910	19.105	15.795	26.288	9.827	12.694
	(0.000)	(0.027)	(0.000)	(0.000)	(0.000)	(0.002)	(0.000)
Joint Test	46.750	12.350	25.363	23.245	31.058	14.750	21.532
	(0.000)	(0.002)	(0.000)	(0.000)	(0.000)	(0.001)	(0.000)
Lower Tail	176.480	5.981	28.031	20.838	63.014	13.762	18.770
	(0.000)	(0.113)	(0.000)	(0.000)	(0.000)	(0.003)	(0.000)

This table reports the backtest evaluation results for the forecasting period from January 2007 until January 2011 for all copula models applied to the commodity futures index portfolio. The copula models are abbreviated with G (Gaussian), T (Student-t), F (Frank), C (Clayton), FC (Frank-Clayton mixture), GC (Gaussian-Clayton mixture) and TC (Student-t-Clayton mixture). The prefixed D denotes dynamic copulas. The confidence level of VaR is $\alpha = 99\%$. The hit ratio reflects the percentage of times when the portfolio return exceeds VaR(α). ES ratio shows whether the mean of the returns when VaR(α) is violated corresponds to the average expected shortfall in these weeks. The traffic light is the model classification of the Basel regulatory framework. Independence (unconditional) coverage is abbreviated with Ind. Cov. (Unc. Cov.). Joint Test is the test for conditional coverage. Lower Tail is the test of Christoffersen (2012) assessing the models' ability to forecast the lowest decile of the P&L distribution. P-values are given in parentheses.

Table 7.16
Financial crisis: Multi asset classes portfolio backtest results for $\alpha = 99\%$

Static	F	C	G	T	FC	GC	TC
Hit Ratio	0.086	0.038	0.062	0.057	0.191	0.196	0.196
ES Ratio	0.741	1.025	0.923	0.967	0.748	0.747	0.762
Traffic Light	Red	Yellow	Red	Red	Red	Red	Red
Ind. Cov.	0.268	1.104	0.046	0.137	0.050	0.157	0.237
	(0.605)	(0.293)	(0.830)	(0.711)	(0.823)	(0.692)	(0.626)
Unc. Cov.	46.951	9.827	26.288	22.608	167.731	174.070	174.070
	(0.000)	(0.002)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Joint Test	47.220	10.931	26.334	22.746	167.781	174.227	174.307
	(0.000)	(0.004)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Lower Tail	139.926	11.826	42.921	22.845	328.814	340.536	331.005
	(0.000)	(0.008)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Regime Switching	G/G	G/T	T/T	T/GC	G/TC	G/T/C	G/C/F
Hit Ratio	0.062	0.062	0.057	0.182	0.201	0.081	0.127
ES Ratio	0.901	0.911	0.894	0.765	0.775	0.897	0.782
Traffic Light	Red	Red	Red	Red	Red	Red	Red
Ind. Cov.	0.046	0.046	0.137	0.130	0.318	0.141	0.377
	(0.830)	(0.830)	(0.711)	(0.718)	(0.573)	(0.707)	(0.539)
Unc. Cov.	26.288	26.288	22.608	155.240	180.470	42.547	298.212
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Joint Test	26.334	26.334	22.746	155.370	180.788	42.688	298.589
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Lower Tail	43.060	43.748	25.358	292.533	336.211	59.567	600.885
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Dynamic	DF	DC	DG	DT	DFC	DGC	DTC
Hit Ratio	0.086	0.029	0.057	0.057	0.043	0.043	0.053
ES Ratio	0.783	0.946	0.893	0.951	0.875	0.899	0.975
Traffic Light	Red	Yellow	Red	Red	Yellow	Yellow	Red
Ind. Cov.	0.140	0.356	0.137	0.137	0.756	0.814	0.282
	(0.708)	(0.550)	(0.711)	(0.711)	(0.385)	(0.367)	(0.595)
Unc. Cov.	46.951	4.910	22.608	22.608	12.694	12.694	19.105
	(0.000)	(0.027)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Joint Test	47.092	5.266	22.746	22.746	13.450	13.509	19.387
	(0.000)	(0.072)	(0.000)	(0.000)	(0.001)	(0.001)	(0.000)
Lower Tail	125.660	7.677	23.042	22.034	25.498	22.778	12.247
	(0.000)	(0.053)	(0.000)	(0.000)	(0.000)	(0.000)	(0.007)

This table reports the backtest evaluation results for the forecasting period from January 2007 until January 2011 for all copula models applied to the multi asset classes index portfolio. The copula models are abbreviated with G (Gaussian), T (Student-t), F (Frank), C (Clayton), FC (Frank-Clayton mixture), GC (Gaussian-Clayton mixture) and TC (Student-t-Clayton mixture). The prefixed D denotes dynamic copulas. The confidence level of VaR is $\alpha = 99\%$. The hit ratio reflects the percentage of times when the portfolio return exceeds VaR(α). ES ratio shows whether the mean of the returns when VaR(α) is violated corresponds to the average expected shortfall in these weeks. The traffic light is the model classification of the Basel regulatory framework. Independence (unconditional) coverage is abbreviated with Ind. Cov. (Unc. Cov.). Joint Test is the test for conditional coverage. Lower Tail is the test of Christoffersen (2012) assessing the models' ability to forecast the lowest decile of the P&L distribution. P-values are given in parentheses.

Chapter 8

Conclusion

A proper specification of financial assets' multivariate distribution is essential to forecast the risk of a portfolio. A large body of empirical evidence suggests that the dependence structure between financial variables is neither symmetric nor time-stable. Consequentially, forecasts of portfolio risks which neglect asymmetries and time variation in the interdependence of the portfolio constituents might yield misleading results. This thesis investigated the importance of modeling time variation and asymmetries in the dependence structure to forecast the risk of a portfolio of international equity indices, a commodity futures index portfolio and a multi asset classes index portfolio.

The use of copula theory allowed to separately specify and estimate the dependence structure and the marginal distributions in order to model the conditional multivariate distribution function. To automatically select the best fitting univariate model for each of the heteroscedastic time series, the Akaike and the Bayesian information criteria were employed, which identified asymmetric volatility models to be most adequate for virtually all return series. Hansen's skewed-t distribution was used to account for excess kurtosis and skewness in the distribution of the univariate series' standardized residuals.

The analysis of the three portfolios' dependence structures in an in-sample setting documented asymmetric exceedance correlations among the demeaned returns which persist after filtering the series with the asymmetric GARCH models. This points to a limited role of the marginal models in capturing multivariate asymmetries and highlights the need for copula models which are capable of accounting for asymmetric dependencies. To amplify the selection of asymmetric dependence models for multivariate applications, mixture copulas were constructed.

The level of dependence as well as the amount of lower tail dependence was found to vary substantially over the sample period for all three portfolios. As a first approach to model this time variation in the interdependence of the portfolio constituents, regime switching copula models were introduced. To enhance the flexibility of this set-up, multivariate mixture copulas were employed to characterize states and the model was extended to three regimes. As a second approach to account for time-instable dependencies, fully dynamic copula models were presented. Enhancing the dynamic Student-t copula's capability to adapt both level of dependence and strength of tail dependence, its degrees of freedom parameter was also allowed to vary over time. Finally, the dynamic copulas were combined into dynamic mixture structures.

The in-sample comparison of the time-varying copulas with their static counterparts exhibited that for each of the three portfolios, the best fitting copulas are dynamic models which allow for tail dependence. In particular, the dynamic Student-t copula and the dynamic Student-t-Clayton mixture copula were found to have a superior in-sample fit.

A comprehensive backtest of the models' out-of-sample portfolio return forecasts was conducted. Since the marginals are identical for all copula models, their differences in forecast accuracy are solely attributable to the differences of the dependence models. The results revealed, that the choice of copula model has a large impact on the forecast accuracy of the portfolio profit and loss distribution's lowest decile. The differences are particularly pronounced for risk measures on the 99% confidence level. Among the static models, the Clayton copula yields by far the best results for all three portfolios. In contrast, the symmetric Frank copula model with asymptotically independent tails was found to be inappropriate to characterize any of the portfolios' dependence structures. This underpins the importance of modeling interdependence asymmetries and accounting for lower tail dependence.

The comparison of the static Gaussian with the multivariate normal benchmark model showed, that allowing for fat tails and asymmetries in the marginal models improves the forecast accuracy of the models for the equity and the multi asset classes portfolio, but has virtually no influence for the commodity portfolio.

Failing to model time variation in the multivariate mixture copulas results in entirely inaccurate forecasts. Accounting for time instable dependencies by employing the dynamic versions of the Student-t-Clayton mixture and the Gaussian-Clayton mixture copulas, however, yields the second and third best VaR(99%) forecasts for the equity index portfolio. Evidence from the in-sample analysis suggests, that the crucial part is to account for time variation in the mixture's copula parameters, while a static mixture weight might be sufficient.

Accounting for time variation by switching between different static copulas as in the regime switching models led to an improvement of some models' forecasts. Particularly, allowing for a second Gaussian state as in the Gaussian/Gaussian regime switching copula improves the forecast accuracy for all three portfolios on almost all VaR confidence levels. Furthermore, the density test results indicate an increase in accuracy of the entire forecasted return distribution when allowing for two instead of only one Gaussian copula regime.

Modeling one of two states in the regime switching framework with a static mixture copula does not provide satisfactory results due to the inaccuracy of the mixture copula with static parameters. Furthermore, adding a third state to the regime switching copula model yields no improvement of the forecast accuracy compared to the two-regime models. To the contrary, extending the two-state Gaussian/Student-t model by adding the Clayton as a separate regime yields worse forecast results for all portfolios. The accuracy of this three-state model is also inferior to the two-state models where the Clayton is mixed with either one of the elliptical copula regimes as in the Student-t/Gaussian-Clayton mixture and Gaussian/Student-t-Clayton mixture models. This evidence suggests, that multivariate regime switching copulas are best set up with two states each consisting of one standalone copula.

The fully dynamic copulas have generally been found to produce forecasts of superior accuracy compared to both static and regime switching copula models. Specifically, the predictive power of the dynamic Student-t and all dynamic mixture copula models is stronger compared to their static versions for each of the portfolios on all three significance levels. The best forecasts for the lower tail of the profit and loss distribution for all three portfolios are produced by the dynamic Clayton copula model, which highlights the importance of modeling both positive lower tail dependence and time variation in the dependence structure. This stands in contrast to the in-sample model fit results, where the dynamic Clayton copula was ranked in the rearmost positions. Generally, the goodness of in-sample fit was not found to provide dependable indications of predictive power. However, the in-sample ranking did point to the superiority of the time varying models compared to the static copulas.

Finally, the forecast accuracy of the models during the last financial crisis was analyzed, which underpinned the superiority of the Clayton copula model. For the equity index portfolio, the multivariate dynamic Clayton copula accurately forecasted the VaR(99%) with a hit ratio of 1% followed by the dynamic Student-t-Clayton mixture copula with 1.4%. Their superiority also reflected in their hit ratios showing the least reaction to the outbreak of the financial crisis, remaining closest to the expected value. The employment of the dynamic Clayton copula model instead of the multivariate normal model would have scaled down the number of VaR(99%) violations for the equity index portfolio during the financial crisis by more than 82%.

The most accurate forecasts for the commodity futures index portfolio during the financial crisis were produced by the static Clayton copula, followed by the dynamic Clayton model. Both yield the single most accurate forecasts among all models for the commodity data at all times during the crisis. Their hit ratios only displayed a minor increase in reaction to the outbreak of the global financial crisis, while the hit ratios of most other models surged drastically.

The dynamic Clayton copula model's forecasts for the risks of the multi asset classes portfolio are by far the most accurate during the financial crisis. It is further the only model to pass the Lower Tail test for an accurate prediction of the profit and loss distribution's lowest decile. Using a static Gaussian copula for the multi asset classes portfolio during the last financial crisis results in more than twice as many VaR(99%) violations as with the dynamic Clayton model. This emphasizes the crucial importance of accounting for time variation and asymmetries in the dependence structure, which is underpinned by the general finding that asymmetric dynamic copula models dominate the top rankings of both crisis and overall forecast performance.

Several aspects, which are beyond the scope of this thesis, deserve to become a topic for future research. While the employed regime switching models allow to account for shifts in the dependence structure, the transition probabilities were assumed to remain constant. If time varying transition probabilities would increase the forecast accuracy and, given they do, how their performance would compare to the fully dynamic copulas is left for future research to determine. Also, if an amalgamation of the two presented approaches to account for time varying dependencies to attain a multivariate regime switching copula with fully dynamic states would yield any improvement, is to be investigated.

Utilizing three different portfolios in the empirical analysis demonstrated the models' abilities of capturing diverse dependence characteristics. However, the application to a wider selection of portfolios would further contribute to the robustness of the findings.

The eight dimensional models employed in this document are a significant step closer to cover real world portfolio sizes compared to bivariate models. Given the scalability of the models developed in this thesis, they are capable of handling portfolios of arbitrary dimensions. However, the impact of the portfolio size on the models' performance would be interesting to explore. Specifically, as the multivariate Clayton copula model has been found to yield risk forecasts of superior accuracy in static and particularly in dynamic form, its capability of capturing the dependencies of very high dimensional portfolios with its only copula parameter θ_C should be subject of future research.

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Experience

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